Constructive martingale representation of nonsmooth path-dependent exponential Brownian functionals

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Abstract. In this paper, we consider some classes of Brownian functionals, including non-smooth functionals (therefore, it is impossible to use the well-known Clark-Ocone formula), depending both on the trajectory and on the last moment of time, and propose a new method for obtaining a constructive stochastic integral representation in the case of both smooth and non-smooth functionals.

Keywords. Brownian functional, Malliavin derivative, stochastic integral representation, martingale representation, Clark–Ocone formula.

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1. Introduction and auxiliary results

The martingale representation theorem is a cornerstone of stochastic calculus and is of great importance because of its profound implications for various areas of financial mathematics (option pricing, risk management), for understanding stochastic processes (structure of martingales, relation to Brownian motion), and for solving problems of control theory and filtration theory. It provides a powerful framework for understanding and managing stochastic processes governed by Brownian motion. Its implications extend far beyond mathematical finance, influencing fields such as physics, engineering, and economics. It serves as a fundamental tool for researchers and practitioners dealing with uncertainty and stochastic phenomena.

After Clark ([1]) obtained the formula for the stochastic integral representation for Brownian Motion functionals, many authors tried to explicitly find the integrand. The works [2]–[8] are especially important in this direction. In the 80s of the last century (see [9]), it became clear that martingale representation theorems (along with Girsanov's absolutely continuous change of measure theorem) play an important role in modern financial mathematics.

Let a Brownian Motion $B = (B_t)$, $t \in [0, T]$, be given on a probability space $(\Omega, \Im, P)$, and let $\Im_t^B = \sigma\{B_u : 0 \le u \le t\}$.

Let $C_p^\infty(R^n)$ be the set of all infinitely differentiable functions $f : R^n \longrightarrow R$ such that f and all its partial derivatives have polynomial growth. Denote by Sm the class of so-called smooth random variables F of the form

$$F = f(B_{t_1}, B_{t_2}, \dots, B_{t_n}), \;\; f \in C_p^\infty(R^n), \;\; t_i \in [0, T].$$

Moreover, we denote by Pol the class of random variables F with polynomial f. It is known (see [15], Chapter 1) that Sm and Pol are dense in $L_2(\Omega)$.

The stochastic derivative (derivative in the Malliavin sense) of a smooth random variable F is defined as a random process D_tF defined by the relation (see [10])

$$D_tF = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(B_{t_1}, B_{t_2}, \dots, B_{t_n}) I_{[0,t_i]}(t).$$

D is closable as an operator from $L_2(\Omega)$ to $L_2(\Omega; L_2([0,T]))$. Denote its domain by $D_{1,2}$. This means that $D_{1,2}$ is equal to the closure of the class of smooth random variables in the norm

$$||F||_{1,2} := \{E[F^2] + E[||DF||^2_{L_2([0,T])}]\}^{1/2}.$$

Theorem 1.1 (Clark-Ocone formula, see [3])**.** *If F is differentiable in the sense of Malliavin, $F \in D_{1,2}$, then the following integral representation holds*

$$F = E[F] + \int_0^T E[D_tF|\Im_t^B]\mathrm{d}B_t \quad (P - a.s.). \tag{1.1}$$

Shiryaev and Yor ([6]) and Shiryaev, Yor and Graversen ([7]) proposed another method for finding the integrand based on the Itô formula and Levy's theorem for the Levy martingale $M_t = E[F|\Im_t^B]$ associated with the considered functional F (as F they considered the so-called "maximal" type functionals of Brownian Motion). Later, using the Clarke-Ocone formula, Renaud and Remillard ([8]) established an explicit martingale representation for Brownian functionals, which also depend on the trajectory (in particular, here F is a continuously differentiable function of three smooth quantities: from the Brownian Motion with drift and processes of its maximum and minimum).

It is clear that the class of functionals to which the Clark-Ocone formula can be applied is limited by the condition that they must be Malliavin differentiable. We study questions of the stochastic integral representation of stochastically non-smooth functionals. Glonti and Purtukhia ([11]) proposed a method for obtaining an integral representation for a non-smooth Brownian functional of a special form using the Trotter-Meyer theorem, which establishes a connection between the predictable quadratic characteristic of a semimartingale and its local time (see also papers [12]–[14]).

