Compactness criterion in weighted Musielak-Orlicz sequence spaces and applications

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Abstract. We study relatively compact sets in weighted Musielak-Orlicz sequence spaces. The characterization of such sets is given in the case of weighted Musielak-Orlicz sequence spaces. As an application, we establish a necessary and sufficient condition on weight functions for the compactness of the discrete Hardy operator on weighted Musielak-Orlicz sequence spaces. In particular, we get similar results for the dual operator of the discrete Hardy operator. The results are illustrated by a number of corollaries.

Keywords: discrete Hardy operator, dual operator, weight function, weighted Musielak-Orlicz sequence spaces, embedding theorems, compactness.

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1 INTRODUCTION

Compactness results in the usual Lebesgue spaces are often vital in existence proofs for nonlinear partial differential equations. A necessary and sufficient condition for a subset of usual Lebesgue spaces to be compact is given in what is often called the Kolmogorov compactness theorem, or Frechet-Kolmogorov theorem. Furthemore we trace out the historical roots of the Kolmogorov compactness theorem, which originated in [20].

Let us mention some generalizations of the Riesz-Kolmogorov theorem. H. Rafeiro characterized precompact sets in variable exponent Lebesgue spaces on Euclidean spaces in [33]. The further characterizations of the relatively compact sets are studied in variable Lebesgue spaces on metric measure spaces. In [31] Kolmogorov theorem for $p = 2$ in terms of the Fourier transform is given (see also [13] and [14]).

In the literature, many authors considered the following standard form of discrete Hardy's inequality in discrete Lebesgue space with a constant exponent. Let $p > 1$,

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$p' = \dfrac{p}{p-1}$ and let $\{x_k\}_{k=1}^{\infty}$ be an arbitrary sequence of non-negative real numbers. Then

$$\left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}x_k\right)^p\right)^{\frac{1}{p}} \leq p'\left(\sum_{n=1}^{\infty}x_k^p\right)^{\frac{1}{p}}. \tag{1.1}$$

The constant p' in (1.1) is sharp. The first result towards a weight characterization of (1.1) was proved by K.F. Andersen and H.P. Heinig in [1]. A full weight characterization of discrete Hardy inequality was proved by G. Bennett in [6] and so on. It is well known that an essential development for Hardy-type inequalities in the discrete case is given by C.A. Okpoti, L.-E. Persson, and A. Wedestig in [30]. There has been a similar development for Hardy-type inequalities in the discrete case given by A.A. Kalybay, L-E. Persson and A.M. Temirkhanova [18]. For a history of Hardy type inequalities on the cones of monotone functions and sequences and for references to related results, we refer to the papers of M.L. Gol'dman [12] and A. Gogatishvili and V.D. Stepanov [10], [11].

It is well known that the variable Lebesgue sequence space was first studied by W. Orlicz [31] in 1931. In [31], Hölder's inequality for variable Lebesgue sequence space was proved. W. Orlicz also considered the variable Lebesgue space on the real line, and proved the Hölder inequality in this setting. However, this paper is essentially the only contribution by Orlicz to the study of the variable Lebesgue spaces. The Musielak-Orlicz space was originally introduced by J. Musielak and W. Orlicz in [27]. The Musielak-Orlicz spaces are closely connected with modular spaces. Basing on the modular theory, Musielak and Orlicz founded in 1959 a theory of Musielak-Orlicz spaces [27]. These spaces have been studied for almost sixty years and there are a large set of applications of such spaces in various fields of analysis. They were also generalized in many directions. For example, many authors have considered their generalizations to the vector-valued functions and spaces generated by families of Musielak-Orlicz modulars. Later, a more explicit version of these spaces, namely modular function spaces, was investigated by many mathematicians. The study of variable exponent spaces has been stimulated by problems of elasticity, fluid dynamics, calculus of variations and differential equations with non-standard growth conditions (see, [8] and [19]). We are strongly convinced that these more general spaces will become increasingly important in the modeling of modern materials. Different characterizations of the mapping properties, such as boundedness and compactness of continuous version of the Hardy operator in the variable Lebesgue spaces were studied in [3]-[5], [7], [9], [17], [21]-[26], [29] and so on. We especially note that the characterization of the compactness of the continuous version of the Hardy operator on weighted Lebesgue spaces was proved in [9], [23] and [26]. A similar problem for the discrete Hardy operator on weighted variable Lebesgue sequence spaces was considered in [4].

In this paper, we obtain a necessary and sufficient condition of the pre-compact sets in the Musielak-Orlicz sequence spaces. We find sufficient conditions on the generalized

Φ-functions to guarantee continuous embeddings between weighted Musielak-Orlicz sequence spaces. Depending on the sufficient conditions on the generalized Φ-functions, the weighted Musielak-Orlicz sequence spaces coincide. As applications, we establish criteria for the compactness of the discrete Hardy operator and its dual operator defined on weighted Musielak-Orlicz sequence spaces. The results are relied on the paper [16].

The remainder of the paper is structured as follows. Section 2 contains some preliminaries along with the standard ingredients used in the proofs. We also recall standard results from the theory of Musielak-Orlicz sequence spaces. In particular, in Section 2 we proved continuous embeddings between weighted Musielak-Orlicz sequence spaces. Our principal assertions are formulated and proved in Section 3. In Section 3 we obtained a necessary and sufficient conditions on subsets of Musielak-Orlicz spaces in order to be relatively compact. In the next section 4, we establish a necessary and sufficient condition on weight functions for the compactness of the discrete Hardy operator on weighted Musielak-Orlicz sequence spaces. A similar problem is also being studied for the dual operator of the discrete Hardy operator.

2 Preliminaries

Let $\mathbb{N}$ be the set of natural number and let $p = \{p_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that $1 \leq \underline{p} \leq p_n \leq \overline{p} < \infty$, where $\underline{p} = \inf\limits_{n\geq 1} p_n$ and $\overline{p} = \sup\limits_{n\geq 1} p_n$. The conjugate exponent function of p_n is defined as $\dfrac{1}{p_n} + \dfrac{1}{p_n'} = 1$ for all $n \in \mathbb{N}$. Denote by χ_A the characteristic function of $A \subset \mathbb{N}$. Throughout this paper $\underline{p}' = \dfrac{\underline{p}}{\underline{p} - 1}$. Let $w = \{w_n\}_{n=1}^{\infty}$ be a sequence of non-negative numbers, i.e., w is a weight function defined on $\mathbb{N}$.

Next we give the definition of Musielak-Orlicz space.

