

CHARACTERIZATION OF LIE HOMOMORPHISMS ON A CLASS OF BANACH ALGEBRAS

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ABSTRACT. In this article, we study the continuous linear maps $f : A \rightarrow B$ between Banach algebras behaving like Lie homomorphisms at zero or idempotent products. More precisely, we consider the subsequent condition $f([a, b]) = [f(a), f(b)]$ for all $a, b \in A$ with $ab = 0$ or $ab = p$, where p is a nontrivial idempotent, and show that if A is a zero product determined with a bounded approximate identity, then f is a Lie homomorphism. As an application we describe the same results for continuous Lie derivations δ from A into a Banach A -bimodule X .

1. INTRODUCTION AND PRELIMINARIES

Let A and B be two associative algebras over \mathbb{C} . A linear map $f : A \rightarrow B$ is called a *Lie homomorphism* if for all $a, b \in A$,

$$f([a, b]) = [f(a), f(b)], \quad (1)$$

where, $[a, b]$ denotes the commutator $ab - ba$.

A simple example of a Lie homomorphism $f : A \rightarrow B$ is a map of the form $f = \theta + \mu$ where $\theta : A \rightarrow B$ is a homomorphism or the negative of an anti-homomorphism, and μ is a linear map of A into $Z(B)$, the center of B , which maps commutators into zero. By [12, Theorem 2.4], every Lie isomorphism between nest algebras is of such a form. The analogous result was obtained for factor von Neumann algebras [15, Theorem 4].

Let A be an algebra and X be an A -bimodule. A linear map $\delta : A \rightarrow X$ is called a *Lie derivation* if for all $a, b \in A$,

$$\delta([a, b]) = [\delta(a), b] + [a, \delta(b)]. \quad (2)$$

Miers in [16, Theorem 3], proved that if $\delta : A \rightarrow A$ is a Lie derivation, where A is a von Neumann algebra, then there exists an operator $T \in A$ such that $\delta(a) = [a, T] + \mu(a)$, where μ is a linear map of A into $Z(A)$ which maps commutators into zero. Note that if $d : A \rightarrow A$ defined by $d(a) = [a, T]$, then d is derivation and $\delta = d + \mu$.

In 1996, Johnson showed [10] that every continuous Lie derivation δ from a C^* -algebra A into a Banach A -bimodule X can be decomposed as $\delta = d + \mu$, where $d : A \rightarrow X$ is a derivation and μ is a linear map from A into the centre of X , vanishing at commutators $[a, b]$ for all $a, b \in A$, see also [13].

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Recall that the centre of A -bimodule X is defined as

$$Z(X) = \{x \in X : ax = xa \text{ for all } a \in A\}.$$

A linear map $f : A \rightarrow B$ is called a Lie homomorphism at a given point $w \in A$ if the equality (1) holds for all $a, b \in A$ with $ab = w$. A Lie derivation at a given point $w \in A$ can be defined analogously.

This kind of maps was discussed by several authors. For example, it is proved that every linear map preserving Lie products at zero points on any finite nest algebra is a Lie homomorphism [14]. Let $B(X)$ be the algebra of all bounded linear operators on Banach space X with $\dim X \geq 3$. Then it is proved in [11] that if δ is a Lie derivation at $w = 0$ ($w = p$, where p is a fixed nontrivial idempotent of $B(X)$), then δ can be decomposed as $\delta = d + \mu$, where d and μ are as above. In [9], the authors gave a similar result for factor von Neumann algebras with dimension greater than 4.

Recently, many researcher have considered various maps on rings or algebras through local actions, and different results have been obtained. For example, we refer the reader to [2, 6, 7, 8, 13, 17, 18] and the references therein.

In this paper, we investigate continuous linear maps from important classes of Banach algebras (that include C^* -algebras, group algebras, and Banach algebras generated by idempotents) into other Banach algebras, where the maps act as Lie homomorphisms at zero or on idempotent products. As a consequence, we also study continuous linear maps that behave as Lie derivations at zero or on idempotent products.

2. CHARACTERIZATION OF LIE HOMOMORPHISMS

Obviously, every homomorphism (derivation) is a Lie homomorphism (derivation), but the converse fails, in general. The following examples illustrate this fact.

