ON THE TANGENT FIBER SURFACE OF THE SPACE $T(Vn)$

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Abstract. The paper considers the theory of the surface of metric tangent fibers for the space $T(Vn)$. Analogues of the Gauss–Weingarten derivation formulas, as well as analogues of the generalized Gauss, Peterson–Codazzi–Mainardi equations, are found.

Let us consider the tangent fiber space $T(Vn)$, where the local coordinates of a point transform by the law

$$\overline{x}^i = \overline{x}^i \left(x^k \right), \quad \overline{y}^i = \overline{x}_k^i y^k, \quad i, j, k, \overline{i}, \overline{j}, \overline{k} = 1, 2, \ldots, n.$$

Assume that the tensor field G_{AB} $(A, B, C = 1, 2, \ldots, 2n)$ is given on the space $T(Vn)$, i. e.,

$$\overline{G_{AB}} = \overset{*}{\mathfrak{X}}_A^C \overset{*}{\mathfrak{X}}_B^D G_{CD},$$

where

$$\mathfrak{X}_B^A = \left\| \frac{\partial \overline{\mathfrak{X}}^A}{\partial \overline{\mathfrak{X}}^B} \right\| = \left\| \begin{matrix} \frac{\partial \overline{x}^i}{\partial x^j} & \frac{\partial \overline{x}^i}{\partial y^j} \\ \frac{\partial \overline{y}^i}{\partial x^j} & \frac{\partial \overline{y}^i}{\partial y^j} \end{matrix} \right\| = \left\| \begin{matrix} x_j^i & 0 \\ x_{pj}^i y^p & x_j^i \end{matrix} \right\|.$$

An inverse matrix of the matrix has the form

$$\mathfrak{X}_B^A = \left\| \frac{\partial \mathfrak{X}^A}{\partial \overline{\mathfrak{X}}^B} \right\| = \left\| \begin{matrix} \frac{\partial x^i}{\partial \overline{x}^k} & \frac{\partial x^i}{\partial \overline{y}^k} \\ \frac{\partial y^i}{\partial \overline{x}^k} & \frac{\partial y^i}{\partial \overline{y}^k} \end{matrix} \right\| = \left\| \begin{matrix} \overset{*}{x}_k^i & 0 \\ \overset{*}{x}_{jk}^i \overset{*}{x}_p^j y^p & \overset{*}{x}_k^i \end{matrix} \right\|.$$

Since $G^{\overline{ij}} = x_k^i x_p^j G^{\overline{kp}}$, where $G_{\overline{ij}} G^{\overline{jk}} = \delta_i^k$ and $G_{\overline{l}i} = \overset{*}{x}_l^m \overset{*}{x}_i^k G_{\overline{m}k} + \overset{*}{x}_l^m \overset{*}{x}_{qi}^h \overset{*}{x}_s^q y^s G_{\overline{m}h}$, we can use them to construct the values Γ_i^k as follows: $\Gamma_i^k = G^{\overline{k}j} G_{\overline{j}i}$.

Furthermore,

$$G^{\overline{j}\overline{p}} G_{\overline{p}i} = x_q^j x_i^p G^{\overline{q}l} \overset{*}{x}_p^s \overset{*}{x}_i^k G_{\overline{s}k} + x_p^j x_q^s G^{\overline{pq}} \overset{*}{x}_s^l \overset{*}{x}_{mi}^k x_h^m y^h G_{\overline{lk}}.$$

Since $x_j^i \overset{*}{x}_k^j = \delta_k^i$, $x_{pk}^j \overset{*}{x}_q^p + x_l^j \overset{*}{x}_{qi}^l x_k^i = 0$, $-\overset{*}{x}_i^k x_{pk}^j \overset{*}{x}_s^p = x_l^j \overset{*}{x}_{si}^l$, we observe that the values Γ_i^k form a linearly connected object with the following transformation law:

$$\overset{*}{\Gamma}_i^j = x_p^j \overset{*}{x}_i^k \Gamma_k^p - \overset{*}{x}_i^k x_{pk}^j y^p,$$

and the values

$$g_{ij} = G_{ij} - \Gamma_i^k G_{\overline{k}j} - \Gamma_j^k G_{\overline{k}i} + \Gamma_i^p \Gamma_j^q G_{\overline{pq}}$$

are a double covariant symmetric tensor, so we can construct an affine connection object Γ_{jk}^i in the following manner:

$$\Gamma_{jk}^i = \frac{1}{2} g^{ip} \left(\nabla_k g_{pj} + \nabla_j g_{kp} - \nabla_p g_{jk} \right).$$

