Functional (co)homological groups of completely regular spaces and their applications

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Abstract

Using the set of functionally open finite covers of completely regular spaces in the paper are constructed Čech type functional homology functor $\check{H}^F_n(-,-;G) : \mathbf{Top^2_{ce}} \to \mathbf{Ab}$ and functional cohomology functor $\hat{H}^n_F(-,-;G) : \mathbf{Top^2_{ce}} \to \mathbf{Ab}$ from the category of pairs of completely regular spaces and their CE-subspaces to the category of abelian groups, defined Bokshtein-Nowak type functional coefficient of cyclicity η^F_G and functional cohomological dimension dim^F_G of completely regular space with values in the set $N \cup \{0, \infty\}$, and proved the equalities $\check{H}^F_n(X, A; G) = \check{H}_n(\beta X, \overline{A}^{\beta X}; G)$, $\hat{H}^n_F(X, A; G) = \hat{H}^n(\beta X, \overline{A}^{\beta X}; G)$, $\eta^F_G(X) = \eta_G(\beta X)$, $dim^F_G(X) = dim_G(\beta X)$, where $N, \check{H}_n(\beta X, \overline{A}^{\beta X}; G)$, $\hat{H}^n(\beta X, \overline{A}^{\beta X}; G), \eta_G(\beta X)$ and $dim_G(\beta X)$ are the set of natural numbers, Čech homology group, Čech cohomology group, Bokshtein-Nowak coefficient of cyclisity and cohomological dimension of Stone-Čech compactification of pair $(X, A) \in \mathrm{ob}(\mathbf{Top^2_{ce}})$ and space $X \in ob(\mathbf{Top_{ce}})$, respectively.

Keywords: Čech homology group, Čech cohomology group, Stone-Čech compactification, Coefficient of cyclicity, cohomological dimension.
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Introduction

The problems of compactification theory of topological spaces also lead to the necessity of the creation and development of such new (co)homological theories, whose methods effectively can be used in the study of specific problems of algebraic topology.

The investigation presented in this paper is centered around the following **Problem: Find necessary and sufficient conditions under which the spaces of the given class has the compactifications with the given properties.**

This problem for various compactifications and properties was studied by several authors (cf. [A-N], [B_1],[B_2], [B-T], [Ba], [Bo], [C], [E-S], [En_1], [En_2], [I], [K], [M-S], [Mi], [N], [P], [S], [Sk], [Sm_1], [Sm_2], [Z], [V]).

The present paper is devoted to the study of this problem for property:

n-dimensional Čech (co) homology group [E-S], coefficient of cyclicity ([Bo],[N]) and cohomological dimension of Stone-Čech compactification of completely regular space is a given group and integer number, respectively.

A special aspect of this topic was considered in [E-S], where was proved that if $(\tilde{X}, \tilde{A})$ is the Tychonoff compactification of closed pair (X, A) of normal spaces, then

$$\check{H}_n^f(X, A; G) = \check{H}_n(\tilde{X}, \tilde{A}; G)$$

and

$$\hat{H}_f^n(X, A; G) = \hat{H}^n(\tilde{X}, \tilde{A}; G),$$

where $\check{H}_n^f(X, A; G)$ and $\hat{H}_f^n(X, A; G)$, $\check{H}_n(\tilde{X}, \tilde{A}; G)$ and $\hat{H}^n(\tilde{X}, \tilde{A}; G)$ respectively are n-dimensional Čech homology groups and n-dimensional Čech cohomology groups of pairs (X, A) and $(\tilde{X}, \tilde{A})$, based on finite open covers. Also note that in papers [B_1], [B-T] and [Mi] the obtained results include the characterizations of other homology and cohomology groups of extensions of spaces from some classes of spaces.

We say that subspace A of X has CE-property, if each continuous map $f : A \to [0, 1]$ is continuously extendable over X (cf. [En_1]). The sat A we also call CE-subspace.

By $\mathbf{Top^2_{ce}}$ denote the category of pairs consisting of completely regular spaces and their CE-subspaces.

In this paper we define the Čech type functional homology and cohomology ∂-functor and δ-cofunctor (see [E-S]) $\check{H}_n^F(-,-;G) : \mathbf{Top^2_{ce}} \to \mathbf{Ab}$ and $\widehat{H}_F^n(-,-;G) : \mathbf{Top^2_{ce}} \to \mathbf{Ab}$ from the category $\mathbf{Top^2_{ce}}$ to the category $\mathbf{Ab}$ of abelian groups. The definitions of these functors are based on the set of functionally open finite covers of pairs $(X, A) \in ob\left(\mathbf{Top^2_{ce}}\right)$.

The main results of the paper are the following theorems.

Theorem 3.1 *For each pair $(X, A) \in ob(\mathbf{Top^2_{ce}})$, one has*

$$\check{H}_n^F(X, A; G) = \check{H}_n(\beta X, \overline{A}^{\beta X}; G)$$

and

$$\hat{H}_F^n(X, A; G) = \hat{H}^n(\beta X, \overline{A}^{\beta X}; G).$$

In the paper also is defined the functional coefficient of cyclicity $\eta_G^F(X)$ and functional cohomological dimension $dim_G^F(X)$ of completely regular space X and proved the following theorems.

Theorem 3.4 *For each completely regular space X holds the equality*

$$\eta_G^F(X) = \eta_G(\beta X).$$

Theorem 3.7. *For each completely regular space X holds the relation*

$$dim_G^F(X) = dim_G(\beta X).$$

Note that, here η_G and dim_G are the classical coefficient of cyclisity and cohomological dimension studied by several autors.