Example 2.1. Let

$$A = \left\{ \begin{bmatrix} 0 & z_1 & z_2 \\ 0 & 0 & z_3 \\ 0 & 0 & 0 \end{bmatrix} : z_1, z_2, z_3 \in \mathbb{C} \right\}.$$

Define $f : A \rightarrow A$ by

$$f\left(\begin{bmatrix} 0 & z_1 & z_2 \\ 0 & 0 & z_3 \\ 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & -z_3 & -z_2 \\ 0 & 0 & -z_1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then $f([a, b]) = [f(a), f(b)]$ for all $a, b \in A$, hence f is a Lie homomorphism, but f is not a homomorphism.

Example 2.2. Let A be a Banach algebra and $f : A \rightarrow \mathbb{C}$ be a non-zero multiplicative linear functional. Define the linear map $\delta : A \rightarrow A^*$ by $(\delta(a))(x) = f(ax)$, where A^* is the dual of A equipped with the usual module structure. Then

$$\delta([a, b])(x) = f(abx) - f(bax) = 0, \quad a, b, x \in A.$$

On the other hand,

$$\begin{aligned}
([\delta(a), b] + [a, \delta(b)])(x) &= (\delta(a)b - b\delta(a))(x) + (a\delta(b) - \delta(b)a)(x) \\
&= \delta(a)(bx) - \delta(a)(xb) + \delta(b)(xa) - \delta(b)(ax) \\
&= f(abx) - f(axb) + f(bxa) - f(bax) \\
&= 0.
\end{aligned}$$

Therefore, $\delta([a, b]) = [\delta(a), b] + [a, \delta(b)]$ for all $a, b \in A$ and hence δ is a Lie derivation. However, δ is not a derivation, in general.

Recall that a bounded net $\{e_\alpha\}_{\alpha \in I}$ in A is called a *bounded approximate identity* if for each $a \in A$,

$$\|e_\alpha a - a\| \longrightarrow 0, \quad \|ae_\alpha - a\| \longrightarrow 0.$$

It is known that C^* -algebras and the group algebra $L^1(G)$ for a locally compact group G has a bounded approximate identity bounded by one, see [5].

The Banach algebra A is said to be *zero product determined* if every continuous bilinear mapping $\phi : A \times A \longrightarrow X$, where X is an arbitrary Banach space, satisfying

$$a, b \in A, \quad ab = 0 \quad \implies \quad \phi(a, b) = 0, \quad (3)$$

can be written as

$$\phi(a, b) = T(ab), \quad a, b \in A,$$

for some continuous linear map $T : A \longrightarrow X$. This concept appeared as a byproduct of the so-called property (\mathbb{B}) introduced in [1].

The Banach algebra A has *property (\mathbb{B})* if for every continuous bilinear map ϕ from $A \times A$ into an arbitrary Banach space X , the condition (3) implies that $\phi(ab, c) = \phi(a, bc)$, for all $a, b, c \in A$. Note that if A has a bounded approximate identity, then A is zero product determined if and only if A has property (\mathbb{B}) , [3, Proposition 5.5].

It turns out in [1] that every C^* -algebra, as well as every group algebra $L^1(G)$ of locally compact group G , has the property (\mathbb{B}) .

We now formulate the first main result.

Theorem 2.3. *Let A be a Banach algebra with property (\mathbb{B}) and having a bounded approximate identity and B be a Banach algebra. If $f : A \longrightarrow B$ is a continuous linear map such that condition*

$$a, b \in A, \quad ab = 0 \quad \implies \quad f([a, b]) = [f(a), f(b)], \quad (4)$$

holds, then f is a Lie homomorphism.

Proof. Define a continuous bilinear map $\phi : A \times A \longrightarrow B$ by

$$\phi(a, b) = f([a, b]) - [f(a), f(b)], \quad a, b \in A.$$

Then $ab = 0$ implies that $\phi(a, b) = 0$, and so the property (\mathbb{B}) gives

$$\begin{aligned} f([ab, c]) - [f(ab), f(c)] &= \phi(ab, c) \\ &= \phi(a, bc) \\ &= f([a, bc]) - [f(a), f(bc)], \end{aligned}$$

for all $a, b, c \in A$. Letting $c = e_\alpha$ in the above equality, we get

$$- [f(ab), f(e_\alpha)] = f([a, b]) - [f(a), f(b)], \quad a, b \in A. \quad (5)$$

Similarly, for $a = e_\alpha$ we obtain

$$f([b, c]) - [f(b), f(c)] = -[f(e_\alpha), f(bc)], \quad b, c \in A. \quad (6)$$

It follows from (5) and (6) that

$$[f(ab), f(e_\alpha)] = [f(e_\alpha), f(ab)] = -[f(ab), f(e_\alpha)],$$

and hence $[f(ab), f(e_\alpha)] = 0$ for all $a, b \in A$.