Further, it turned out that the requirement for the smoothness of a functional can be weakened by the requirement for the smoothness of only its conditional mathematical expectation. It is known that if a random variable is stochastically differentiable in the sense of Malliavin, then its conditional mathematical expectation is also differentiable (see Proposition 1.2.8 [15])[1]. On the other hand, the conditional mathematical expectation may be smooth even if the random variable is not stochastically smooth. For example, it is known that $I_{\{B_T \le c\}} \notin D_{1,2}$[2], but for all $t \in [0,T)$:

$$E[I_{\{B_T \le c\}}|\Im_t^B] = \Phi\left(\frac{c - B_t}{\sqrt{T-t}}\right) \in D_{1,2},$$

where c is some real constant and Φ is the standard normal distribution function.

Glonti and Purtukhia ([16]) generalized the Clark-Ocone formula to the case when the functional is not stochastically smooth, but its conditional mathematical

[1] In particular, if $F \in D_{1,2}$, then $E(F|\Im_s^B) \in D_{1,2}$ and $D_t[E(F|\Im_s^B)] = E(D_tF|\Im_s^B)I_{[0,s]}(t)$.

[2] The event indicator I_A is Malliavin differentiable if and only if the probability $P(A)$ equals zero or one (see Proposition 1.2.6 [15]).

expectation is stochastically differentiable, and proposed a method for finding the integrand.

In this paper, we consider some classes of Brownian functionals, including non-smooth functionals (therefore, it is impossible to use the well-known Clark-Ocone formula (1.1)), depending both on the trajectory and on the last moment of time, and propose a new method for obtaining a constructive stochastic integral representation in the case of both smooth and non-smooth functionals.

2. Constructive martingale representations

Let $h_s \equiv h_s(\omega) \equiv h_s(B_s(\omega))$ be an integrable process adapted to the flow of σ-algebras $\Im_s^B$. Let's consider the functional

$$F(a,b) = \exp\{\int_a^b h_s ds\},$$

where $0 \le a \le b \le T$.

Theorem 2.1. *If the function $V(t,x) = E[F(t,T)|B_t = x]$ satisfies the requirements of the classical Itô formula (i.e. $V(\cdot,\cdot) \in C^{1,2}([0,T]\times R)$), then the following stochastic integral representation is fulfilled*

$$F(0,T) = EF(0,T) + \int_0^T F(0,t) \cdot V_x^{'}(t,B_t) dB_t \quad (P-a.s.). \tag{2.1}$$

Proof. According to Itô's formula, we have

$$V(t,B_t) = V(0,B_0) + \int_0^t [V_s^{'}(s,B_s) + \frac{1}{2} V_{xx}^{''}(s,B_s)]ds +$$

$$+\int_0^t V_x^{'}(s,B_s)dB_s \quad (P-a.s.). \tag{2.2}$$

Due to the Markov property of the Brownian Motion

$$V(t,B_t) = E[F(t,T)|B_t = x]|_{x=B_t} =$$

$$= E[F(t,T)|B_t] = E[F(t,T)|\Im_t^B] \quad (P-a.s.)$$

and hence, under the conditions of the theorem, the process

$$F(0,t)\cdot V(t,B_t) = F(0,t)\cdot E[F(t,T)|\Im_t^B] =$$

$$= E[F(0,t)\cdot F(t,T)|\Im_t^B] = E[F(0,T)|\Im_t^B] := M_t$$

is a martingale. Since the martingale M_t is adapted to the filtration generated to the Brownian filtration, it will be a continuous martingale.

Further, from (2.2), taking into account that the product of functions of finite variation is again finite variation, we can write

$$M_t = F(0,t)\cdot V(0,B_0) + F(0,t)\cdot \int_0^t [V_s^{'}(s,B_s) + \frac{1}{2} V_{xx}^{''}(s,B_s)]ds +$$

$$+F(0,t)\cdot \int_0^t V_x^{'}(s,B_s)dB_s = F(0,t)\cdot V(0,B_0) +$$

$$+F(0,t)\cdot \int_0^t [V_s^{'}(s,B_s) + \frac{1}{2} V_{xx}^{''}(s,B_s)]ds +$$

$$(2.3) \qquad +\int_0^t F(0,s)\cdot V_x^{'}(s,B_s)dB_s - \int_0^t \Big(\int_0^s V_x^{'}(\theta,B_\theta)dB_\theta\Big)dF(0,s).$$

On the other hand, a continuous martingale of bounded variation starting from 0 is identically equal to 0 (see Theorem 39 [17]). Therefore, in equality (2.3) the term of bounded variation is equal to zero and, taking into account the equalities

$$EF(0,T) = E[F(0,T)|\Im_0^B] = M_0 = F(0,0)\cdot V(0,B_0) = V(0,B_0) \quad (P-a.s.)$$

and

$$M_T = E[F(0,T)|\Im_T^B] = F(0,T) \quad (P-a.s.)$$

the proof of the theorem is easily completed. □

Remark 2.1. It should be noted that the result of Theorem 2.1 is especially interesting for nonsmooth functionals, although it is also useful in the case of smooth functionals.