Thus, the functional (co)homology groups intrinsically, in terms of functionaly open sets of completely regular spaces describe the Čech (co)homology groups, coefficients of cyclicity and coohomological dimensions of Stone-Čech compactifications of completely regular spaces. In particular, a pair $(X, A) \in ob(\mathbf{Top^2_{ce}})$ has the Stone-Čech compactification with Čech (co)homology group isomorphic to the abelian group M if and only if (X, A) has functional (co)homology group isomorphic to abelian group M. Besides, completely regular space X has the Stone-Čech compactification with coefficient of cyclicity or cohomological dimension equal to the integer $n \geq 0$ if and only if X has functional coefficient of cyclicity or cohomological dimension equal to the integer $n \geq 0$.

In view of this results the relations $\check{H}_n^F(X, A; G) = M$, $\hat{H}^n_F(X, A; G) = M$, $\eta_G^F(X) = n$ and $dim_G^F(X) = n$ are the internal necessary and sufficient conditions on (X, A) and X so that (X, A) and X have the compactification with $\check{H}_n(\beta X, \overline{A}^{\beta X}; G) = M$, $\hat{H}^n(\beta X, \overline{A}^{\beta X}; G) = M$, $\eta_G(\beta X) = n$ and $dim_G(\beta X) = n$.

Finally, let us note that without any further reference in the case of necessity we will use definitions and results from the books General Topology [En_1], Algebraic Topology [E-S] and dimension theory [En_2], [N].

1 Čech type functional (co)homology groups

In this section of the paper using the methods developed in [E-S] and [En_1] we will give the construction of Cech Type functional homology and cohomology functors on the category $\mathbf{Top^2_{ce}}$ and deduce some of their consequences.

Let (X, A) be a pair consisting of completely regular space X and its CE subset A. By $\mathrm{cov}_F(X)$ denote the set of functionally open finite covers of space X. Let $\alpha = \{\alpha_v\}_{v \in V_\alpha} \in \mathrm{cov}_F(X), |V_\alpha| < \chi_0$ and $s \subset V_\alpha$ be a subset of V_α. By $\mathrm{car}_\alpha(s)$ denote the carrier of s. By definition, $\mathrm{car}_\alpha(s) = \bigcap_{v \in s} \alpha_v$. Consider the nerve X_α of α and its subcomplex A_α consisting of all finite subsets $s \subset V_\alpha$ such that $\mathrm{car}_\alpha(s) \cap A \neq \emptyset$

The chain and cochain groups $C_n(X_\alpha, A_\alpha; G)$ and $C^n(X_\alpha, A_\alpha; G)$ with coefficients in abelian group G of simplicial pair (X_α, A_α) are defined in the usual way [E-S].

Now we assign to the simplicial pair (X_α, A_α) homology and cohomology groups $H_n(X_\alpha, A_\alpha; G)$ and $H^n(X_\alpha, A_\alpha; G)$ with coefficients in group G (see[E-S], ch. V1, 3.9).

Let $C_n(X_\alpha)$ be the free abelian group generated by all ordered n-simplices of X_α and $C_n(A_\alpha)$ be the free abelian subgroup generated by all ordered n-simplices of A_α. By $C_n(X_\alpha, A_\alpha)$ denote the quotient group $C_n(X_\alpha)/C_n(A_\alpha)$.

For each generator $(v_0, , v_1, \ldots, v_n)$ of $C_n(X_\alpha)$ is defined its boundary

$$\partial(v_0, v_1, \ldots, v_n) = \sum_{i=0}^{n} (-1)^i (v_0, v_1, \ldots, \check{v}_i, \ldots, v_n)\,.$$

Hence, there exists boundary homomorphism

$$\partial_n : C_n(X_\alpha) \to C_{n-1}(X_\alpha)$$

with properties

$$\partial_n(C_n(X_\alpha)) \subset C_{n-1}(X_\alpha)$$

and

$$\partial_n(C_n(X_\alpha, A_\alpha)) \subset C_{n-1}(X_\alpha, A_\alpha)\,.$$

Thus, we have ordered chain complexes

$$C(X_\alpha) = \{C_n(X_\alpha), \partial_n\},\ C(A_\alpha) = \{C_n(A_\alpha), \partial_n\}$$

and

$$C(X_\alpha, A_\alpha) = \{C_n(X_\alpha, A_\alpha), \partial_n\}.$$

Let

$$C_n(X_\alpha; G) = C_n(X_\alpha) \otimes G, C_n(A_\alpha; G) = C_n(A_\alpha) \otimes G$$

and

$$C_n(X_\alpha, A_\alpha; G) = C_n(X_\alpha, A_\alpha) \otimes G,$$

where symbol $\otimes$ denotes the tensor product of groups. Consequently, we have obtained the chain complexes

$$C(X_\alpha; G) = \{C_n(X_\alpha; G), \partial_n \otimes 1_G\}, C(A_\alpha; G) = \{C_n(A_\alpha; G), \partial_n \otimes 1_G\}$$

$$C(X_\alpha, A_\alpha; G) = \{C_n(X_\alpha, A_\alpha; G), \partial_n \otimes 1_G\}.$$

For simplicity the homomorphisms $\partial_n \otimes 1_G$ again denote by ∂_n.

As in ([E-S], ch.5) we can describe cochain complexes $\{C^n(X_\alpha; G), \delta^n\}$, $\{C^n(A_\alpha; G), \delta^n\}$ and $\{C^n(X_\alpha, A_\alpha; G), \delta^n\}$.

Let $H_n(X_\alpha, A_\alpha; G) = ker\partial_n / im\partial_{n+1}$ and $H^n(X_\alpha, A_\alpha; G) = ker\delta^n / Im\delta^{n-1}$. The boundary homomorphism $\partial : H_n(X_\alpha, A_\alpha; G) \to H_{n-1}(A_\alpha; G)$ and coboundary homomorphism $\delta : H^n(A_\alpha; G) \to H^{n+1}(X_\alpha, A_\alpha; G)$ are defined in the usual way. Also note that each simplicial map $f_\beta : (X_\alpha, A_\alpha) \to (Y_\beta, B_\beta)$ induces the homomorphisms $f_{\beta *} : H_n(X_\alpha, A_\alpha; G) \to H_n(Y_\beta, B_\beta; G)$ and $f_\beta^* : H^n(Y_\beta, B_\beta; G) \to H^n(X_\alpha, A_\alpha; G)$.