Definition 1. *[8, 27] A convex, left-continuous function $\varphi : [0, \infty) \mapsto [0, \infty]$ with $\varphi(0) = 0$, $\lim\limits_{t\to+0} \varphi(t) = 0$ and $\lim\limits_{t\to\infty} \varphi(t) = \infty$ is called a Φ-function. It is called positive if $\varphi(t) > 0$ for all $t > 0$.*

Definition 2. *[8, 27] Let (A, Σ, μ) be a σ-finite, complete measure space. A real function $\varphi : A \times [0, \infty) \mapsto [0, \infty]$ is called a generalized Φ-function on (A, Σ, μ) if it satisfies the following conditions:*

(a) $\varphi(x, \cdot)$ *is a Φ-function for all* $x \in A$.

(b) $\varphi(\cdot, t)$ *is measurable for all* $t \geq 0$.

If φ is a generalized Φ-function on (A, Σ, μ), we shortly write $\varphi \in \Phi(A, \mu)$.

We observe that any generalized Φ-function $\varphi(x,\cdot)$ is non-decreasing on $[0,\infty)$ for all $x \in A$.

Remark 1. *Let $\varphi \in \Phi(A,\mu)$. Then, as a convex function, $\varphi(x,\cdot)$ is continuous if and only if $\varphi(x,\cdot)$ is finite on $[0,\infty)$ for all $x \in A$ (see, [8]).*

Definition 3. *[8, 27] Let $\varphi \in \Phi(A,\mu)$ and let ρ_φ be given by*

$$\rho_\varphi(f) := \int_A \varphi(x, |f(x)|)\, d\mu(x) \ \ \text{for all } f \in L_0(\Omega).$$

We put $L_\varphi(A,\mu) = \{f \in L_0(A,\mu) : \ \rho_\varphi(\lambda_0 f) < \infty \text{ for some } \lambda_0 > 0\}$ and

$$\|f\|_{L_\varphi(A,\mu)} = \inf\left\{\lambda > 0 : \quad \rho_\varphi\left(\frac{f}{\lambda}\right) \le 1\right\}$$

The space $L_\varphi(A,\mu)$ is called the Musielak-Orlicz space.

Let μ be the counting measure on $\mathbb{N}$ and let $\ell_0(\mathbb{N})$ be the space of all sequences $\mathbf{x} = \{x_n\}_{n=1}^\infty$. Let $w = \{w_n\}_{n=1}^\infty$ be a weight function defined on $\mathbb{N}$. Then we have the definition of weighted Musielak-Orlicz sequence space $\ell_{\varphi,w}(\mathbb{N})$. We put

$$\ell_{\varphi,w}(\mathbb{N}) = \left\{\mathbf{x} \in \ell_0(\mathbb{N}) : \ \sum_{n=1}^\infty \varphi\left(n, \frac{|x_n|\, w_n}{\lambda_0}\right) < \infty \text{ for some } \lambda_0 > 0\right\}.$$

The functional

$$\|\mathbf{x}\|_{\ell_{\varphi,w}(\mathbb{N})} = \inf\left\{\lambda > 0 : \ \sum_{n=1}^\infty \varphi\left(n, \frac{|x_n|\, w_n}{\lambda}\right) \le 1\right\}.$$

is defined a norm in $\ell_{\varphi,w}(\mathbb{N})$.

Suppose that $1 \le p_n < \infty$, $n \in \mathbb{N}$. In case $\varphi(n,t) = t^{p_n}$, $\ell_{\varphi,w}(\mathbb{N})$ is the variable exponent weighted Lebesgue sequence space $\ell_w^{p_n}(\mathbb{N})$. There are some examples on the weighted Musielak-Orlicz sequence space $\ell_{\varphi,w}(\mathbb{N})$. We give some examples on Φ-functions:

$\varphi_1(n,t) = t^{p_n}\log(1+t)$, $\varphi_2(n,t) = t^{p_n} + a_n\, t^{q_n}$, $1 \le q_n < \infty$ and $a_n \ge 0$, $\varphi_3(n,t) = e^t\, t^{p_n}$ for all $n \in \mathbb{N}$ and so on.

An ε-cover of a metric space is a cover of the space consisting of sets of diameter at most ε. A metric space is called totally bounded if it admits a finite ε-cover for all $\varepsilon > 0$. It is well known that a metric space is compact if and only if it is complete and totally bounded(or precompact) (see [16]). Since we are interested in compactness results for subsets of Banach spaces, we concentrate our attention on total boundedness.

Next, we need the following lemmas.

Lemma 1. *[16] Let (X, d_X) be a metric space. Suppose that for any $\varepsilon > 0$ there exists $\delta > 0$, metric space (W, d_W) and function $\Phi : X \mapsto W$ satisfying the following conditions:*

(a) $\Phi[X]$ is totally bounded,

(b) if $x, y \in X$ and $d_W(\Phi(x), \Phi(y)) < \delta$, then $d_X(x, y) < \varepsilon$.

Then X is totally bounded.

Lemma 2. *Let μ be the counting measure on $\mathbb{N}$ and let φ, $\psi \in \Phi(\mathbb{N}, \mu)$. Suppose that $v = \{v_n\}_{n=1}^{\infty}$ is a weight function defined on $\mathbb{N}$. Suppose that $\psi(n, t)$ is finite on $[0, \infty)$ for all $n \in \mathbb{N}$. Suppose that there exists $C > 0$ such that*

$$\varphi\left(n, \frac{t}{C}\right) \le \psi(n, t) \tag{2.1}$$

for all $n \in \mathbb{N}$ and $0 \le t \le \sup\limits_{n\in\mathbb{N}} \psi^{-1}(n, 1) < \infty$.

