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A NOTION OF n-NORM ON ORLICZ SPACE

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Abstract. We introduce an n-norm in the Orlicz space. We show that the Orlicz space furnished with the n-norm is a generalization the Lebesgue space furnished with its n-norm. We discuss several properties of Orlicz space with n-norm. Moreover we define a new norm that is obtained using the n-norm in the Orlicz space and prove the connection between the convergence in the Orlicz space furnished with the n-norm and the convergence in the Orlicz space furnished with norm.
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1. Introduction and preliminaries

Initial assessments focused on the axioms of an abstract n-dimensional metric, the key advancements in the definition of 2-metric, 2-normed spaces, and their topological characteristics,The theory of n-normed space with $n \geq 2$ was introduced by Gähler in the last 1960's [3–5]. Since that time, numerous researchers have made advancements and achieved the latest results related to this space. For instance, in [7, 16, 18]. In 2015, Ekariani *et al.* [1] have examined the n-normed space of the Lebesgue space. Geometrically, the n-norm speak about volume of parallelepiped which spanned by n elements of vector. B. Hazarika et al. in [8] discussed statistical convergence on n-norm spaces. Recently Sibel Ersan [2], discussed a quasi-Cauchy sequence and ward continuity of a function to an s-quasi-Cauchy sequence and s-ward continuity of a function in an n-normed space, for any given positive integer s. Their work also offers intriguing theorems pertaining to compactness, s-ward continuity, uniform continuity, and ordinary continuity.

The Lebesgue integration theory was created by Hendri Lebesgue in 1904.In order to suggest a generalised space of L_p, which was later called as Orlicz space, Z.W. Birnbaum and W. Orlicz were inspired by this idea and the function x^p in the description of L_p space. Orlicz himself went on to develop this area. The basic characteristics of Orlicz space with Lebesgue measure are discovered in [15]. C. Léonard discuss [14] discussed basic results about Orlicz spaces. Author proved their results in easier way. Using estimates for characteristic functions of balls in $\mathbb{R}^n$, Ifronika et al. in [10] provided sufficient and necessary conditions for generalised Hölder's inequality in Orlicz spaces and weak Orlicz spaces. In [9], B. Hazarika et al. extend Orlicz spaces in the framework of the Henstock-Kurzweil integral, a non-absolute integrable function. B. Hazarika et al. proved that C_0^∞, which is dense in their Orlicz space is naturally a dense subspace of their version of Orlicz space. One can see [11–13] for several properties of this version of Orlicz space. In 2024, Nur *et al* [17] has introduced the norm-2 in $L_\phi(X)$.

The work of [1] motivated us to construct this work for Orlicz spaces.

The structure of the manuscript is as follows: In Section 1, we recall several important definitions, and results that are important for our next section. In Section 2, introduce the n-norm for $n \geq 2$ in the Orlicz space $L_\phi(X)$ which generalizes the 2-norm. We also discuss that $L_\Phi(X)$ with n norm can be seen as a generalization of $L_p(X)$ with the n-norm. Further In Section 4,, we introduce a new norm that is derived from the n-norm and prove the connection between the convergence in the norm of the Orlicz space.

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Throughout the manuscript we denote (X, Σ, μ) be a measurable space and $1 \leq p < \infty$. The space $L_p(X)$ consists of equivalence classes of measurable functions $f : X \to \mathbb{R}$ such that $\int |f|^p d\mu < \infty$ where two measurable functions are equivalent if they equal μ-a.e.. In L_p-norm of $L_p(X)$ is defined by $||f||_{L_p} = \left(\int |f|^p d\mu \right)^{\frac{1}{p}}$. Next we recall in general, an n-norm is a function $\|\cdot, \dots, \cdot\| : X^n \to \mathbb{R}$ which meets the following requirements:

(1) $\|x_1, \dots, x_n\| = 0$ if and only if $x_1, \dots, x_n$ are vectors of linearly dependent;
(2) $\|x_1, \dots, x_n\|$ is unchanged by permutation;
(3) $\|\gamma x_1, \dots, x_n\| = |\gamma| \|x_1, \dots, x_n\|$ for any $x_1, \dots, x_n \in X$ and for any $\gamma \in \mathbb{R}$;
(4) $\|x_1, \dots, x_{n-1}, y + z\| \leq \|x_1, \dots, x_{n-1}, y\| + \|x_1, \dots, x_{n-1}, z\|$ for any $x, y, z \in X$.