Let $\alpha' \in cov_F(X)$ be a refinement of $\alpha \in cov_F(X)$ (notation $\alpha < \alpha'$). Note that the set $cov_F(X)$ is a directed set with respect to the refinement relation $<$.

Any two refinement projection maps induce contiguous simplicial maps from pair $(X_{\alpha'}, A_{\alpha'})$ to pair (X_α, A_α). Consequently, any two refinement projection maps $p_{\alpha\alpha'}$ and $p'_{\alpha\alpha'}$ induce the unique homomorphisms

$$p_{\alpha\alpha' *} = p'_{\alpha\alpha' *} : H_n(X_{\alpha'}, A_{\alpha'}; G) \to H_n(X_\alpha, A_\alpha; G)$$

and

$$p^*_{\alpha\alpha'} = p^{'*}_{\alpha\alpha'} : H^n(X_\alpha, A_\alpha; G) \to H^n(X_{\alpha'}, A_{\alpha'}; G).$$

For each $\alpha \in cov_F(X)$ the homomorphisms $p_{\alpha\alpha *}$ and $p^*_{\alpha\alpha}$ are identity homomorphisms and for each triple $\alpha < \alpha' < \alpha^{''}$,

$$p_{\alpha\alpha'' *} = p_{\alpha\alpha' *} \cdot p_{\alpha'\alpha'' *}$$

and

$$p^*_{\alpha\alpha''} = p^*_{\alpha'\alpha''} \cdot p^*_{\alpha\alpha'}.$$

Thus, there exists the inverse system $\{H_n(X_\alpha, A_\alpha; G), p_{\alpha\alpha' *}, cov_F(X, A)\}$ and the direct system $\{H^n(X_\alpha, A_\alpha; G), p^*_{\alpha\alpha'}, cov_F(X, A)\}$ for pair $(X, A) \in ob(\mathbf{Top^2_{ce}})$.

We have the following

Proposition 1.1. Let A be a CE-subspace of space $X \in ob(\mathbf{Top_{ce}})$. The set of covers $\{\alpha_v \cap A | \alpha_v \in \alpha \in cov_F(X)\}$ is cofinal subset of the set $cov_F(A)$.

Proof. Let $\alpha = \{\alpha_{v_i}\}_{v_i \in V_\alpha}, i = 1, 2, \ldots, n$ be a functionally open cover of A. There exists a functionally closed shrinking $\beta = \{\beta_{v_i}\}_{v_i \in V_\alpha}, i = 1, 2, \ldots, n$ of the cover α (see [En1], Theorem 7.1.5). For each pair $(A \setminus \alpha_{v_i}, \beta_{v_i}), v_i \in V_\alpha$ there exists a continuous map $f_{v_i} : A \to [0, 1]$ such that $f_{v_i}(A \setminus \alpha_{v_i})) = \{0\}$ and $f_{v_i}(\beta_{v_i}) = \{1\}$. By condition of proposition each map f_{v_i} has continuous extension $\tilde{f}_{v_i} : X \to [0, 1]$. The family

$$\{\tilde{f}^{-1}_{v_1}((1/2, 1]), \tilde{f}^{-1}_{v_2}((1/2, 1]), \ldots, \tilde{f}^{-1}_{v_n}((1/2, 1]), \bigcap_{i=1}^{n} \tilde{f}^{-1}_{v_i}([0, 1))\}$$

is a functionally open finite cover of X. Note that $A \cap \tilde{f}^{-1}_{v_i}((1/2, 1]) \subset \alpha_{v_i}$ for each $i = 1, 2, \ldots, n$ and $A \cap \bigcap_{i=1}^{n} \tilde{f}^{-1}_{v_i}([0, 1)) = \emptyset$. Thus, The family

$$\{A \cap \tilde{f}^{-1}_{v_1}((1/2, 1]), A \cap \tilde{f}^{-1}_{v_2}((1/2, 1]), \ldots, A \cap \tilde{f}^{-1}_{v_n}((1/2, 1])\}$$

is a functionally open finite cover of A and it is a refinement of α.

□

The limit groups

$$\check{H}^F_n(X, A; G) = \varprojlim \{H_n(X_\alpha, A_\alpha; G), p_{\alpha\alpha' *}, cov_F(X, A)\}$$

and

$$\hat{H}^n_F(X, A; G) = \varinjlim \{H^n(X_\alpha, A_\alpha; G), p^*_{\alpha\alpha'}, cov_F(X, A)\}$$

are called n-dimensional functional homology and cohomology groups of pair $(X, A) \in ob(\mathbf{Top^2_{ce}})$ with coefficients in abelian group G, respectively.

Let

$$(f : (X, A) \to (Y, B)) \in Mor_{\mathbf{Top^2_{ce}}}((X, A), (Y, B))$$

and

$$\beta = \{\beta_v\}_{v \in V_\beta} \in cov_F(Y, B).$$

The family $\alpha = \{\alpha_v\}_{v \in V_\alpha}$,where $\alpha_v = f^{-1}(\beta_v)$ for each $v \in V_\alpha = V_\beta$, is functionally open cover of X. Note that X_α is a subcomplex of Y_β and A_α is a subcomplex of B_β. Hence, there exists the inclusion simplicial map $f_\beta : (X_\alpha, A_\alpha) \to (Y_\beta, B_\beta)$. Let $\beta' \in cov_F(Y)$ and $\beta < \beta'$. Then any refinement projection $p_{\beta\beta'}$ of β' into β induces a refinement projection $p_{\alpha\alpha'}$ of α' into α. From equality $f_\beta . p_{\alpha\alpha'} = p_{\beta\beta'} . f'_\beta$ it follows that