Then $\ell_{\psi,v}(\mathbb{N}) \hookrightarrow \ell_{\varphi,v}(\mathbb{N})$ and the following inequality holds

$$\|\mathbf{x}\|_{\ell_{\varphi,v}(\mathbb{N})} \le C \, \|\mathbf{x}\|_{\ell_{\psi,v}(\mathbb{N})}.$$

Proof. Let $\mathbf{x} \in \ell_{\psi,v}(\mathbb{N})$ and let $\|\mathbf{x}\|_{\ell_{\psi,v}(\mathbb{N})} \le 1$. For a generalized Φ-function $\psi \in \Phi(\mathbb{N}, \mu)$ we define $\psi^{-1}(n, 1) = \sup\{t \ge 0 : \;\; \psi(n, t) \le 1\}$. So, we have that

$$\psi\left(n, \frac{|x_n| \, v_n}{\|\mathbf{x}\|_{\ell_{\psi,v}(\mathbb{N})}}\right) \le \sum_{k=1}^{\infty} \psi\left(k, \frac{|x_k| \, v_k}{\|\mathbf{x}\|_{\ell_{\psi,v}(\mathbb{N})}}\right) \le 1 \quad \text{for all } n \in \mathbb{N}.$$

Consequently, $\psi\left(n, \frac{|x_n| v_n}{\|\mathbf{x}\|_{\ell_{\psi,v}(\mathbb{N})}}\right)$ is a bounded function for all $n \in \mathbb{N}$. Next, one has $\dfrac{|x_n| \, v_n}{\|\mathbf{x}\|_{\ell_{\psi,v}(\mathbb{N})}} \le \psi^{-1}(n, 1)$ and therefore $|x_n| \, v_n \le \sup\limits_{n\in\mathbb{N}} \psi^{-1}(n, 1) < \infty$. Thus, by (2.1), we have

$$\sum_{k=1}^{\infty} \varphi\left(k, \frac{|x_k| \, v_k}{C \, \|\mathbf{x}\|_{\ell_{\psi,v}(\mathbb{N})}}\right) \le \sum_{k=1}^{\infty} \psi\left(k, \frac{|x_k| \, v_k}{\|\mathbf{x}\|_{\ell_{\psi,v}(\mathbb{N})}}\right) \le 1 \quad \text{for all } n \in \mathbb{N}.$$

This completes the proof of Lemma 2.

Lemma 3. *Let μ be the counting measure on $\mathbb{N}$ and let φ, $\psi \in \Phi(\mathbb{N}, \mu)$. Suppose that $v = \{v_n\}_{n=1}^{\infty}$ is a weight function defined on $\mathbb{N}$. Suppose that there exist $C > 0$ and $\mathbf{h} = \{h_n\}_{n=1}^{\infty} \in \ell_1(\mathbb{N})$ with $\|\mathbf{h}\|_{\ell_1} \le 1$ such that*

$$\psi\left(n, \frac{t}{C}\right) \le \varphi(n, t) + h_n \tag{2.2}$$

for all $n \in \mathbb{N}$ *and* $t \geq 0$.

Then $\ell_{\varphi,v}(\mathbb{N}) \hookrightarrow \ell_{\psi,v}(\mathbb{N})$ *and the following inequality holds*

$$\|\mathbf{x}\|_{\ell_{\psi,v}(\mathbb{N})} \leq 2C \|\mathbf{x}\|_{\ell_{\varphi,v}(\mathbb{N})}.$$

Proof. Let $\mathbf{x} \in \ell_{\varphi,v}(\mathbb{N})$ and let $\|\mathbf{x}\|_{\ell_{\varphi,v}(\mathbb{N})} \leq 1$. By the unit ball property we have that $\sum_{n=1}^{\infty} \varphi(n, |x_n| v_n) \leq 1$. So, by (2.2), we have

$$\sum_{n=1}^{\infty} \psi\left(n, \frac{|x_n| v_n}{2C \|\mathbf{x}\|_{\ell_{\varphi,v}}}\right) \leq \frac{1}{2} \sum_{n=1}^{\infty} \psi\left(n, \frac{|x_n| v_n}{C \|\mathbf{x}\|_{\ell_{\varphi,v}}}\right)$$

$$\leq \frac{1}{2}\left(\sum_{n=1}^{\infty} \varphi\left(n, \frac{|x_n| v_n}{\|\mathbf{x}\|_{\ell_{\varphi,v}}}\right) + \sum_{n=1}^{\infty} |h_n|\right) \leq 1.$$

Thus, by definition 3, one has $\|\mathbf{x}\|_{\ell_{\psi,v}} \leq 2C \|\mathbf{x}\|_{\ell_{\varphi,v}}$.

This completes the proof of Lemma 3.

From Lemma 2 and Lemma 3, we have the following theorem.

Theorem 1. *Let* μ *be the counting measure on* $\mathbb{N}$ *and let* φ, $\psi \in \Phi(\mathbb{N}, \mu)$. *Suppose that* $v = \{v_n\}_{n=1}^{\infty}$ *is a weight function defined on* $\mathbb{N}$. *Let* $\psi(n,t)$ *be finite on* $[0, \infty)$ *for all* $n \in \mathbb{N}$. *Suppose that there exist constants* $C_1, C_2 > 0$ *such that conditions (2.1) and (2.2) of Lemma 2 and Lemma 3 are satisfied, respectively.*

Then $\ell_{\varphi,v}(\mathbb{N}) = \ell_{\psi,v}(\mathbb{N})$ *and the following inequality holds*

$$\frac{1}{C_1} \|\mathbf{x}\|_{\ell_{\varphi,v}(\mathbb{N})} \leq \|\mathbf{x}\|_{\ell_{\psi,v}(\mathbb{N})} \leq 2C_2 \|\mathbf{x}\|_{\ell_{\varphi,v}(\mathbb{N})}.$$

Let $1 \leq p < \infty$ and $\varphi \in \Phi(\mathbb{N}, \mu)$. Suppose that there exists $C_1 > 0$ such that

$$\varphi\left(n, \frac{t}{C_1}\right) \leq t^p \tag{2.3}$$

for all $n \in \mathbb{N}$ and $0 \leq t \leq 1$. Also, suppose that there exist $C_2 > 0$ and $\mathbf{h} = \{h_n\}_{n=1}^{\infty} \in \ell_1(\mathbb{N})$ with $\|\mathbf{h}\|_{\ell_1} \leq 1$ such that

$$\left(\frac{t}{C_2}\right)^p \leq \varphi(n,t) + h_n \tag{2.4}$$

for all $n \in \mathbb{N}$ and $t \geq 0$.