A pair $(X, \|\cdot, \dots, \cdot\|)$ is called an *n-normed space*. As an example, the space L_p of p-integrable functions, where $1 \leq p < \infty$ with the n-norm [6] as follows.

$$\|f_1, \dots, f_n\|_{L_p(X)} = \left[\frac{1}{n!} \int_X \cdots \int_X \left| \det \begin{pmatrix} f_1(x_1) & \cdots & f_1(x_n) \\ \vdots & \ddots & \vdots \\ f_n(x_1) & \cdots & f_n(x_n) \end{pmatrix} \right|^p dx_1 \cdots dx_n \right]^{\frac{1}{p}}. \tag{1.1}$$

Definition 1. [15] A function $\phi : \mathbb{R} \to [0, \infty]$ is a Young function if

(1) ϕ is convex lower semi continuous $[0, \infty]$-valued function on $\mathbb{R}$;
(2) ϕ is even and $\phi(0) = 0$;
(3) ϕ is non-trivial; it is different from the constsnt function $\phi(s) = 0, \ s \in \mathbb{R}$ and its convex conjugate

$$\phi^*(s) = \begin{cases} 0, \ \text{if } s = 0 \\ \\ +\infty, \ \text{otherwise.} \end{cases}$$

The literature occasionally permits Young functions to be trivial in the sense of condition (iii) or does not require them to be lower semicontinuous. These comments include these limitations in order to avoid annoying trivial examples and make working with convex conjugacy painless. One does not lose generality in the process.

In normed space, Orlicz spaces $(L_\phi(X))$ is a generalization Lebesgue spaces $(L_p(X))$.

Definition 2. [15] Let $\phi : [0, \infty) \to [0, \infty)$ is a Young function. $L_\phi(X)$ is the set of functions that are measurable $f : X \to \mathbb{R}$ in such a way that

$$\int_X \phi(k|f(x)|) dx < \infty$$

for a scalar $k > 0$.

The usual norm in $L_\phi(X)$ is

$$\|f\|_{L_\phi(X)} := \inf \left\{ b > 0 : \int_X \phi\left(\frac{|f(x)|}{b} \right) dx \leq 1 \right\}$$

(see [10, 14, 15, 19]). As a note one can easily find if, $\phi(u) := u^p$ for $p \geq 1$ then $L_\phi(X) = L_p(X)$.

2. n-norm for Orlicz spaces for $n \geq 2$

In this Section of the manuscript, we introduce n-norm on Orlicz space. We start this Section with the following.

Let ϕ be a Young function with $\phi : [0,\infty) \to [0,\infty)$. We define the function $\|\cdot,\dots,\cdot\|_{L_\phi(X)} : L_\phi(X) \times \dots \times L_\phi(X) \to \mathbb{R}$ as follows.

$$\|f_1,\dots,f_n\|_{L_\phi(X)} := \tag{2.1}$$
$$\inf\left\{ b>0 : \frac{1}{n!}\int\limits_X \cdots \int\limits_X \phi\left(\frac{1}{b}\left|\det\begin{pmatrix} f_1(x_1) & \cdots & f_1(x_n) \\ \vdots & \ddots & \vdots \\ f_n(x_1) & \cdots & f_n(x_n)\end{pmatrix}\right|\right) dx_1\cdots dx_n \le 1 \right\}.$$

We will demonstrate that the mapping in (2.1) defines an n-norm on $L_\phi(X)$. To accomplish this, we will utilize the following lemmas.