$$f_{\beta *} \cdot p_{\alpha\alpha' *} = p_{\beta\beta' *} \cdot f_{\beta' *}$$

and

$$p^*_{\alpha\alpha'} \cdot f^*_\beta = f^*_{\beta'} \cdot p^*_{\beta\beta'}.$$

Consequently, the map $f^{-1} : cov_f(Y, B) \to cov_F(X, A)$ and the homomorphisms

$$f_{\beta *} : H_n(X_\alpha, A_\alpha; G) \to H_n(Y_\beta, B_\beta; G)$$

and

$$f^*_\beta : H^n(Y_\beta, B_\beta; G) \to H^n(X_\alpha, A_\alpha; G)$$

form morphisms

$$(f_{\beta *}, f^{-1}) : \{H_n(X_\alpha, A_\alpha; G), p_{\alpha\alpha' *}\,, cov_F(X, A)\} \to \{H_n(Y_\beta, B_\beta; G), p_{\beta\beta' *}, cov_F(Y, B)\}$$

and

$$(f^*_\beta, f^{-1}) : \{H^n(Y_\beta, B_\beta; G), p^*_{\beta\beta'}, cov_F(Y, B)\} \to \{H^n(X_\alpha, A_\alpha; G), p^*_{\alpha\alpha'}, cov_F(X, A)\}.$$

Let

$$f_* = \varprojlim (f_{\beta *}, f^{-1}) : \check{H}^F_n(X, A; G) \to \check{H}^F_n(Y, B; G)$$

and

$$f^* = \varinjlim (f^*_\beta, f^{-1}) : \hat{H}^n_F(Y, B; G) \to \hat{H}^n_F(X, A; G).$$

The homomorphisms f_* and f^* are called the homomorphisms induced by the map $f : (X, A) \to (Y, B)$.

2 Čech type functional (co)homology functors

In this section are constructed Čech type functional homology and cohomology ∂ and δ founctors

$$\check{H}^F_n(-, -; G) : \mathbf{Top^2_{ce}} \to \mathbf{Ab}$$

and

$$\hat{H}^n_F(-, -; G) : \mathbf{Top^2_{ce}} \to \mathbf{Ab}.$$

By definition,

$$\check{H}^F_n(-, -; G)((X, A)) = \check{H}^F_n(X, A; G), (X, A) \in ob(\mathbf{Top^2_{ce}}),$$

$$\check{H}^F_n(f) = f_*, f \in Mor_{\mathbf{Top^2_{ce}}}((X, A), (Y, B)),$$

$$\hat{H}^n_F(-,-;G)((X,A)) = \hat{H}^n_F(X,A;G), (X,A) \in ob(\mathbf{Top^2_{ce}}),$$
$$\hat{H}^n_F(f) = f^*, f \in Mor_{\mathbf{Top^2_{ce}}}((X,A),(Y,B)).$$

This implies that for each continuous maps

$$f \in Mor_{\mathbf{Top^2_{ce}}}((X,A),(Y,B)), g \in Mor_{\mathbf{Top^2_{ce}}}((Y,B),(Z,C))$$

and identity map

$$1_{(X,A)} \in Mor_{\mathbf{Top^2_{ce}}}((X,A),(X,A))$$

hold the following equalities

$$\check{H}^F_n(g \circ f) = \check{H}^F_n(g) \circ \check{H}^F_n(f),$$
$$\hat{H}^n_F(g \circ f) = \hat{H}^n_F(f) \circ \hat{H}^n_F(g),$$
$$\hat{H}^F_n(1_{(X,A)}) = 1_{\check{H}^F_n(X,A;G)},$$
$$\hat{H}^n_F(1_{(X,A)}) = 1_{\hat{H}^n_F(X,A;G)}.$$

Thus we have the following

Theorem 2.1. *There exist functional homology functor and functional cohomology functor*

$$\check{H}^F_n(-,-;G) : \mathbf{Top^2_{ce}} \to \mathbf{Ab}$$

and

$$\hat{H}^n_F(-,-;G) : \mathbf{Top^2_{ce}} \to \mathbf{Ab},$$

respectively.

Let $(X,A) \in ob(\mathbf{Top^2_{ce}})$. Consider a cover $\alpha \in cov_F(X,A)$ which is indexed by pair (V_α, V^A_α).

Let $V^A_\alpha = \{v \in V_\alpha | \alpha_v \cap A \neq \varnothing\}$. Now define a cover $\alpha' = \{\alpha_v \cap A\}_{v \in V^A_\alpha}$ of A and an order preserving map $\varphi : cov_F(X,A) \to cov_F(A,\varnothing)$. By definition, $\varphi(\alpha) = \alpha'$. The correspondence $\alpha_v \to \varphi(\alpha)_v$ induces the identification of simplicial complexes A_α and $A_{\varphi(\alpha)}$, i.e. $A_\alpha = A_{\varphi(\alpha)}$. Note that, hold the following equalities $p_{\alpha\alpha'*} \cdot \varphi_{\alpha'*} = \varphi_{\alpha*} \cdot p_{\varphi(\alpha)\varphi(\alpha')*}$ and $\varphi^*_{\alpha'} \cdot p^*_{\alpha\alpha'} = p^*_{\varphi(\alpha)\varphi(\alpha')} \cdot \varphi^*_\alpha$ for each pair $\alpha < \alpha'$ of $cov_F(X,A)$.