Theorem 2.2. *Let $F(0,t)\in D_{1,2}$ for almost all t. Then the Clark-Ocone representation (1.1) for the functional $F(0,T)$ follows from the representation (2.1).*

Proof. If, in addition to the conditions of this theorem, the functional F (0,T) satisfies the conditions of Theorem 2.1, then, according to Propositions 1.2.3 and 1.2.8 [15], taking into account the Corollary 1.2.1 [15], equality (2.1) can be rewritten as follows

$$F(0,T) = EF(0,T) + \int_0^T F(0,t)\cdot V_x^{'}(t,B_t)dB_t =$$

$$= EF(0,T) + \int_0^T F(0,t)\cdot D_t[V(t,B_t)]dB_t =$$

$$= EF(0,T) + \int_0^T F(0,t)\cdot D_t\{E[F(t,T)|\Im_t^B]\}dB_t =$$

$$= EF(0,T) + \int_0^T D_t\{F(0,t)\cdot E[F(t,T)|\Im_t^B]\}dB_t =$$

$$= EF(0,T) + \int_0^T D_t\{E[F(0,T)|\Im_t^B]\}dB_t =$$

$$(2.4) \qquad = EF(0,T) + \int_0^T E[D_tF(0,T)|\Im_t^B]dB_t \quad (P-a.s.).$$

Let us now consider a sequence of smooth random variables $F_n(0,T)\in Pol$ converging to $F(0,T)$ in $D_{1,2}$. Due to well-known properties of Brownian Motion and conditional mathematical expectation, it is not difficult to see that the conditional mathematical expectation of polynomial $f(B_{t_1},B_{t_2},\dots,B_{t_n})$ with respect to σ-algebras $\Im_t^B$ is a polynomial again, but already relative only to B_t itself and those B_s, where $s<t$. To test the validity of this proposition, it is easy to see that it is sufficient to consider the case $t<u<v$. In this case we have

$$E[B_u^m B_v^n|\Im_t^B] = E\{E[B_u^m B_v^n|\Im_u^B]|\Im_t^B\} =$$

$$-E\{B_u^m E[(B_v-B_u+B_u)^n|\Im_u^B]|\Im_t^B\} =$$

$$= E\{B_u^m \sum_{i=0}^n C_n^i E(B_v-B_u)^i B_u^{n-i}|\Im_t^B\} =$$

$$= \sum_{i=0}^{[n/2]} C_n^{2i} E(B_v - B_u)^{2i} E\{B_u^{m+n-2i} | \Im_t^B\} =$$

$$= \sum_{i=0}^{[n/2]} C_n^{2i} (2i-1)!!(v-u)^i E\{(B_u - B_t + B_t)^{m+n-2i} | \Im_t^B\} =$$

$$= \sum_{i=0}^{[n/2]} C_n^{2i} (2i-1)!!(v-u)^i \times$$

$$\times \sum_{j=0}^{[(m+n-2i)/2]} C_{m+n-2i}^{2j} (2j-1)!!(u-t)^j B_t^{m+n-2i-2j}.$$

Therefore, it is not difficult to see that smooth random variables $F_n(0,T) \in Pol$ satisfy the condition of Theorem 2.1. Consequently, due to the relation (2.4), we have

$$F_n(0,T) = EF_n(0,T) + \int_0^T E[D_t F_n(0,T) | \Im_t^B] dB_t \quad (P - a.s.). \tag{2.5}$$

Next, by assumption

$$\lim_{n \longrightarrow \infty} \{||F_n(0,T) - F(0,T)||_{L_2(\Omega)} + |||DF_n(0,T) - DF(0,T)|||_{L_2(\Omega; L_2([0,T]))}\} = 0.$$

From here, based on the inequality of Jensen, we have

$$\lim_{n \longrightarrow \infty} \{|||E[DF_n(0,T)|\Im_t^B] - E[DF(0,T)|\Im_t^B]|||_{L_2(\Omega; L_2([0,T]))}\} \leq$$

$$\leq \lim_{n \longrightarrow \infty} \{|||DF_n(0,T) - DF(0,T)|||_{L_2(\Omega; L_2([0,T]))}\} = 0.$$

Therefore, taking the limit in (2.4), we obtain the Clark-Ocone representation (1.1). □

Remark 2.2. Note that to satisfy the condition of Theorem 2.2 it is sufficient, for example, that the process h be smooth, and the process h itself and its stochastic derivative be bounded.