Now, the equality (5) implies that $f([a, b]) = [f(a), f(b)]$ for all $a, b \in A$. \square

As a consequence of Theorem 2.3, we obtain the following results.

Corollary 2.4. *Let A be a C^* -algebra and B be a Banach algebra. Let $f : A \rightarrow B$ be a continuous linear map satisfying (4). Then f is a Lie homomorphism.*

Corollary 2.5. *Let G be a locally compact group. Then every continuous linear map f from $L^1(G)$ into a Banach algebra B satisfying (4) is a Lie homomorphism.*

It is known that the matrix algebra $M_n(A)$ of $n \times n$ matrices over a unital Banach algebra A has property (\mathbb{B}) . Thus, from Theorem 2.3 we have

Corollary 2.6. *Every continuous linear map f from $M_n(A)$ into a Banach algebra B satisfying (4) is a Lie homomorphism.*

A von Neumann algebra A is called a *factor von Neumann algebra* if $Z(A) = \mathbb{C}e_A$, where e_A is the unit of A . It should be pointed out that factor von Neumann algebras are unital prime algebras.

From Theorem 2.3 and [15, Theorem 4], we have the next result.

Corollary 2.7. *Let A and B be factor von Neumann algebras and $f : A \rightarrow B$ be a continuous bijective linear map. Then f satisfies the condition (4) if and only if $f = \theta + \mu$, where θ is an isomorphism or the negative of an anti-isomorphism, and μ is a linear functional such that $\mu([a, b]) = 0$ for all $a, b \in A$.*

We denote by $\mathfrak{J}(A)$ the subalgebra of A generated algebraically by all idempotents in A . A Banach algebra A is generated by idempotents, if $A = \overline{\mathfrak{J}(A)}$. Since every Banach algebra with the property that $A = \overline{\mathfrak{J}(A)}$ has property (\mathbb{B}) (see [1]), therefore the next result follows.

Corollary 2.8. *Let A be a Banach algebra with a bounded approximate identity and the property that $A = \overline{\mathfrak{J}(A)}$. Let f be a continuous linear map from A into a Banach algebra B such that condition (4) holds. Then f is a Lie homomorphism.*

Corollary 2.8 can be compared with [8, Theorem 2.1], which is the analogous conclusion under the assumption that A and B are locally convex algebras.

Some examples of Banach algebras with the property that $A = \overline{\mathfrak{J}(A)}$, are listed below. For further examples of such Banach algebras we refer the reader to [1].

- (1) The Banach algebra $C(X)$, where X is a compact Hausdorff space.
- (2) The group algebra $L^1(G)$ for a compact group G .
- (3) Topologically simple Banach algebras containing a non-trivial idempotent.
- (4) Simple C^* -algebras that are linearly spanned by their projections.
- (5) $B(\mathcal{H})$, the operator algebra of all bounded linear operators on Hilbert space \mathcal{H} .

Consequently, the conclusion of Corollary 2.8 holds for such algebras.

The following result follows from [18, Theorem 2.1]. However, we prove it for sake of completeness.

Theorem 2.9. *Let A be a unital Banach algebra and X be a Banach space. Suppose that $\phi : A \times A \rightarrow X$ is a continuous bilinear mapping such that*

$$a, b \in A, \quad ab = p \implies \phi(a, b) = 0, \quad (7)$$

where p is an idempotent in A . Then, for all $a \in A$,

$$\phi(e_A, ap) = \phi(a, p) \quad \text{and} \quad \phi(e_A, a^2p) = \phi(a, ap).$$