Corollary 1. *Let $1 \le p < \infty$ and let $\psi(n,t) = t^p$ and $\varphi \in \Phi(\mathbb{N},\mu)$. Suppose that $v = \{v_n\}_{n=1}^{\infty}$ is a weight function defined on $\mathbb{N}$. Let us assume that there exist constants $C_1, C_2 > 0$ such that conditions (2.3) and (2.4) are satisfied, respectively.*

Then $\ell_{\varphi,v}(\mathbb{N}) = \ell_v^p(\mathbb{N})$ and the following inequality holds

$$\frac{1}{C_1}\,\|\mathbf{x}\|_{\ell_{\varphi,v}(\mathbb{N})} \le \|\mathbf{x}\|_{\ell_v^p(\mathbb{N})} \le 2C_2\,\|\mathbf{x}\|_{\ell_{\varphi,v}(\mathbb{N})}.$$

Corollary 2. *Let $1 \le p_n \le q_n \le \overline{q} < \infty$ and let $\Omega_1 = \{n \in \mathbb{N} :\ p_n < q_n\}$ and $\Omega_2 = \{n \in \mathbb{N} :\ p_n = q_n\}$. Assume that $\dfrac{1}{r_n} = \dfrac{1}{p_n} - \dfrac{1}{q_n}$ and let $\|1\|_{\ell^{r_n}(\mathbb{N})} < \infty$. Let $\psi(n,t) = t^{p_n}$ and $\varphi(n,t) = t^{q_n}$. Suppose that $v = \{v_n\}_{n=1}^{\infty}$ is a weight function defined on $\mathbb{N}$.*

Then $\ell_v^{p_n}(\mathbb{N}) = \ell_v^{q_n}(\mathbb{N})$ and the following inequality holds

$$\|\mathbf{x}\|_{\ell_v^{p_n}(\mathbb{N})} \le \|\mathbf{x}\|_{\ell_v^{q_n}(\mathbb{N})} \le \Big(A + B + \|\chi_{\Omega_2}\|_{\ell^\infty(\mathbb{N})}\Big)^{\frac{1}{p}}\ \|1\|_{\ell^{r_n}(\mathbb{N})}\,\|\mathbf{x}\|_{\ell_v^{p_n}(\mathbb{N})},$$

where $A = \sup\limits_{n\in\Omega_1} \dfrac{p_n}{q_n}$ and $B = \sup\limits_{n\in\Omega_1} \dfrac{q_n - p_n}{q_n}$.

Remark 2. *We observe that the embeddings between variable Lebesgue sequence spaces was proved by A. Nekvinda in [28]. The next step in the development of the embeddings between variable Lebesgue spaces with measure was studied in [2] and in the monograph [8].*

We need the following lemma.

Lemma 4. *Let $1 \le q < \infty$, $B_n = \displaystyle\sum_{k=n}^{\infty} \left(\frac{\omega_k}{k}\right)^q$, and let $\omega_n > 0$ for all $n \in \mathbb{N}$. Then the following statements holds:*

(a) If $0 < \gamma < 1$, then

$$\gamma\, B_n^{\gamma-1} \left(\frac{\omega_n}{n}\right)^q \le B_n^\gamma - B_{n+1}^\gamma \le \gamma\, B_{n+1}^{\gamma-1} \left(\frac{\omega_n}{n}\right)^q \quad \text{for all } n \in \mathbb{N}.$$

(b) If $\gamma < 0$ or $\gamma \ge 1$, then

$$\gamma\, B_{n+1}^{\gamma-1} \left(\frac{\omega_n}{n}\right)^q \le B_n^\gamma - B_{n+1}^\gamma \le \gamma\, B_n^{\gamma-1} \left(\frac{\omega_n}{n}\right)^q \quad \text{for all } n \in \mathbb{N}.$$

Proof. Let $f(x) = x^\gamma$ and let $0 < \gamma < 1$. Then by mean value theorem, one has

$$\gamma\, x^{\gamma-1}(x-y) \le x^\gamma - y^\gamma \le \gamma\, y^{\gamma-1}(x-y) \ \text{ for } \ 0 < y < x. \tag{2.5}$$

By inequality (2.5), we get

$$\gamma\, B_n^{\gamma-1}\,(B_n - B_{n+1}) \le B_n^{\gamma} - B_{n+1}^{\gamma} \le \gamma\, B_{n+1}^{\gamma-1}\,(B_n - B_{n+1})$$

So, we have

$$B_n^{\gamma-1}\left(\frac{\omega_n}{n}\right)^q \le \frac{1}{\gamma}\left(B_n^{\gamma} - B_{n+1}^{\gamma}\right) \le B_{n+1}^{\gamma-1}\left(\frac{\omega_n}{n}\right)^q.$$

It is obvious that

$$\sum_{m=n}^{\infty} B_m^{\gamma-1}\left(\frac{\omega_m}{m}\right)^q \le \frac{1}{\gamma}\sum_{m=n}^{\infty}\left(B_m^{\gamma} - B_{m+1}^{\gamma}\right) = \frac{1}{\gamma} B_n^{\gamma}. \tag{2.6}$$

In the similar way we can prove statements (b). Therefore we omit the proof.

3 Precompactness in $\ell_{\varphi,v}(\mathbb{N})$

In this section we give a characterization of relatively compact sets in $\ell_{\varphi,v}(\mathbb{N})$.

Theorem 2. *Suppose that $v = \{v_n\}_{n=1}^{\infty}$ is a weight function defined on $\mathbb{N}$. Let $\mathcal{F} \subset \ell_{\varphi,v}(\mathbb{N})$ and let $\varphi(n,t) < \infty$ for all $n \in \mathbb{N}$ and $t > 0$. Then the set $\mathcal{F} = \{a^i\}_{i\in I}$ is precompact in $\ell_{\varphi,v}(\mathbb{N})$ if and only if the following conditions are satisfied:*

(i) $\mathcal{F}$ is bounded, i.e. there exists $M > 0$ such that for every $a^i \in \mathcal{F}$

$$\left\|a^i\right\|_{\ell_{\varphi,v}(\mathbb{N})} \le M,$$

(ii) for every $\varepsilon > 0$ there exists some $n \in \mathbb{N}$ such that for every $a^i \in \mathcal{F}$

$$\left\|a_k^i\right\|_{\ell_{\varphi,v}(k>n)} < \varepsilon.$$

Proof. Let $\mathcal{F} \subset \ell_{\varphi,v}(\mathbb{N})$ and let conditions $(i)-(ii)$ be satisfied. Let us fix $\varepsilon > 0$. We choose $n \in \mathbb{N}$ from condition (ii) for $\frac{\varepsilon}{3}$ and define a mapping $\Phi : \mathcal{F} \mapsto \mathbb{R}^n$ by

$$\Phi\left(a^i\right) = \left(a_1^i, \ldots, a_n^i\right).$$

We observe that boundedness of $\mathcal{F}$ implies boundedness $\Phi[\mathcal{F}]$. Moreover, $\Phi[\mathcal{F}]$ is totally bounded since $\Phi[\mathcal{F}] \subset \mathbb{R}^n$.