Let us now consider another form of path-dependent and non-smooth Brownian functional

$$G(a,b) = I_{\{\max_{s \in [a,b]} B_s \leq c\}}.$$

Since $max_{s \in [0,T]} B_s = max\{max_{s \in [0,t]} B_s, max_{s \in [t,T]} B_s\}\}$ $(0 < t < T)$ and for any pair of events A and B: $I_{A \cap B} = I_A \cdot I_B$, it is obvious that the functional $G(a,b)$, like the functional $F(a,b)$ considered above, has the property of multiplicative expansion: $G(0,T) = G(0,t) \cdot G(t,T)$.

Therefore, following statements similar to those we used in proving Theorem 2.1, it is not difficult to prove an analogue of Theorem 2.1 for the functional $G(0,T)$.

Proposition 2.1. If $G(0,T) = I_{\{\max_{s \in [0,T]} B_s \leq c\}}$, then the function $V(t,x) = E[G(t,T)|B_t = x]$ satisfies the requirements of the Ito's formula and the following stochastic integral representation is valid

$$G(0,T) = 2\Phi(\frac{c}{\sqrt{T}}) - \int_0^T G(0,t) \cdot \frac{1}{\sqrt{T-t}} \varphi(\frac{c - B_t}{\sqrt{T-t}}) dB_t \quad (P - a.s.),$$

where φ is the density of the standard normal distribution.

Proof. According to the Reflection Principle the probability that the maximum of the Brownian motion over the interval $[0,T]$ is less than or equal to x is equal to twice the probability that a standard normal variable is less than or equal to $x/\sqrt{T}$, i.e.

$$P\{\max_{s\in[0,T]} B_s \le x\} = 2\Phi\big(\frac{x}{\sqrt{T}}\big).$$

Hence, we have

$$E[I_{\{\max_{s\in[0,T]} B_s \le c\}}] = P\{\max_{s\in[0,T]} B_s \le c\} = 2\Phi\big(\frac{c}{\sqrt{T}}\big).$$

Next, we recall that the distribution function of the maximum value of the increment of Brownian motion is equal to

$$P\{\max_{s\in[t,T]} (B_s - B_t) \le x\} = 1 - 2\Phi\big(-\frac{x}{\sqrt{T-t}}\big).$$

Therefore, we easily conclude that

$$V(t,x) = E[I_{\{\max_{s\in[t,T]} B_s \le c\}} | B_t = x] =$$

$$= E[I_{\{\max_{s\in[t,T]} (B_s - B_t) \le c-x\}} | B_t = x] = E[I_{\{\max_{s\in[t,T]} (B_s - B_t) \le c-x\}}] =$$

$$= P\{\max_{s\in[t,T]} (B_s - B_t) \le c - x\} = 1 - 2\Phi\big(-\frac{c-x}{\sqrt{T-t}}\big) = 2\Phi\big(\frac{c-x}{\sqrt{T-t}}\big) - 1.$$

It is easy to see that there are continuous derivatives

$$V_x^{'}(t,x) = -\frac{2}{\sqrt{T-t}}\varphi\big(\frac{c-x}{\sqrt{T-t}}\big)$$

and

$$V_{xx}^{''}(t,x) = -\frac{2(c-x)}{\sqrt{(T-t)^3}}\varphi\big(\frac{c-x}{\sqrt{T-t}}\big).$$

Let us now check the continuously differentiability with respect to t. Suppose that

$$\theta \in \big[\frac{c-x}{\sqrt{T-t}}, \frac{c-x}{\sqrt{T-t-\Delta t}}\big].$$