Now consider the map $\psi : cov_F(X,A) \to cov(X,\varnothing)$ given by formula $\psi(\alpha)_v = \alpha_v$. Alsou note that the nerves X_α and $X_{\psi(\alpha)}$ are isomorphical simplicial complex, i.e. $X_\alpha = X_{\psi(\alpha)}$. There exist the maps of inverse systems

$$\{H_n(A_\alpha;G), p_{\alpha\alpha'*}, cov_F(A,\varnothing)\} \to \{H_n(A_\alpha;G), p_{\alpha\alpha'*}, cov_F(X,A)\}$$

and

$$\{H_n(X_\alpha;G), p_{\alpha\alpha'*}, cov_F(X,\varnothing)\} \to \{H_n(X_\alpha;G), p_{\alpha\alpha'*}, cov_F(X,A)\}.$$

It is clear that the limit homomorphisms

$$\varphi_\infty : \check{H}^F_n(A;G) \to \check{H}^F_n(A;G)_{(X,A)}$$

and

$$\psi_\infty : \check{H}_n^F(X;G) \to \check{H}_n^F(X;G)_{(X,A)}$$

are isomorphisms.

For cohomology we have isomorphisms

$$\varphi^\infty : \hat{H}_F^n(A;G)_{(X,A)} \to \hat{H}_F^n(A;G)$$

and

$$\psi^\infty : \hat{H}_F^n(X;G)_{(X,A)} \to \hat{H}_F^n(X;G).$$

Let $\alpha \in cov_F(X,A)$. Consider the homology and cohomology sequences of pair (X_α, A_α)

$$\ldots \leftarrow H_n(X_\alpha, A_\alpha; G) \xleftarrow{j_{\alpha *}} H_n(X_\alpha; G) \xleftarrow{i_{\alpha *}} H_n(A_\alpha; G) \xleftarrow{\partial_\alpha} H_{n+1}(X_\alpha, A_\alpha; G) \leftarrow \ldots$$

and

$$\ldots \to H^n(X_\alpha, A_\alpha; G) \xrightarrow{j_\alpha^*} H^n(X_\alpha; G) \xrightarrow{i_\alpha^*} H^n(A_\alpha; G) \xrightarrow{\delta_\alpha} H^{n+1}(X_\alpha, A_\alpha; G) \to \ldots \; .$$

There exist the following limit sequences

$$\ldots \leftarrow \check{H}_n^F(X,A) \xleftarrow{j'_*} \check{H}_n^F(X)_{(X,A)} \xleftarrow{i'_*} \check{H}_n^F(A)_{(X,A)} \xleftarrow{\partial'} \check{H}_{n+1}^F(X,A) \leftarrow \ldots$$

and

$$\ldots \to \hat{H}_F^n(X,A) \xrightarrow{j'^*} \hat{H}_F^n(X)_{(X,A)} \xrightarrow{i'^*} \hat{H}_F^n(A)_{(X,A)} \xrightarrow{\delta'} \hat{H}_F^{n+1}(X,A) \to \ldots \quad ,$$

respectively.

Let $\partial : \check{H}_n^F(X,A;G) \to \check{H}_{n-1}^F(A;G)$ and $\delta : \hat{H}_F^n(A;G) \to \hat{H}_F^{n+1}(X,A;G)$ are homomorphisms given by formulas $\partial = (\varphi_\infty)^{-1} \cdot \partial'$ and $\delta = \delta' \cdot (\varphi^\infty)^{-1}$.

Thus we obtained the following

Theorem 2.2. *For every map $f : (X,A) \to (Y,B)$, integer n and abelian group G the following rectangles*

$$\begin{array}{ccc}
\check{H}_n^F(X,A;G) & \xrightarrow{\partial} & \check{H}_{n-1}^F(A;G) \\
\downarrow f_* & & \downarrow (f_{|A})_* \\
\check{H}_n^F(Y,B;G) & \xrightarrow[\partial]{} & \check{H}_{n-1}^F(B;G)
\end{array}$$

and

$$\begin{array}{ccc} \hat{H}_F^{n-1}(B;G) & \xrightarrow{\delta} & \hat{H}_F^{n}(Y,B;G) \\ \downarrow{\scriptstyle (f_{|A})^*} & & \downarrow{\scriptstyle f^*} \\ \hat{H}_F^{n-1}(A;G) & \xrightarrow[\delta]{} & \hat{H}_F^{n}(X,A;G) \end{array}$$

are commutative.

Note that $\varphi_\infty \cdot \partial = \varphi_\infty \cdot \varphi_\infty^{-1} \cdot \partial' = \partial'$. Now prove that holds equality $i'_* \cdot \varphi_\infty = \psi_\infty \cdot i_*$. Consider the maps $\tau, \eta : cov_F(X,A) \to cov_F(A, \varnothing)$ defined by formulas

$$\tau\alpha = \varphi\alpha, (\tau\alpha)_v = A \cap \alpha_v, \alpha \in cov_F(X,A),$$

$$\eta\alpha = i^{-1}\psi\alpha, (\eta\alpha)_v = A \cap \alpha_v, \alpha \in cov_F(X,A).$$

Let $\tau_\alpha : H_n(A_{\tau\alpha};G) \to H_n(X_\alpha;G)$ and $\eta_\alpha : H_n(A_{\eta\alpha};G) \to H_n(X_\alpha;G)$ are homomorphisms induced by simplicial embedding maps of complex $A_{\tau\alpha}$ into complex X_α and complex $A_{\eta\alpha}$ into complex X_α, respectively.

For each finite functionally open cover $\alpha \in cov_F(X,A)$ we have $\eta\alpha < \tau\alpha$. There exists a homomorphism $p_{\eta\alpha,\tau\alpha*} : H_n(A_{\tau\alpha};G) \to H_n(A_{\eta\alpha};G)$ such that $\tau_\alpha = \eta_\alpha \cdot p_{\eta\alpha,\tau\alpha*}$.