Note that the linear connectedness Γ_i^j induces the vertical affine connectedness defined by the object $\nabla_{\overline{k}} \Gamma_i^j \equiv \Gamma_{\overline{k}i}^j$ with the following transformation law:

$$\overline{\Gamma_{\overline{p}i}^j} = \overset{*}{x}_i^k \overset{*}{x}_p^q x_l^j \Gamma_{\overline{q}k}^l - x_{pi}^j.$$

Structural equations of the space $T(Vn)$ have the form:

$$\begin{cases} D\omega^i = \omega^k \wedge \widetilde{\omega}_k^i, \\ D\widetilde{\theta^l} = \widetilde{\theta^k} \wedge \widetilde{\omega k}^{\bar{l}} + R_{ik}^{\bar{l}}\omega^i \wedge \omega^k, \\ D\widetilde{\omega}_{\bar{p}}^{\bar{l}} = \widetilde{\omega}_{\bar{p}}^{\bar{l}} \wedge \widetilde{\omega}_{\bar{l}}^{\bar{l}} + R_{\bar{p}ik}^{\bar{l}}\omega^i \wedge \omega^k + R_{\bar{p}i\bar{l}}^{\bar{l}}\omega^i \wedge \widetilde{\theta}^l, \\ D\widetilde{\omega}_{\bar{j}}^i = \widetilde{\omega}_{\bar{j}}^k \wedge \widetilde{\omega}_k^i + R_{jpq}^i\omega^p\omega^q + R_{jp\bar{k}}^i\omega^p \wedge \widetilde{\theta}^k. \end{cases}$$

Assume that a hypersurface $\mathfrak{M}$ is given on the space $T(Vn)$,

$$\omega^i = W_a^i \Psi^a, \tag{1}$$

and the 1-forms Ψ^a are such that

$$\begin{cases} D\Psi^a = \Psi^b \wedge \Psi_b^a, \\ D\Psi_b^a = \Psi_b^c \wedge \Psi_c^a + \Psi^c \wedge \Psi_{bc}^a. \end{cases}$$

Note that

$$D\widetilde{\theta^j} = \widetilde{\theta^l} \wedge \widetilde{\omega}_{\bar{l}}^{\bar{j}} + R_{ik}^{\bar{j}}\omega^i \wedge \omega^k = \widetilde{\theta^l} \wedge \widetilde{\omega}_{\bar{l}}^{\bar{j}} + R_{jpq}^i\omega^p\omega^q + R_{ik}^{\bar{j}}W_a^i\Psi^a \wedge W_a^k\Psi^a = R_{ab}^{\bar{j}}\Psi^a \wedge \Psi^b,$$

where $R_{ab}^{\bar{j}} = R_{ik}^{\bar{j}}W_a^iW_b^k$.

The extension of system (1) is given by

$$\begin{cases} \nabla W_a^i = W_{ab}^i\Psi^b, \quad \nabla W_{ab}^i + W_c^i\Psi_{ab}^c = W_{abc}^i\Psi^c, \\ \nabla W_{abc}^i + 2W_{(a|d|}^i\Psi_{b)c}^d - W_d^i\Psi_{bc}^d = W_{abd}^i\Psi^d, \end{cases}$$

where

$$\begin{cases} \nabla W_{[a,b]}^i = 0, \quad \nabla W_{a[bc]}^i = -R_{jpq}^iW_a^pW_b^qW_c^j, \\ \nabla W_{ab[cd]}^i = -R_{pqj}^iW_{ab}^pW_c^qW_d^j. \end{cases}$$

The values W_a^i, W_{ab}^i and W_{abc}^i form a fundamental third-order difference-geometric object of the surface $\mathfrak{M}$.

The normal vector of the hypersurface $\mathfrak{M}$ at the point T satisfies the equations

$$g_{ij}n^iW_a^i = 0, \quad g_{ij}n^in^j = 1.$$

A metric tensor of the hypersurface $\mathfrak{M}$ is written in the form

$$g_{ab} = g_{ij}W_a^iW_b^j \quad \text{and} \quad \nabla g_{ab} = g_{abc}\Psi^c,$$

where $g_{abc} = g_{ij}W_a^iW_{bc}^j + g_{ij}W_{ac}^iW_b^j$.