Then, by the mean value theorem for integrals, it is easy to verify that for some θ the sequence of relations is true

$$V_t'(t,x) = 2 \lim_{\Delta t \longrightarrow 0} \frac{\Phi\big(\frac{c-x}{\sqrt{T-t-\Delta t}}\big) - \Phi\big(\frac{c-x}{\sqrt{T-t}}\big)}{\Delta t} =$$

$$= 2 \lim_{\Delta t \longrightarrow 0} \frac{\varphi(\theta)\big[\frac{c-x}{\sqrt{T-t-\Delta t}} - \frac{c-x}{\sqrt{T-t}}\big]}{\Delta t} =$$

$$= 2 \lim_{\Delta t \longrightarrow 0} \frac{\varphi(\theta)(c-x)\big[T-t-(T-t-\Delta t)\big]}{\Delta t\sqrt{T-t-\Delta t}\sqrt{T-t}(\sqrt{T-t}+\sqrt{T-t-\Delta t})} =$$

$$= \frac{c-x}{\sqrt{(T-t)^3}}\varphi\big(\frac{c-x}{\sqrt{T-t}}\big).$$

Therefore, the continuity with respect to t of the function $V_t'(t,x)$ is obvious on the interval $[0,T)$.

On the other hand, for any positive numbers α and β, we have that

$$\lim_{s\longrightarrow 0} \frac{e^{-\alpha/s}}{s^\beta} = \lim_{1/s\longrightarrow\infty} \big(\frac{1}{s}\big)^\beta e^{-\alpha\frac{1}{s}} = \lim_{x\longrightarrow\infty} x^\beta / e^{\alpha x} = 0,$$

which, together with the above relation, shows the continuously differentiability with respect to t of the functions $V(t,x)$ on the segment $[0,T]$.

Moreover, it is obvious that

$$V_t'(t,x) + \frac{1}{2}V_{xx}''(t,x) = 0.$$

Combining now all of the above on the basis of an analogue of Theorem 2.1, we complete the proof of the proposition. □

Remark 2.3. Note that an approach similar to Theorem 2.1 can also be used for functionals depending on the last moment of time. There is the following statement.

Proposition 2.2. If $F(T) = \exp\{I_{\{B_T \le c\}}\}$, then the function $V(t,x) = E[F(T)|B_t = x]$ satisfies the requirements of the Ito's formula and the following stochastic integral representation is valid

$$\exp\{I_{\{B_T \le c\}}\} = 1 + (e-1)\Phi(\frac{c}{\sqrt{T}}) - \int_0^T \frac{e-1}{\sqrt{T-t}}\varphi(\frac{c-B_t}{\sqrt{T-t}})dB_t \quad (P-a.s.),$$

where e is the Euler number.

Proof. We have

$$V(t,x) = E[\exp\{I_{\{B_T \le c\}}\}|B_t = x] = E[\exp\{I_{\{B_T - B_t \le c-x\}}\}|B_t = x] =$$

$$= E[\exp\{I_{\{B_T - B_t \le c-x\}}\}] = \frac{1}{\sqrt{2\pi(T-t)}} \times$$

$$\times \int_{-\infty}^{\infty} \exp\{I_{y \le c-x}\} \exp\{-\frac{y^2}{2(T-t)}\}dy = \frac{1}{\sqrt{2\pi(T-t)}} \times$$

$$\times \big[e\int_{-\infty}^{c-x} \exp\{-\frac{y^2}{2(T-t)}\}dy + \int_{c-x}^{\infty} \exp\{-\frac{y^2}{2(T-t)}\}dy\big] =$$

$$= \frac{1}{\sqrt{2\pi}}\big[e\int_{-\infty}^{\frac{c-x}{\sqrt{T-t}}} \exp\{-\frac{u^2}{2}\}du + \int_{\frac{c-x}{\sqrt{T-t}}}^{\infty} \exp\{-\frac{u^2}{2}\}du\big] =$$

$$= e\Phi(\frac{c-x}{\sqrt{T-t}}) + 1 - \Phi(\frac{c-x}{\sqrt{T-t}}) = 1 + (e-1)\Phi(\frac{c-x}{\sqrt{T-t}}).$$

Next, following statements similar to those we used to prove Proposition 2.1, and taking into account relationships

$$E[\exp\{I_{\{B_T \le c\}}\}] = 1 + (e-1)\Phi(\frac{c}{\sqrt{T}})$$

and

$$V_x'(t,x) = -\frac{e-1}{\sqrt{T-t}}\varphi(\frac{c-x}{\sqrt{T-t}}),$$

it is not difficult to complete the proof of the proposition. □

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