The homomorphisms $i'_* \cdot \varphi_\infty$ and $\psi_\infty \cdot i_*$ are limits of morphisms

$$(\tau_\alpha, \tau), (\eta_\alpha, \eta) : \{H_n(A_\alpha;G), p_{\alpha\alpha'*}, cov_F(A,\varnothing)\} \to \{H_n(X_\alpha;G), p_{\alpha\alpha'*}, cov_F(X,A)\}.$$

Since $\tau_\infty = \eta_\infty$, we have $i'_* \cdot \varphi_\infty = \psi_\infty \cdot i_*$

Let $\sigma, \rho : cov_F(X,A) \to cov_F(X,\varnothing)$ be functions given by formulas

$$\sigma\alpha = \alpha.$$

$$\rho\alpha = \alpha.$$

Note that $X_{\sigma\alpha} = X_{\rho\alpha} = X_\alpha$. Let homomorphisms $\sigma_\alpha, \rho_\alpha : H_n(X_\alpha;G) \to H_n(X_\alpha, A_\alpha;G)$ are induced by inclusion map $X_\alpha \to (X_\alpha, A_\alpha)$. We have morphisms

$$(\sigma_\alpha, \sigma), (\rho_\alpha, \rho) : \{H_n(X_\alpha;G), p_{\alpha\alpha'*}, cov_F(X,\varnothing)\} \to \{H_n(X_\alpha, A_\alpha;G), p_{\alpha\alpha'*}, cov_F(X,A)\}.$$

Note that for each $\alpha \in cov_F(X,A), \rho\alpha < \sigma\alpha, \sigma\alpha = \rho_\alpha \cdot p_{\rho\alpha,\sigma\alpha*}$. Besides, $\sigma_\infty = \rho_\infty$.

The homomorphisms $j'_* \cdot \psi_\infty$ and j_* are limits of morphisms (σ_α, σ) and (ρ_α, ρ), respectiveli. Hence, $j'_* \cdot \psi_\infty = j_*$.

Consequenty, commutativity relations hold in the diagram

$$\begin{array}{ccccccc} \check{H}_{n+1}^F(X,A;G) & \xrightarrow{\partial} & \check{H}_n^F(A;G) & \xrightarrow{i_*} & \check{H}_n^F(X;G) & \xrightarrow{j_*} & \check{H}_n^F(X,A;G) \\ || & & \varphi_\infty\downarrow & & \psi_\infty\downarrow & & || \\ \check{H}_{n+1}^F(X,A;G) & \xrightarrow{\partial'} & \check{H}_n^F(A;G)_{(X,A)} & \xrightarrow{i'_*} & \check{H}_n^F(X;G)_{(X,A)} & \xrightarrow{j'_*} & \check{H}_n^F(X,A;G) \end{array}$$

For cohomology exists similar diagram.

Therefore, we have the following

Theorem 2.3. *For any pair* $(X, A) \in ob(\mathbf{Top^2_{ce}})$ *the homology sequence*

$$\cdots \to \check{H}^F_n(A;G) \to \check{H}^F_n(X;G) \to \check{H}^F_n(X,A;G) \to \check{H}^F_{n-1}(A;G) \to \ldots$$

is of order 2 and the cohomology sequence

$$\cdots \to \hat{H}^{n-1}_F(A;G) \to \hat{H}^n_F(X,A;G) \to \hat{H}^n_F(X;G) \to \hat{H}^n_F(A;G) \to \ldots$$

is exact.

Therefore the functional homology and cohomology functors

$$\check{H}^F_n(-,-;G) : \mathbf{Top^2_{cr}} \to \mathbf{Ab}$$

and

$$\hat{H}^n_F(-,-;G) : \mathbf{Top^2_{cr}} \to \mathbf{Ab}$$

are ∂-functor and δ-cofunctor in the sence of [E − S], respectively. This is a direct consequence of Theorems 2.1, 2.2 and 2.3.

Let α be the cover of X consisting of the single point . The inverse system of nerves of functionally open covers of X has a single term $X_\alpha = N(\alpha)$ consisting of a single vertex. Consequently, $\check{H}^F_n(X;G) = H_n(N(\alpha);G)$ and $\hat{H}^n_F(X;G) = H^n(N(\alpha);G)$. Hence, $\check{H}^F_0(X;G) \approx G$ and $\check{H}^F_n(X;G) = 0$ for $n \neq 0$. Analogously, $\hat{H}^0_F(X;G) \approx G$ and $\hat{H}^n_F(X;G)$=0 for $n \neq 0$.

Thus we have the following

Theorem 2.4. *For the distinguished singleton space* $X = \{*\}$

$$\check{H}^F_n(X;G) = \begin{cases} 0, & n \neq 0 \\ G, & n = 0 \end{cases}$$

and

$$\hat{H}^n_F(X;G) = \begin{cases} 0, & n \neq 0 \\ G, & n = 0. \end{cases}$$

3 Characterizations of Čech (co)homology groups, coefficients cyclicity and cohomological dimensions of Stone-Čech compactifications

Now we are mainly interested in the following question: How the Čech (co)homology groups, coefficients of cyclicity and cohomological dimensions of Stone-Čech compactifications of completely regular spaces can be characterized intrinsically, in terms of functionally open subsets of completely regular spaces.

Consider the inclusion maps $i : A \to \bar{A}^{\beta X}$, $j : A \to \beta X$, $k : \bar{A}^{\beta X} \to \beta X$ and homeomorphism $\varphi : \bar{A}^{\beta X} \to \beta A$ (see [En$_1$]) for which hold equalities $\varphi \cdot i = j$

and $\beta(f)\cdot j = f$, where $\beta(f)$ is an extension of map f. For arbitrary extension $\psi : \bar{A}^{\beta X} \to [0,1]$ of map f we have $\beta(f)\cdot\varphi_{|A} = \psi_{|A}$. It is clear that $\psi = \beta(f)\cdot\varphi$.

Let $l = k\cdot i$. Besides, there exists an extension $\tilde{f} : \beta X \to [0,1]$ of composition $\beta(f)\cdot\varphi$, i.e. $\tilde{f}\cdot k = \beta(f)\cdot\varphi$.