Let $a^i = \{a_k^i\}_{k=1}^{\infty}$, $b^i = \{b_k^i\}_{k=1}^{\infty} \in \mathcal{F}$ with

$$\left|\Phi\left(a^i\right) - \Phi\left(b^i\right)\right|_{\phi,v} = \left\|a_k^i - b_k^i\right\|_{\ell_{\varphi,v}(k\le n)} < \frac{\varepsilon}{3}.$$

So, we have

$$\left\|a^i - b^i\right\|_{\ell_{\varphi,v}(\mathbb{N})} \le \left\|a^i_k - b^i_k\right\|_{\ell_{\varphi,v}(k\le n)} + \left\|a^i_k - b^i_k\right\|_{\ell_{\varphi,v}(k>n)}$$

$$\le \left|\Phi\left(a^i\right) - \Phi\left(b^i\right)\right|_{\phi,v} + \left\|a^i_k\right\|_{\ell_{\varphi,v}(k>n)} + \left\|b^i_k\right\|_{\ell_{\varphi,v}(k>n)} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Thus, by Lemma 1, $\mathcal{F}$ is totally bounded.

For the converse, suppose that the family F is a totally bounded subset of $\ell_{\varphi,v}(\mathbb{N})$. We shall show that conditions (i) and (ii) are satisfied.

(i) The existence of a finite ε-cover for $\mathcal{F}$, for any $\varepsilon > 0$, clearly implies the boundedness of $\mathcal{F}$.

(ii) We fix $\varepsilon > 0$, and we choose $\{a^1, a^2, \ldots, a^m\} \subset \mathcal{F}$ an $\frac{\varepsilon}{2}$-net of $\mathcal{F}$. For any $j = 1, \ldots, m$ we choose $n_j \in \mathbb{N}$ satisfying inequality

$$\left\|a^j_k\right\|_{\ell_{\varphi,v}(k>n_j+1)} \le \frac{\varepsilon}{2}.$$

Let $n = \max\{n_j :\ j = 1, \ldots, m\}$. For any fixed $a^0 \in \mathcal{F}$ there exists a^j from the such that

$$\left\|a^0_k - a^j_k\right\|_{\ell_{\varphi,v}(\mathbb{N})} \le \frac{\varepsilon}{2}.$$

Thus, we have

$$\left\|a^0_k\right\|_{\ell_{\varphi,v}(k>n)} \le \left\|a^0_k - a^j_k\right\|_{\ell_{\varphi,v}(k\le n)} + \left\|a^j_k\right\|_{\ell_{\varphi,v}(k>n)} < \varepsilon.$$

This completes the proof of Theorem 2.

Corollary 3. *[3] Suppose that $v = \{v_n\}_{n=1}^{\infty}$ is a weight function defined on $\mathbb{N}$. Let $1 \le p_n \le \overline{p} < \infty$ and $\varphi(n,t) = t^{p_n}$ for all $n \in \mathbb{N}$ and $t \ge 0$. Then the set $\mathcal{F} = \{a^i\}_{i\in I} \subset \ell_v^{p_n}(\mathbb{N})$ is precompact in $\ell_v^{p_n}(\mathbb{N})$ if and only if the following conditions are satisfied:*

(i) $\mathcal{F}$ is bounded, i.e. there exists $M > 0$ such that for every $a^i \in \mathcal{F}$

$$\sum_{n=1}^{\infty} \left(\left|a^i_n\right| v_n\right)^{p_n} \le M,$$

(ii) for every $\varepsilon > 0$ there exists some $k \in \mathbb{N}$ such that for every $a^i \in \mathcal{F}$

$$\sum_{n=k+1}^{\infty} \left(\left|a^i_n\right| v_n\right)^{p_n} < \varepsilon.$$

Remark 3. *We observe that Corollary 3 in the case $v = 1$ was proved in [15]. Also, for $p_n = p = const$, $n \in \mathbb{N}$ and $v = 1$ Corollary 3 was proved in [16].*

4 Applications

Let $\{x_n\}_{n=1}^{\infty} \in \ell_{v_n}^{p_n}(\mathbb{N})$ be an arbitrary sequence of real numbers. Suppose that $H_n = \dfrac{1}{n}\sum_{k=1}^{n} x_k$ and $H_n^{\star} = \sum_{k=n}^{\infty} \dfrac{x_k}{k}$.

In this section we give a compactness characterization of discrete Hardy operator from $\ell_{\phi,v}(\mathbb{N})$ into $\ell_{\psi,w}(\mathbb{N})$.

First we give the boundedness characterization of discrete Hardy operator on weighted Musielak-Orlicz sequence spaces.

Theorem 3. *[5] Let $1 < p < \infty$ and $0 < \alpha < 1$. Let μ be the counting measure on $\mathbb{N}$ and let $\psi\left(n, t^{\frac{1}{p}}\right) \in \Phi(\mathbb{N}, \mu)$. Let $\varphi \in \Phi(\mathbb{N}, \mu)$ be function satisfies conditions (2.3) and (2.4), respectively. Suppose that $v = \{v_n\}_{n=1}^{\infty}$ and $\omega = \{\omega_n\}_{n=1}^{\infty}$ are sequences of positive numbers. Then the inequality*

$$\|H_n\|_{\ell_{\psi,\omega}(\mathbb{N})} \leq C\|\mathbf{x}\|_{\ell_{\varphi,v}(\mathbb{N})} \tag{4.1}$$

holds if and only if

$$R(\alpha) = \sup_{k\geq 1}\left(\sum_{n=1}^{k} v_n^{-p'}\right)^{\frac{\alpha}{p'}} \left\| \frac{\left(\sum\limits_{m=1}^{n} v_m^{-p'}\right)^{\frac{1-\alpha}{p'}}}{n} \right\|_{\ell_{\psi,\omega}(n\geq k)} < \infty. \tag{4.2}$$

Moreover, if $C > 0$ is the best possible constant in (4.1), then

$$\frac{1}{C_1}\sup_{0<\alpha<1}\left(\frac{\alpha(p-1)}{\alpha(p-1)+1}\right)^{\frac{1}{p}} R(\alpha) \leq C \leq 2\,C_2 \inf_{0<\alpha<1} \frac{R(\alpha)}{(1-\alpha)^{\frac{1}{p'}}}.$$

Theorem 4. *Let $1 < p < \infty$ and $0 < \alpha < 1$. Let μ be the counting measure on $\mathbb{N}$ and let $\psi\left(n, t^{\frac{1}{p}}\right) \in \Phi(\mathbb{N}, \mu)$. Let $\varphi \in \Phi(\mathbb{N}, \mu)$ be function satisfies conditions (2.3) and (2.4), respectively. Suppose that $v = \{v_n\}_{n=1}^{\infty}$ and $\omega = \{\omega_n\}_{n=1}^{\infty}$ are sequences of positive numbers.*