Lemma 1. *Let $L_\phi(X)$ be Orlicz space. If $0 \le \|f_1,\dots,f_n\|_{L_\phi(X)} < \infty$ then*

$$\frac{1}{n!}\int\limits_X \cdots \int\limits_X \phi\left(\frac{1}{\|f_1,\dots,f_n\|_{L_\phi(X)}}\left|\det\begin{pmatrix} f_1(x_1) & \cdots & f_1(x_n) \\ \vdots & \ddots & \vdots \\ f_n(x_1) & \cdots & f_n(x_n)\end{pmatrix}\right|\right) dx_1\cdots dx_n \le 1.$$

Proof. Suppose that

$$\begin{aligned}&\|f_1,\dots,f_n\|_{L_\phi(X)}\\ &= \inf\left\{ b>0 : \frac{1}{n!}\int\limits_X \cdots \int\limits_X \phi\left(\frac{1}{b}\left|\det\begin{pmatrix} f_1(x_1) & \cdots & f_1(x_n) \\ \vdots & \ddots & \vdots \\ f_n(x_1) & \cdots & f_n(x_n)\end{pmatrix}\right|\right) dx_1\cdots dx_n \le 1 \right\}\\ &= \inf \mathbb{B}.\end{aligned}$$

Foe every $\epsilon > 0$, there exist $b_\epsilon \in \mathbb{B}$ such that $\|f_1,\dots,f_n\|_{L_\phi(X)} \le b_\epsilon \le \|f_1,\dots,f_n\|_{L_\phi(X)} + \epsilon$. As consequence

$$\frac{\left|\det\begin{pmatrix} f_1(x_1) & \cdots & f_1(x_n) \\ \vdots & \ddots & \vdots \\ f_n(x_1) & \cdots & f_n(x_n)\end{pmatrix}\right|}{\|f_1,\dots,f_n\|_{L_\phi(X)} + \epsilon} \le \frac{\left|\det\begin{pmatrix} f_1(x_1) & \cdots & f_1(x_n) \\ \vdots & \ddots & \vdots \\ f_n(x_1) & \cdots & f_n(x_n)\end{pmatrix}\right|}{b_\epsilon}.$$

for every $x_1,\dots,x_n \in X$. By utilizing the properties of the Young function, we obtain.

$$\begin{aligned}&\int\limits_X \cdots \int\limits_X \phi\left[\frac{1}{\|f_1,...,f_n\|_{L_\phi(X)}+\epsilon}\left|\det\begin{pmatrix} f_1(x_1) & \cdots & f_1(x_n) \\ \vdots & \ddots & \vdots \\ f_n(x_1) & \cdots & f_n(x_n)\end{pmatrix}\right|\right] dx_1\cdots dx_n\\ &\le \int\limits_X \cdots \int\limits_X \phi\left[\frac{1}{b_\epsilon}\left|\det\begin{pmatrix} f_1(x_1) & \cdots & f_1(x_n) \\ \vdots & \ddots & \vdots \\ f_n(x_1) & \cdots & f_n(x_n)\end{pmatrix}\right|\right] dx_1\cdots dx_n.\end{aligned}$$

Therefore

$$\int\limits_X \cdots \int\limits_X \phi\left[\frac{1}{\|f_1,...,f_n\|_{L_\phi(X)}+\epsilon}\left|\det\begin{pmatrix} f_1(x_1) & \cdots & f_1(x_n) \\ \vdots & \ddots & \vdots \\ f_n(x_1) & \cdots & f_n(x_n)\end{pmatrix}\right|\right] dx_1\cdots dx_n \le n!.$$