The vectors $W_{ab}^ie_i$ and $n_a^ie_i$ admit representations in the form of a linear combination of vectors of the reference point $\{T, W_a, n\}$:

$$W_{ab}^ie_i = \mathbb{U}_{ab}^c + \mathcal{L}_{ab}n, \tag{2}$$

$$n_a^ie_i = \mathcal{L}_a^bW_b + n_an, \tag{3}$$

where

$$\mathbb{U}_{ab}^c = g^{cd}g_{ik}W_{ab}^iW_d^k, \quad \mathcal{L}_{ab} = g_{ki}n^kW_{ab}^i, \quad \mathcal{L}_a^b = -g^{cb}\mathcal{L}_{ca}, \quad n_a = g_{ki}n^kn_a^i. \tag{4}$$

Hence the following theorem follows.

Theorem 1. *If a hypersurface $\mathfrak{M}$ (1) is defined on the space $T(V_n)$, then the Gauss–Weingarten derivation formulas have the form (2) and (3).*

We call equations (2) and (3) the Gauss–Weingarten formulas of the hypersurface $\mathfrak{M}$. From (4), we obtain

$$\nabla\mathbb{U}_{ab}^c + W_{ab}^c = \mathbb{U}_{abd}\Psi^d, \tag{5}$$

$$\nabla\mathcal{L}_{abc}\Psi^c, \tag{6}$$

where

$$\begin{cases} \mathbb{U}^c_{abd} = g^{ec}_{\ldots,d} g_{ik} W^i_{ab} W^k_e + g^{ei} g_{ik} W^i_{abd} W^k_e + g^{ec} g_{ik} W^i_{ab} W^k_{ed}, \\ \mathcal{L}_{abc} = g_{ki} n^k_c W^i_{ab} + g_{ki} n^k W^i_{abc}, \\ g^{ec}_{\ldots,d} = -g^{ea} g^{bc} g_{abd}. \end{cases}$$

From (5) and (6) it follows that $\mathbb{U}^c_{ab}$ is the object of affine connectedness and $\mathcal{L}_{ab}$ is the tensor.

We call the object $\mathbb{U}^c_{ab}$ the object of the induced affine connectedness of the hypersurface $\mathfrak{M}$. It is easy to prove that the induced affine connectedness and the internal affine connectedness coincide. The 1-forms of this connectedness have the form:

$$\widetilde{\Psi}^a_b = \Psi^a_b + \mathbb{U}^a_{bc}\Psi^c.$$

Obviously,

$$D\Psi^a = \Psi^b \wedge \widetilde{\Psi}^a_b, \quad D\widetilde{\Psi}^a_b = \widetilde{\Psi}^c_b \wedge \widetilde{\Psi}^a_c + W^a_{bcd}\Psi^c \wedge \Psi^d,$$

where

$$W^a_{bcd} = \mathbb{U}^a_{b[cd]} - \mathbb{U}^a_{e[c}\mathbb{U}^e_{|b|d]}.$$

The values W^a_{bcd} form the tensor which we call the curvature tensor of the hypersurface $\mathfrak{M}$.

Expanding equation (2), we obtain

$$R^i_{jpq} W^k_a W^p_b W^q_c = \left(W^d_{abc} - \mathcal{L}_{a[b}\mathcal{L}^d_{c]} \right) - \left(\overset{k}{\nabla}_{[c}\mathcal{L}_{|a|b]} - W^d_{bc}\mathcal{L}_{ad} + \mathcal{L}_{a[b}n_{c]} \right) n^i,$$

where $\overset{k}{\nabla}_c$ is the symbol of nonholomorphic covariant differentiation.

From the above equalities, we obtain the generalized Gauss equations

$$R_{jpq} W^j_a W^p_b W^q_c W^i_e = W^i_{abce} + \mathcal{L}_{a[b}\mathcal{L}_{c]d} \tag{7}$$

and the generalized Peterson–Codazzi–Mainardi equations

$$R_{kpqr} W^k_a W^q_b W^r_c n^p = W^d_{bc}\mathcal{L}_{ad} - \overset{k}{\nabla}_{[c}\mathcal{L}_{|a|b]} - \mathcal{L}_{a[b}n_{c]}, \tag{8}$$

where

$$R_{ipqr} = g_{ij}R^j_{pqr}, \quad W_{abce} = g_{de}W^d_{abc}.$$

Hence the following theorem holds.

Theorem 2. *If a hypersurface $\mathfrak{M}$ (1) is defined on the space $T(Vn)$, then the generalized Gauss equations and the generalized Peterson–Kodashi–Maitardi equations have the form* (7) *and* (8).

E quations (7) and (8) establish the connection between the curvature tensor of the space $T(Vn)$ and the curvature tensor of the hypersurface $\mathfrak{M}$ in $T(Vn)$ [1–3, 7].

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