Then H_n is compact from $\ell_{\varphi,v}(\mathbb{N})$ into $\ell_{\psi,\omega}(\mathbb{N})$ if and only if

$$\lim_{k\to\infty}\left(\sum_{n=1}^{k} v_n^{-p'}\right)^{\frac{\alpha}{p'}} \left\| \frac{\left(\sum\limits_{m=1}^{n} v_m^{-p'}\right)^{\frac{1-\alpha}{p'}}}{n} \right\|_{\ell_{\psi,\omega}(n\geq k)} = 0. \tag{4.3}$$

Proof. *Sufficiency.* Let the condition (4.3) be provided. Then the condition (4.2) of Theorem 3 is valid. Therefore, by Theorem 3, the operator H_n is bounded from $\ell_{\varphi,v}(\mathbb{N})$ into $\ell_{\psi,\omega}(\mathbb{N})$. Assume that $\mathcal{A} = \{f^m\}_{m\in I} \subset \ell_{\varphi,v}(\mathbb{N})$ and $M = \sup\limits_{m\geq 1} \|f_m\|_{\ell_{\varphi,v}(\mathbb{N})}$. Let us show that the set $\{H_n f_m\}$ is precompact in $\ell_{\psi,\omega}(\mathbb{N})$. By Theorem 3, we have

$$\|H_n f_m\|_{\ell_{\psi,\omega}(\mathbb{N})} \leq C\ \|f_m\|_{\ell_{\varphi,v}(\mathbb{N})} \leq C\, M.$$

So the set $\{H_n f_m\}$ is uniformly bounded in $\ell_{\psi,\omega}(\mathbb{N})$.

For $s > 1$ we set $\overline{\omega}_s = \{\overline{\omega}_{s,i}\}_{i=1}^{\infty}$, $\overline{\omega}_{s,i} = \begin{cases} 0, & \text{if} \quad 1 \leq i \leq s-1, \\ \omega_i, & \text{if} \quad i \geq s. \end{cases}$

By Theorem 3, the inequality

$$\|H_n\|_{\ell_{\psi,\overline{\omega}_s}(\mathbb{N})} = \|H_n\|_{\ell_{\psi,\omega}(n\geq s)} \leq C\|f\|_{\ell_{\varphi,v}(\mathbb{N})} \tag{4.4}$$

holds if and only if $R_s(\alpha) < \infty$, where

$$R_s(\alpha) = \sup_{k\geq s} \left(\sum_{n=1}^{k} v_n^{-p'}\right)^{\frac{\alpha}{p'}} \left\| \frac{\left(\sum\limits_{m=1}^{n} v_m^{-p'}\right)^{\frac{1-\alpha}{p'}}}{n} \right\|_{\ell_{\psi,\omega}(n\geq k)} = \sup_{k\geq s} F_k(\alpha).$$

Obviously, if $C > 0$ is the best possible constant in (4.4), then $C \leq 2\, C_2 \, \dfrac{R_s(\alpha)}{(1-\alpha)^{\frac{1}{p'}}}$ for all $0 < \alpha < 1$. Then, by (4.4), we have

$$\sup_{\|f_m\|_{\ell_{\phi,v}(\mathbb{N})}\leq M} \|H_n f_m\|_{\ell_{\psi,\overline{\omega}_s}(n\geq s)} = \sup_{\|f_m\|_{\ell_{\phi,v}(\mathbb{N})}\leq M} \|\overline{\omega}_{s,n}\, H_n f_m\|_{\ell_{\psi}(\mathbb{N})}$$

$$\leq C \sup_{\|f_m\|_{\ell_{\phi,v}(\mathbb{N})\leq M}} \|f_m\|_{\ell_{\phi,v}(\mathbb{N})} \leq 2\, M\, C_2\, \frac{R_s(\alpha)}{(1-\alpha)^{\frac{1}{p'}}} \quad \text{for all } 0 < \alpha < 1. \tag{4.5}$$

So, by (4.5), we get

$$\lim_{s\to\infty} \left(\sup_{\|f_m\|_{\ell_{\phi,v}(\mathbb{N})}\leq M} \|\omega_{s,n}\, H_n f_m\|_{\ell_{\psi}(n\geq s)} \right)$$

$$\leq 2\, M\, C_2\, \frac{1}{(1-\alpha)^{\frac{1}{p'}}} \lim_{s\to\infty} R_s(\alpha)$$

$$= 2\, M\, C_2\, \frac{1}{(1-\alpha)^{\frac{1}{p'}}} \lim_{s\to\infty} \sup_{k\geq s} F_k(\alpha)$$

$$= 2\, M\, C_2\, \frac{1}{(1-\alpha)^{\frac{1}{p'}}} \overline{\lim_{s\to\infty}} F_s(\alpha)$$

$$= 2\,M\,C_2 \frac{1}{(1-\alpha)^{\frac{1}{p'}}} \lim_{s\to\infty} F_s(\alpha) = 0.$$

Necessity. Let the operator H_n be compact from $\ell_{\phi,v}(\mathbb{N})$ into $\ell_{\psi,\omega}(\mathbb{N})$. For $s \geq 1$ we introduce the following sequence $f_s = \{f_{s,j}\}_{j=1}^{\infty}$,

$$f_{s,j} = \begin{cases} v_j^{-p'}, & \text{if} \quad 1 \leq j \leq s, \\ 0, & \text{if} \quad j > s. \end{cases}$$

Let $g_s = \left\{ \frac{f_{s,j}}{\|f_s\|_{\ell_{\phi,v}(\mathbb{N})}} \right\}_{j=1}^{\infty}$. It is obvious that $\|g_s\|_{\ell_{\phi,v}(\mathbb{N})} = 1$ for all $1 \leq j \leq s$. Since the operator H_n is compact from $\ell_{\phi,v}(\mathbb{N})$ into $\ell_{\psi,\omega}(\mathbb{N})$, it implies that the set $\left\{H_n\varphi,\ \|\varphi\|_{\ell_{\phi,v}(\mathbb{N})=1}\right\}$ is precompact in $\ell_{\psi,\omega}(\mathbb{N})$. Thus, by the criteria on precompactness of the sets in $\ell_{\psi,\omega}(\mathbb{N})$ (Theorem 2), we have

$$\lim_{s\to\infty} \left(\sup_{\|\varphi\|_{\ell_{\phi,v}(\mathbb{N})}=1} \|\omega_n\, H_n \varphi\|_{\ell_\psi(n\geq s)} \right) = 0. \tag{4.6}$$