Since $\epsilon > 0$ is arbitrary, we have

$$\frac{1}{n!}\int\limits_X \cdots \int\limits_X \phi\left[\frac{1}{\|f_1,\dots,f_n\|_{L_\phi(X)}}\left|\det\begin{pmatrix} f_1(x_1) & \cdots & f_1(x_n) \\ \vdots & \ddots & \vdots \\ f_n(x_1) & \cdots & f_n(x_n)\end{pmatrix}\right|\right] dx_1\cdots dx_n \le 1,$$

as desired. □

Lemma 2. $\|f_1,\ldots,f_n\|_{L_\phi(X)}=0$ *if and only if for every* $\epsilon>0$,

$$\int\limits_X\cdots\int\limits_X \phi\left(\frac{1}{\epsilon}\left|\det\begin{pmatrix} f_1(x_1) & \cdots & f_1(x_n)\\ \vdots & \ddots & \vdots\\ f_n(x_1) & \cdots & f_n(x_n)\end{pmatrix}\right|\right)dx_1\cdots dx_n\le n!.$$

Proof. ($\Leftarrow$) It is obvious that

$$\frac{1}{n!}\int\limits_X\cdots\int\limits_X \phi\left(\frac{1}{\epsilon}\left|\det\begin{pmatrix} f_1(x_1) & \cdots & f_1(x_n)\\ \vdots & \ddots & \vdots\\ f_n(x_1) & \cdots & f_n(x_n)\end{pmatrix}\right|\right)dx_1\cdots dx_n\le 1$$

for every $\epsilon>0$ then $\|f_1,\ldots,f_n\|_{L_\phi(X)}=0$.

($\Rightarrow$) Assume, for the sake of contradiction, that there exists some $\epsilon_0>0$ such that

$$\frac{1}{n!}\int\limits_X\cdots\int\limits_X \phi\left(\frac{1}{\epsilon_0}\left|\det\begin{pmatrix} f_1(x_1) & \cdots & f_1(x_n)\\ \vdots & \ddots & \vdots\\ f_n(x_1) & \cdots & f_n(x_n)\end{pmatrix}\right|\right)dx_1\cdots dx_2> 1.$$

Next, suppose now that

$$\begin{aligned}&\|f_1,\ldots,f_n\|_{L_\phi(X)}\\ &=\inf\left\{b>0:\frac{1}{n!}\int\limits_X\cdots\int\limits_X \phi\left(\frac{1}{b}\left|\det\begin{pmatrix} f_1(x_1) & \cdots & f_1(x_n)\\ \vdots & \ddots & \vdots\\ f_n(x_1) & \cdots & f_n(x_n)\end{pmatrix}\right|\right)dx_1\cdots dx_2\le 1\right\}\\ &=\inf\mathbb{B}.\end{aligned}$$

Take arbirtary $k\in\mathbb{B}$, we have $\epsilon_0\neq k$. We will examine two cases:

Case I: $\epsilon_0>k$. By applying the properties of the Young function, we have

$$\begin{aligned}&\frac{1}{n!}\int\limits_X\cdots\int\limits_X \phi\left(\frac{1}{\epsilon_0}\left|\det\begin{pmatrix} f_1(x_1) & \cdots & f_1(x_n)\\ \vdots & \ddots & \vdots\\ f_n(x_1) & \cdots & f_n(x_n)\end{pmatrix}\right|\right)dx_1\cdots dx_n\\ &\le\frac{1}{n!}\int\limits_X\cdots\int\limits_X \phi\left(\frac{1}{k}\left|\det\begin{pmatrix} f_1(x_1) & \cdots & f_1(x_n)\\ \vdots & \ddots & \vdots\\ f_n(x_1) & \cdots & f_n(x_n)\end{pmatrix}\right|\right)dx_1\cdots dx_n\le 1.\end{aligned}$$

Case II: $k>\epsilon_0$. This implies that $\|f_1,\ldots,f_n\|_{L_\phi(X)}>\epsilon_0>0$. Hence, both cases contradict. □

Lemma 3. $\|f_1,\ldots,f_n\|_{L_\phi(X)}=0$ *if and only if*

$$\frac{1}{n!}\int\limits_X\cdots\int\limits_X \phi\left(\beta\left|\det\begin{pmatrix} f_1(x_1) & \cdots & f_1(x_n)\\ \vdots & \ddots & \vdots\\ f_n(x_1) & \cdots & f_n(x_n)\end{pmatrix}\right|\right)dx_1\ldots dx_n=0$$

for any $\beta>0$.