Note that

$$\tilde{f}\cdot l = \tilde{f}\cdot(k\cdot i) = (\tilde{f}\cdot k)\cdot i = (\beta(f)\cdot\varphi)\cdot i = \beta(f)\cdot(\varphi\cdot i) = \beta(f)\cdot j = f.$$

Now we list a numbers of facts about CE-subspaces, wich are easy consequences of general results of compactifications theory. Note that, if A is CE-subspace of X and X is the CE-subspace of Y, then A is the CE- subspase of Y. Basides, if A is the CE-subspace of completely regular space X, then A and $\overline{A}^{\beta X}$ are CE-subspaces of βX.

The main result about the connection between Čech (co)homology groups of compactifications of completely regular spaces and Čech functional (co)homology groups of completely regular spaces is the following

Theorem 3.1. *For each pair* $(X,A) \in ob(\mathbf{Top}^{\mathbf{2}}_{\mathbf{ce}})$ *one has*

$$\check{H}^F_n(X,A;G) = \check{H}_n(\beta X, \overline{A}^{\beta X}; G)$$

and

$$\hat{H}^n_F(X,A;G) = \hat{H}^n(\beta X, \overline{A}^{\beta X}; G).$$

Proof. Let X be a completely regular space and A be a subspace with CE-property. For each finite open cover $\alpha = \{\alpha_{u_i}\}_{u_i\in V_\alpha}, i = 1,2,...,k$ of βX there exists a functionally open shrinking $\beta = \{\beta_{v_i}\}_{v_i\in V_\beta}, i = 1,2,...,k$ of α such that $\bar{\beta}_{v_i} \subset \alpha_{u_i}, i = 1,2,...,k$. For the cover $\beta\wedge X = \{\beta_{v_i}\cap X\}_{v_i\in V_{\beta\wedge X}}, i = 1,2,...,k$ there exist a functionally open shrinking $\gamma = \{\gamma_{w_i}\}_{w_i\in V_\gamma}, i = 1,2,...,k$.

Let $E_X(\gamma_{w_i}) = \beta X \setminus \overline{(X\setminus\gamma_{w_i})}, i = 1,2,...,k$. From relation $E_X(\gamma_{w_i}) \subset \overline{\gamma_{w_i}} \subset \overline{\beta_{v_i}} \subset \alpha_{u_i}$ it follows that the famili $E_X(\gamma) = \{E_X(\gamma_{w_i})\}_{i=1}^k$ is an open shrinking of $\alpha = \{\alpha_{u_i}\}_{u_i\in V_\alpha}$. Note that $E_X(\gamma)\wedge X = \gamma$ and $E_X(\gamma) > \alpha$

Let $\alpha = \{\alpha_{u_i}\}_{u_i\in(V_\alpha, V_\alpha^A)}$ be a cover of $(X,A) \in ob(\mathbf{Top}^{\mathbf{2}}_{\mathbf{ce}})$. The closure $\overline{A}^{\beta X}$ of A in βX is equivalent to the Stone-Čech compactification βA of A. For finite functionally open cover α

$$E_X(\alpha_{u_1}) \cap E_X(\alpha_{u_2}) \cap ...E_X(\alpha_{u_n}) \cap \overline{A}^{\beta X} \neq \varnothing$$

if and only if

$$\alpha_{u_1} \cap \alpha_{u_2} \cap ... \cap \alpha_{u_n} \cap A \neq \varnothing.$$

The given arguments show that the nerves of functionally open cover of α and $E_X(\alpha)$ of pairs (X,A) and $(\beta X, \overline{A}^{\beta X})$ are isomorphical simplisial complexses.

The set $\{\alpha = \beta\wedge X | \beta \in cov_F(\beta X)\}$ is cofinal subset of set $cov_F(X)$ and the set $\{E_X(\alpha)|\alpha = \beta\wedge X, \beta \in cov(\beta X)\}$ is cofinal subset of $cov(\beta X)$. The simplicial isomorphisms yield the isomorphisms of groups

$$H_n(X_\alpha, A_\alpha; G) \approx H_n((\beta X)_{E_X(\alpha)}, (\overline{A}^{\beta X})_{E_X(\alpha)}; G)$$

for each $\alpha = \beta \wedge X_1,\ \beta \in cov_F\left(\beta X\right)$. Hence, the limit groups

$$\check{H}_n(\beta X, \overline{A}^{\beta X}; G) = \varprojlim (H_n((\beta X)_\beta, (\overline{A}^{\beta X})_\beta; G), p_{\beta\beta'_*}, cov_F(\beta X))$$

and

$$\check{H}_n^F(X, A; G) = \varprojlim \{H_n(X_\alpha, A_\alpha; G), p_{\alpha,\alpha'*}, cov_F(\beta X)\}$$

are isomorfical.

Applying last isomorphisms we conclude that $\check{H}_n^F(X, A; G) = \check{H}_n(\beta X, \overline{A}^{\beta X}; G)$.

We analogously can verify that $\hat{H}_F^n(X, A; G) = \hat{H}^n(\beta X, \overline{A}^{\beta X}; G)$. □

Corollary 3.2. *The compactification* $(\beta X, \overline{A}^{\beta X})$ *of pair* $(X, A) \in ob\left(\mathbf{Top^2_{ce}}\right)$ *has Čech (co)homology group isomorphic to abelian group* M *if and only if* (X, A) *has Čech functional (co)homology group isomorphic to* M.

Now give the following

Definition 3.3. Let G be an abelian group. A functional coefficient of cyclisity $\eta_G^F(X)$ of $X \in ob(\mathbf{Top_{ce}})$ with coefficients group G is equal to n, if $\hat{H}_F^m(X; G) = 0$ for each $m > n$ and $\hat{H}_F^n(X; G) \neq 0$. We put $\eta_G^F(X) = \infty$ if $X \neq \varnothing$ and for every m there is an $n \geq m$ with $\hat{H}_F^n(X; G) \neq 0$.