By Corollary 1, we have

$$\sup_{\|\varphi\|_{\ell_{\phi,v}(\mathbb{N})}=1} \|\omega_n\, H_n \varphi\|_{\ell_\psi(n\geq s)} \geq \|\omega_n\, H_n g_s\|_{\ell_\psi(n\geq s)} = \left\| \frac{\omega_n}{n} \frac{\sum_{j=1}^{n} f_{s,j}}{\|f_s\|_{\ell_{\phi,v}(\mathbb{N})}} \right\|_{\ell_\psi(n\geq s)}$$

$$\geq \frac{1}{C_1} \left\| \frac{\omega_n}{n} \frac{\sum_{j=1}^{n} f_{s,j}}{\|f_s\|_{\ell_v^p(\mathbb{N})}} \right\|_{\ell_\psi(n\geq s)} = \frac{1}{C_1} \left\| \frac{\omega_n}{n} \frac{\sum_{j=1}^{n} v_j^{-p'}}{\left(\sum_{j=1}^{s} v_j^{-p'}\right)^{\frac{1}{p}}} \right\|_{\ell_\psi(n\geq s)}$$

$$= \frac{1}{C_1} \left\| \frac{\omega_n}{n} \left(\frac{\sum_{j=1}^{n} v_j^{-p'}}{\left(\sum_{j=1}^{s} v_j^{-p'}\right)^{\frac{1}{p}}} \right)^{\alpha} \left(\frac{\sum_{j=1}^{n} v_j^{-p'}}{\left(\sum_{j=1}^{s} v_j^{-p'}\right)^{\frac{1}{p}}} \right)^{1-\alpha} \right\|_{\ell_\psi(n\geq s)}$$

$$\geq \frac{1}{C_1} \left(\frac{\left(\sum_{j=1}^{s} v_j^{-p'}\right)}{\left(\sum_{j=1}^{s} v_j^{-p'}\right)^{\frac{1}{p}}} \right)^{\alpha} \left\| \frac{\omega_n}{n} \left(\frac{\left(\sum_{j=1}^{n} v_j^{-p'}\right)}{\left(\sum_{j=1}^{n} v_j^{-p'}\right)^{\frac{1}{p}}} \right)^{1-\alpha} \right\|_{\ell_\psi(n\geq s)}$$

$$= \frac{1}{C_1} \left(\sum_{j=1}^{s} v_j^{-p'} \right)^{\frac{\alpha}{p'}} \left\| \frac{\left(\sum_{j=1}^{n} v_j^{-p'} \right)^{\frac{1-\alpha}{p'}}}{n} \right\|_{\ell_{\psi,\omega}(n \geq s)} . \tag{4.7}$$

So, by (4.6) and (4.7) we have the condition (4.3).

This completes the proof.

A similar theorem holds for the dual operator of the discrete Hardy operator.

Theorem 5. *Let $1 < p < \infty$ and $0 < \beta < 1$. Let μ be the counting measure on $\mathbb{N}$ and let $\psi\left(n, t^{\frac{1}{p}}\right) \in \Phi(\mathbb{N}, \mu)$. Let $\varphi \in \Phi(\mathbb{N}, \mu)$ be function satisfies conditions (2.3) and (2.4), respectively. Suppose that $v = \{v_n\}_{n=1}^{\infty}$ and $\omega = \{\omega_n\}_{n=1}^{\infty}$ are sequences of positive numbers.*

Then $H_n^{\star}$ is compact from $\ell_{\varphi,v}(\mathbb{N})$ into $\ell_{\psi,\omega}(\mathbb{N})$ if and only if

$$\lim_{k \to \infty} \left(\sum_{n=k}^{\infty} \frac{v_n^{-p'}}{n^{p'}} \right)^{\frac{\beta}{p'}} \left\| \left(\sum_{m=n}^{\infty} \frac{v_m^{-p'}}{m^{p'}} \right)^{\frac{1-\beta}{p'}} \right\|_{\ell_{\psi,\omega}(n \leq k)} = 0. \tag{4.8}$$

In particular, we have the following corollaries.

Corollary 4. *[4] Let $\{p_n\}_{n=1}^{\infty}$ and $\{q_n\}_{n=1}^{\infty}$ be sequences of real numbers such that $1 < \underline{p} \leq q_n \leq \overline{q} < \infty$ and let $\varphi(n, t) = t^{p_n}$ and $\psi(n, t) = t^{q_n}$ for all $n \in \mathbb{N}$ and $t \geq 0$. Suppose that $\frac{1}{r_n} = \frac{1}{\underline{p}} - \frac{1}{p_n}$ for all $n \in \mathbb{N}$ and let $\|1\|_{\ell^{r_n}(\mathbb{N})} < \infty$. Let $\omega = \{\omega_n\}_{n=1}^{\infty}$ and $v = \{v_n\}_{n=1}^{\infty}$ are sequences of positive numbers.*

Then H_n is compact from $\ell_v^{p_n}(\mathbb{N})$ into $\ell_\omega^{q_n}(\mathbb{N})$ if and only if

$$\lim_{k \to \infty} \left(\sum_{n=1}^{k} v_n^{-\underline{p}'} \right)^{\frac{1}{\underline{q}\,\underline{p}'}} \left\| \frac{\left(\sum_{m=1}^{n} v_m^{-\underline{p}'} \right)^{\frac{1}{\underline{p}'\,\underline{q}'}}}{n} \right\|_{\ell_\omega^{q_n}(n \geq k)} = 0.$$

Corollary 5. *[4] Let $\{p_n\}_{n=1}^{\infty}$ and $\{q_n\}_{n=1}^{\infty}$ be sequences of real numbers such that $1 < \underline{p} \leq q_n \leq \overline{q} < \infty$ and let $\varphi(n, t) = t^{p_n}$ and $\psi(n, t) = t^{q_n}$ for all $n \in \mathbb{N}$ and $t \geq 0$. Suppose that $\frac{1}{r_n} = \frac{1}{\underline{p}} - \frac{1}{p_n}$ for all $n \in \mathbb{N}$ and let $\|1\|_{\ell^{r_n}(\mathbb{N})} < \infty$. Let $\omega = \{\omega_n\}_{n=1}^{\infty}$ and $v = \{v_n\}_{n=1}^{\infty}$ are sequences of positive numbers.*

Then $H_n^\star$ is compact from $\ell_v^{p_n}(\mathbb{N})$ into $\ell_\omega^{q_n}(\mathbb{N})$ if and only if

$$\lim_{k\to\infty}\left(\sum_{n=k}^{\infty}\frac{v_n^{-\underline{p}'}}{n^{\underline{p}'}}\right)^{\frac{1}{\underline{q}\,\underline{p}'}}\left\|\left(\sum_{m=n}^{\infty}\frac{v_m^{-\underline{p}'}}{m^{\underline{p}'}}\right)^{\frac{1}{\underline{p}'\,\underline{q}'}}\right\|_{\ell_\omega^{q_n}(n\le k)}=0.$$

In the case of classical weighted Lebesgue sequence spaces we have the following lemma.