Proof. For every $0<\epsilon<1$ and $\beta>0$, we obtain

$$\phi\left(\beta\left|\det\begin{pmatrix} f_1(x_1) & \cdots & f_1(x_n)\\ \vdots & \ddots & \vdots\\ f_n(x_1) & \cdots & f_n(x_n)\end{pmatrix}\right|\right)\le\epsilon\phi\left(\frac{\beta}{\epsilon}\left|\det\begin{pmatrix} f_1(x_1) & \cdots & f_1(x_n)\\ \vdots & \ddots & \vdots\\ f_n(x_1) & \cdots & f_n(x_n)\end{pmatrix}\right|\right).$$

Using Lemma 2, we have

$$\frac{1}{n!}\int\limits_X\cdots\int\limits_X \phi\left(\beta\left|\det\begin{pmatrix} f_1(x_1) & \cdots & f_1(x_n)\\ \vdots & \ddots & \vdots\\ f_n(x_1) & \cdots & f_n(x_n)\end{pmatrix}\right|\right)dx_1\cdots dx_n\le\epsilon.$$

Since $0 < \epsilon < 1$ is arbitrary then

$$\frac{1}{n!}\int_X \cdots \int_X \phi\left(\beta \left|\det\begin{pmatrix} f_1(x_1) & \cdots & f_1(x_n) \\ \vdots & \ddots & \vdots \\ f_n(x_1) & \cdots & f_n(x_n)\end{pmatrix}\right|\right) dx_1 \cdots dx_n = 0$$

for any $\beta > 0$. Conversaly, let

$$\frac{1}{n!}\int_X \cdots \int_X \phi\left(\beta \left|\det\begin{pmatrix} f_1(x_1) & \cdots & f_1(x_n) \\ \vdots & \ddots & \vdots \\ f_n(x_1) & \cdots & f_n(x_n)\end{pmatrix}\right|\right) dx_1 \cdots dx_n = 0$$

for every $\beta > 0$. Then

$$\frac{1}{\beta} \in \left\{ b > 0 : \frac{1}{n!}\int_X \cdots \int_X \phi\left(\frac{1}{b} \left|\det\begin{pmatrix} f_1(x_1) & \cdots & f_1(x_n) \\ \vdots & \ddots & \vdots \\ f_n(x_1) & \cdots & f_n(x_n)\end{pmatrix}\right|\right) dx_1 dx_2 \leq 1 \right\}.$$

Hence, $\|f_1, \ldots, f_n\|_{L_\phi(X)} \leq \frac{1}{\beta}$. Since $\beta > 0$ is arbitrary, we conclude that $\|f_1, \ldots, f_n\|_{L_\phi(X)} = 0$. □

Next, we have a result about the new n-norm on $L_\Phi(X)$ as follows.

Theorem 2.1. *The mapping (2.1) defines an n-norm on $L_\Phi(X)$*

Proof. We must verify that $\|\cdot, \ldots, \cdot\|_{L_\phi(X)}$ Fulfills the four properties of the n-norm.