Proof. Since $e_A p = p$, it follows that $\phi(e_A, p) = 0$. Let $a \in A$ be an arbitrary nonzero element. For $\lambda \in \mathbb{C}$ with $|\lambda| < 1/\|a\|$, $e_A - \lambda a$ is invertible and $(e_A - \lambda a)^{-1} = \sum_{n=0}^{\infty} \lambda^n a^n$. Since $(e_A - \lambda a)(e_A - \lambda a)^{-1}p = p$, it follows from condition (7) that

$$\phi(e_A - \lambda a, \sum_{n=0}^{\infty} \lambda^n a^n p) = 0.$$

Therefore,

$$\sum_{n=1}^{\infty} \phi(e_A, \lambda^n a^n p) - \lambda \sum_{n=0}^{\infty} \phi(a, \lambda^n a^n p) = 0,$$

and hence

$$\sum_{n=0}^{\infty} \lambda^{n+1} (\phi(e_A, a^{n+1}p) - \phi(a, a^n p)) = 0.$$

Thus, $\phi(e_A, a^{n+1}p) = \phi(a, a^n p)$ for all $a \in A$. In particular, we have

$$\phi(e_A, ap) = \phi(a, p) \quad \text{and} \quad \phi(e_A, a^2p) = \phi(a, ap),$$

for all $a \in A$. □

In the following theorem, $\mathfrak{L}(A)$ stands for the linear span of all idempotents in A .

Theorem 2.10. *Let A be a unital Banach algebra with the property that $A = \overline{\mathfrak{L}(A)}$. Let f be a continuous linear map from A into a Banach algebra B such that for every non-trivial idempotent p in A ,*

$$a, b \in A, \quad ab = p \quad \implies \quad f([a, b]) = [f(a), f(b)]. \quad (8)$$

Then f is a Lie homomorphism.

Proof. Let p be a non-trivial idempotent in A . As $pe_A = p$, we get $[f(p), f(e_A)] = 0$. Since the linear span of A generated by its idempotents is dense in A , we conclude that

$$[f(a), f(e_A)] = 0, \quad a \in A. \quad (9)$$

Define a bilinear map $\phi : A \times A \longrightarrow B$ via $\phi(a, b) = f([a, b]) - [f(a), f(b)]$. Then $ab = p$ implies that $\phi(a, b) = 0$. Applying Theorem 2.9, we obtain

$$\begin{aligned} f([a, p]) - [f(a), f(p)] &= \phi(a, p) \\ &= \phi(e_A, ap) \\ &= f([e_A, ap]) - [f(e_A), f(ap)]. \end{aligned}$$

Therefore,

$$f([a, p]) - [f(a), f(p)] = [f(ap), f(e_A)],$$

for all $a \in A$ and every idempotent $p \in A$.

As the linear span of A generated by its idempotents is dense in A , we thus get

$$f([a, b]) - [f(a), f(b)] = [f(ab), f(e_A)], \quad a, b \in A. \quad (10)$$

Now it follows from (9) and (10) that

$$f([a, b]) = [f(a), f(b)],$$

for every $a, b \in A$. Thus, f is a Lie homomorphism. \square

If A is a von Neumann algebra or a unital C^* -algebra of real rank zero [4], then the linear span of projections are norm dense in A , hence $A = \overline{\mathfrak{L}(A)}$, so Theorem 2.10 holds for such algebras.

As a consequence of Theorem 2.10 and [15, Theorem 4], we have the next result.

Corollary 2.11. *Let A and B be factor von Neumann algebras and let $f : A \longrightarrow B$ be a continuous bijective linear map. Then f satisfies the condition (8) if and only if $f = \theta + \mu$, where θ is an isomorphism or the negative of an anti-isomorphism, and μ is a linear functional such that $\mu([a, b]) = 0$ for all $a, b \in A$.*

3. CHARACTERIZATION OF LIE DERIVATIONS

In this section, we characterize continuous linear maps which are Lie derivations at zero or idempotent products.

We begin with the next result which is closely related to Theorem 2.3.

Theorem 3.1. *Let A be a Banach algebra with property (\mathbb{B}) and having a bounded approximate identity and X be a Banach A -bimodule. Suppose that $\delta : A \rightarrow X$ is a continuous linear map such that*

$$a, b \in A, \quad ab = 0 \implies \delta([a, b]) = [\delta(a), b] + [a, \delta(b)]. \quad (11)$$

Then δ is a Lie derivation.