Thus, the functional coefficient of cyclicity of X with coefficients group G is a function $\eta_G^F : \mathbf{Top_{ce}} \to \boldsymbol{N} \cup \{\mathbf{0}, \infty\} : \mathbf{X} \to \mathbf{n}$, such that $\eta_G^F(X) = n$.

The following therem is a direct consequence of Theorem 3.1.

Theorem 3.4. *For each completely regular space* X *holds equality*

$$\eta_G^F\left(X\right) = \eta_G\left(\beta X\right).$$

Proof. Assume that $\eta_G(\beta X) = n$. Then for every $m > n$ and $\hat{H}^m(\beta X; G) = 0$ and $\hat{H}^n(\beta X; G) \neq 0$. From the relation $\hat{H}^k(\beta X; G) = \hat{H}_F^k(X; G)$ it follows that $\hat{H}_F^m(X; G) = 0$ for every $m > n$ and $\hat{H}_F^n(X; G) \neq 0$. Thus, $\eta_G^F(X) = n$. Consequently, $\eta_G^F(X) = \eta_G(\beta X)$. □

Corollary 3.5. *For each completely regular space* X *the coefficient of cyclicity* $\eta_G(\beta X)$ *is equal to number* n *if and only if the functional coefficient of cyclicity* $\eta_G^F(X)$ *is equal to* n.

The construction of meaningfull cohomological dimension theory for general topological spaces is one of main problems of modern topology. It is known that the theorems of classical cohomological dimension theory ([A], [Bo], [D], [D-Dy-W], [Dy], [K], [Ku], [M-S], [Mi], [N], [Na], [No], [S], [Sk], [H-Wa]) of "good" (metric compact or Hausdorff compact) spaces often are impossible to generalize for completely regular spaces. In the paper we will also investigate this problem, using Čech functional cohomology theory. For this aim we define functional cohomological dimension and establish its some main properties.

Definition 3.6. The functional cohomological dimension $dim_G^F X$ of a completely regular space X is defined to be the largest integer $n \geq 0$, such that $\hat{H}_F^n(X, A; G) \neq 0$ for CE-subspace A of X.

The functional cohomological dimension of X is function $dim_G^F : \mathbf{Top_{ce}} \rightarrow N \cup \{0, \infty\} : X \rightarrow n$, where $dim_G^F X = n$.

Theorem 3.7. *For each completely regular space X holds the relation*

$$dim_G^F X = dim_G \beta X.$$

This Theorem is consequence of Theorem 3.1.

Corollary 3.8. *For functionally covering dimension of completely regular space X, $\dim X = 0$ [En_1] if and only if $dim_G^F X = 0$.*

Theorem 3.9. *The following propositions are equivalent:*

i) $dim_G^F X \leq n$;

ii) For each $m \geq n$ and CE-subspace A of completelly regular space X the induced homomorphism $\check{H}_F^n(X : G) \rightarrow H_F^n(A; G)$ is an epimorphism.

Proof. The proof follows from Theorem 3.1. and Theorem 37.7. of Skliarenko (see [N]). □

Theorem 3.10. *Let A be a CE-subspace of completely regular space X. Then*

$$dim_G^F A \leq dim_G^F X.$$

Proof. Note that

$$dim_G^F A = dim_G \beta A = dim_G \overline{A}^{\beta X} \leq dim_G \beta X = dim_G^F X.$$

Hence, $dim_G^F A \leq dim_G^F X$. □

Theorem 3.11. *Let completely regular spece X is union of CE-subspaces X_1 and X_2. If $dim_G^F X_1 \leqslant m$ and $dim_G^F X_2 \leqslant n$, then $dim_G^F X \leq \max\{m, n\}$.*

Proof. Note that $\beta X = \overline{X} = \overline{X_1 \cup X_2} = \overline{X}_1 \cup \overline{X}_2$. Since $\overline{X}_1$ and $\beta X_1, \overline{X}_2$ and βX_2 are homeomorphic spaces and holds Theorem 3.6, we have the following relations

$$dim_G \overline{X}_1 = dim_G \beta X_1 = dim_G^F X_1 \leq m$$

and

$$dim_G \overline{X}_2 = dim_G \beta X_2 = dim_G^F X_2 \leq n.$$

From Theorem 38.3 of Kodama (see [N]) it follows that

$$dim_G \beta X = dim_G(\overline{X}_1 \cup \overline{X}_2) \leq \max\{m, n\}.$$

By theorem 3.7 we have $dim_G^F X \leq \max\{m, n\}$. □

From above obtained results are follow the following propositions.

Proposition 3.12. Let $\{G_\alpha, p_{\alpha\alpha'}, A\}$ be a direct system of abelian groups and $G = \varinjlim\{G_\alpha, p_{\alpha\alpha'}, A\}$. If for each $\alpha \in A$ and completely regular space X $dim^F_{G_\alpha} X \leq n$, , then $dim^F_G X \leq n$.

Proposition 3.13. If $G = \underset{\alpha\in A}{\oplus} G_\alpha$, then for each completely regular space X

$$dim^F_G X = \max\{dim^F_{G_\alpha} X | \alpha \in A\}.$$

Using Theorem 3.7 and results of papers [D-Dy-W] and [Mi] we can easily prove the following

Theorem 3.14. *Let*

$$0 \to G' \to G \to G'' \to 0$$

be a short exact sequence of Abelian groups. Then the equalities $dim^F_G X \leq \max\{dim^F_{G'} X, dim^F_{G''} X\}, dim^F_{G'} X \leqslant \max\{dim^F_G X, dim^F_{G''} X{+}1\}$ *and* $dim^F_{G''} X \leq \max\{dim^F_G X, dim^F_{G'} X - 1\}$ *are hold.*

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