Lemma 5. *Let $1<p\le q<\infty$, $\varphi(n,t)=t^p$ and let $\psi(n,t)=t^q$ for all $n\in\mathbb{N}$. Suppose $\{\omega_n\}_{n=1}^\infty$ and $\{v_n\}_{n=1}^\infty$ are sequences of nonnegative numbers. Let $\{x_n\}_{n=1}^\infty\in\ell_v^p(\mathbb{N})$ be an arbitrary sequence of real numbers. Then condition (4.3) is equivalent to the condition*

$$\lim_{k\to\infty}\left(\sum_{n=1}^{k}v_n^{-p'}\right)^{\frac{1}{p'}}\left(\sum_{n=k}^{\infty}\left(\frac{\omega_n}{n}\right)^q\right)^{\frac{1}{q}}=0. \tag{4.9}$$

Proof. Let us assume that condition (4.3) holds. Let $M_k=\left(\sum_{n=1}^{k}v_n^{-p'}\right)^{\frac{1}{p'}}\left(\sum_{n=k}^{\infty}\left(\frac{\omega_n}{n}\right)^q\right)^{\frac{1}{q}}$. It is obvious that

$$F_k(\alpha)=\left(\sum_{n=1}^{k}v_n^{-p'}\right)^{\frac{\alpha}{p'}}\left\|\frac{\left(\sum\limits_{m=1}^{n}v_m^{-p'}\right)^{\frac{1-\alpha}{p'}}}{n}\right\|_{\ell_\omega^q(n\ge k)}\ge\left(\sum_{n=1}^{k}v_n^{-p'}\right)^{\frac{\alpha}{p'}}\left(\sum_{m=1}^{k}v_m^{-p'}\right)^{\frac{1-\alpha}{p'}}\left\|\frac{1}{n}\right\|_{\ell_\omega^q(n\ge k)}$$

$$=\left(\sum_{n=1}^{k}v_n^{-p'}\right)^{\frac{1}{p'}}\left(\sum_{n=k}^{\infty}\left(\frac{\omega_n}{n}\right)^q\right)^{\frac{1}{q}}=M_k.$$

Conversely, suppose that condition (4.9) is satisfied. So, we have that $M=\sup\limits_{k\ge1}M_k<\infty$. Therefore, we get

$$\left(\sum_{n=1}^{k}v_n^{-p'}\right)^{\frac{1}{p'}}\le M\left(\sum_{n=k}^{\infty}\left(\frac{\omega_n}{n}\right)^q\right)^{-\frac{1}{q}}\quad\text{for all }k\in\mathbb{N}.$$

Next, by inequality (2.6), we have

$$\left\|\frac{\left(\sum\limits_{m=1}^{n}v_m^{-p'}\right)^{\frac{1-\alpha}{p'}}}{n}\right\|_{\ell_\omega^q(n\ge k)}\le M^{1-\alpha}\left\|\frac{\left(\sum\limits_{m=n}^{\infty}\left(\frac{\omega_m}{m}\right)^q\right)^{-\frac{1-\alpha}{q}}}{n}\right\|_{\ell_\omega^q(n\ge k)}$$

$$= M^{1-\alpha}\left(\sum_{n=k}^{\infty}\left(\sum_{m=n}^{\infty}\left(\frac{\omega_m}{m}\right)^q\right)^{\alpha-1}\left(\frac{\omega_n}{n}\right)^q\right)^{\frac{1}{q}} \leq \frac{1}{\alpha^{1/q}} M^{1-\alpha}\left(\sum_{n=k}^{\infty}\left(\frac{\omega_n}{n}\right)^q\right)^{\frac{\alpha}{q}} \tag{4.10}$$

for all $k \in \mathbb{N}$. Thus, by (4.10), one has

$$\left(\sum_{n=1}^{k} v_n^{-p'}\right)^{\frac{\alpha}{p'}} \left\| \frac{\left(\sum_{m=1}^{n} v_m^{-p'}\right)^{\frac{1-\alpha}{p'}}}{n} \right\|_{\ell^q_\omega(n\geq k)} \leq \frac{1}{\alpha^{1/q}} M^{1-\alpha} \left(\sum_{n=1}^{k} v_n^{-p'}\right)^{\frac{\alpha}{p'}} \left(\sum_{n=k}^{\infty}\left(\frac{\omega_n}{n}\right)^q\right)^{\frac{\alpha}{q}}$$

$$= \frac{1}{\alpha^{1/q}} M^{1-\alpha} M_k^{\alpha}$$

for all $k \in \mathbb{N}$. Finally, we have that

$$M_k \leq F_k(\alpha) \leq \frac{1}{\alpha^{1/q}} M^{1-\alpha} M_k^{\alpha} \ \text{ for all } k \geq 1 \text{ and } 0 < \alpha < 1.$$

This completes the proof of Lemma 5.

Similarly, we can prove the following lemma.

Lemma 6. *Let $1 < p \leq q < \infty$, $\varphi(n,t) = t^p$ and let $\psi(n,t) = t^q$ for all $n \in \mathbb{N}$. Suppose $\{\omega_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ are sequences of nonnegative numbers. Let $\{x_n\}_{n=1}^{\infty} \in \ell_v^p(\mathbb{N})$ be an arbitrary sequence of real numbers. Then condition (4.8) is equivalent to the condition*

$$\lim_{k\to\infty}\left(\sum_{n=k}^{\infty}\frac{v_n^{-p'}}{n^{p'}}\right)^{\frac{1}{p'}}\left(\sum_{n=1}^{k}\omega_n\right)^{\frac{1}{q}} = 0.$$

Remark 4. *In the case of classical weighted Lebesgue sequence spaces, the problem of compactness of a class of matrix operators was considered in [29] and so on. In particular, a class of matrix operators includes the discrete Hardy operator.*

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All authors contributed the paper and reviewed it.

8 Conflict of interest

The authors declare no conflict of interest.

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