Proof. Let $\delta([a, b]) = [\delta(a), b] + [a, \delta(b)]$ for all $a, b \in A$ with $ab = 0$. Take

$$B = \left\{ \begin{bmatrix} a & x \\ 0 & a \end{bmatrix} : a \in A, x \in X \right\}.$$

Then B becomes a Banach algebra under the usual matrix operations. Define a linear map $f : A \rightarrow B$ by

$$f(a) = \begin{bmatrix} a & \delta(a) \\ 0 & a \end{bmatrix}, \quad a \in A.$$

Then for all $a, b \in A$ with $ab = 0$, we have

$$f([a, b]) = \begin{bmatrix} [a, b] & \delta([a, b]) \\ 0 & [a, b] \end{bmatrix} = \begin{bmatrix} [a, b] & [\delta(a), b] + [a, \delta(b)] \\ 0 & [a, b] \end{bmatrix} = [f(a), f(b)].$$

Now Theorem 2.3 yields that

$$f([a, b]) = [f(a), f(b)],$$

for all $a, b \in A$, and hence f is a Lie homomorphism. According to this fact, it follows that δ is a Lie derivation. \square

Theorem 3.1 can be applied for a large class of Banach algebras which are zero product determined. In particular, we have the following result which generalizes [9, Theorem 2.1].

Corollary 3.2. *Let A be a C^* -algebra and $\delta : A \rightarrow X$ be a continuous linear map such that condition (11) holds. Then $\delta = d + \mu$, where $d : A \rightarrow X$ is a derivation and μ is a linear map from A into the centre of X such that $\mu([a, b]) = 0$ for all $a, b \in A$.*

Proof. By Theorem 3.1, δ is a Lie derivation. By [10, Theorem 9.5], the result follows. \square

Next we present an example of a linear map δ that satisfies the condition (11), but it is not a Lie derivation. This example shows that Corollary 3.2 is not true for Banach algebras, in general.

Example 3.3. Let

$$A = \left\{ \begin{bmatrix} z_1 & z_2 \\ 0 & z_3 \end{bmatrix} : z_1, z_2, z_3 \in \mathbb{C} \right\}.$$

We make $X = \mathbb{C}$ an A -bimodule by defining

$$a\lambda = z_3\lambda, \quad \lambda a = \lambda z_1, \quad \lambda \in \mathbb{C}, \quad a \in A.$$

Let $\delta : A \rightarrow X$ be a linear map defined by $\delta(a) = z_2$. Then $\delta(ab) = \delta(b)a + b\delta(a)$ for all $a, b \in A$. It is easy to see that

$$\delta([a, b]) = [\delta(a), b] + [a, \delta(b)],$$

for all $a, b \in A$ with $ab = 0$. Therefore, δ is a Lie derivation at zero point, but, δ is not a Lie derivation. Consequently, δ can not be decomposed as $\delta = d + \mu$.

Combining Theorem 3.1 and [10, Theorem 9.2], we have

Corollary 3.4. *Let G be a locally compact amenable group and $A = L^1(G)$. Then every continuous linear map δ from A into a Banach A -bimodule X satisfying (11) is of the form $\delta = d + \mu$, where $d : A \rightarrow X$ is a derivation and μ is a linear map from A into the centre of X such that $\mu([a, b]) = 0$ for all $a, b \in A$.*

As a consequence of Theorem 3.1 we obtain the following result.

Corollary 3.5. *Let A be a Banach algebra that is generated by idempotents and having a bounded approximate identity. Let δ be a continuous linear map from A into a Banach A -bimodule X satisfying (11). Then δ is a Lie derivation.*

Theorem 3.6. *Let A be a unital Banach algebra with the property that $A = \overline{\mathfrak{L}(A)}$ and X be a Banach A -bimodule. Suppose that $\delta : A \rightarrow X$ is a continuous linear map such that for every non-trivial idempotent p in A ,*

$$a, b \in A, \quad ab = p \implies \delta([a, b]) = [\delta(a), b] + [a, \delta(b)]. \quad (12)$$

Then δ is a Lie derivation.

In particular, if A is a von Neumann algebra, then $\delta = d + \mu$, where $d : A \rightarrow X$ is a derivation and μ is a linear map from A into the centre of X such that $\mu([a, b]) = 0$ for all $a, b \in A$.

Proof. Let δ satisfy condition (12). Let B and f be as in Theorem 3.1. Then for all $a, b \in A$ with $ab = p$, we have

$$f([a, b]) = [f(a), f(b)].$$

Thus, f is a Lie homomorphism by Theorem 2.10 and hence δ is a Lie derivation.

If A is a von Neumann algebra, then the result follows from [10, Theorem 9.5]. \square

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