(1) Suppose that $\|f_1, \ldots, f_n\|_{L_\phi(X)} = 0$. By Lemma 3, we obtain

$$\int_X \cdots \int_X \phi\left(\alpha \left|\det\begin{pmatrix} f_1(x_1) & \cdots & f_1(x_n) \\ \vdots & \ddots & \vdots \\ f_n(x_1) & \cdots & f_n(x_n)\end{pmatrix}\right|\right) dx_1 \cdots dx_n = 0$$

for every $\alpha > 0$. Since $\phi\left(\alpha \left|\det\begin{pmatrix} f_1(x_1) & \cdots & f_1(x_n) \\ \vdots & \ddots & \vdots \\ f_n(x_1) & \cdots & f_n(x_n)\end{pmatrix}\right|\right) \geq 0$, we conclude that

$$\phi\left(\alpha \left|\det\begin{pmatrix} f_1(x_1) & \cdots & f_1(x_n) \\ \vdots & \ddots & \vdots \\ f_n(x_1) & \cdots & f_n(x_n)\end{pmatrix}\right|\right) = 0.$$

Therefore $\det\begin{pmatrix} f_1(x_1) & \cdots & f_1(x_n) \\ \vdots & \ddots & \vdots \\ f_n(x_1) & \cdots & f_n(x_n)\end{pmatrix} = 0$. Hence, $\{f_1, \ldots, f_n\}$ is linearly dependent set.

Conversaly, suppose $f_1, \ldots, f_n$ are linearly dependent. Observe that

$$\det\begin{pmatrix} f_1(x_1) & \cdots & f_1(x_n) \\ \vdots & \ddots & \vdots \\ f_n(x_1) & \cdots & f_n(x_n)\end{pmatrix} = 0.$$

Then

$$
\begin{aligned}
&\|f_1, \ldots, f_n\|_{L_\phi(X)} \\
&= \inf \left\{ b > 0 : \frac{1}{n!} \int_X \cdots \int_X \phi \left(\frac{1}{b} \left| \det \begin{pmatrix} f_1(x_1) & \cdots & f_1(x_n) \\ \vdots & \ddots & \vdots \\ f_n(x_1) & \cdots & f_n(x_n) \end{pmatrix} \right| \right) dx_1 \cdots dx_n \leq 1 \right\} \\
&= \inf \left\{ b > 0 : \frac{1}{n!} \int_X \cdots \int_X \phi(0)\, dx_1 \cdots dx_n \leq 1 \right\} = \inf \{ b > 0 \} = 0.
\end{aligned}
$$

(2) By employing the properties of determinants, we obtain the invariance of $\|f_1, \ldots, f_n\|_{L_\phi(X)}$ under permutation.

(3) Observe that

$$
\begin{aligned}
&\|\gamma f_1, \ldots, f_n\|_{L_\phi(X)} \\
&= \inf \left\{ b > 0 : \frac{1}{n!} \int_X \cdots \int_X \phi \left(\frac{1}{b} \left| \det \begin{pmatrix} \gamma f_1(x_1) & \cdots & \gamma f_1(x_n) \\ \vdots & \ddots & \vdots \\ f_n(x_1) & \cdots & f_n(x_n) \end{pmatrix} \right| \right) dx_1 \cdots dx_n \leq 1 \right\} \\
&= \inf \left\{ b > 0 : \frac{1}{n!} \int_X \cdots \int_X \phi \left(\frac{1}{\frac{b}{|\gamma|}} \left| \det \begin{pmatrix} f_1(x_1) & \cdots & f_1(x_n) \\ \vdots & \ddots & \vdots \\ f_n(x_1) & \cdots & f_n(x_n) \end{pmatrix} \right| \right) dx_1 \cdots dx_n \leq 1 \right\} \\
&= \inf \left\{ |\gamma|\, \delta > 0 : \frac{1}{n!} \int_X \cdots \int_X \phi \left(\frac{1}{\delta} \left| \det \begin{pmatrix} f_1(x_1) & \cdots & f_1(x_n) \\ \vdots & \ddots & \vdots \\ f_n(x_1) & \cdots & f_n(x_n) \end{pmatrix} \right| \right) dx_1 \cdots dx_n \leq 1 \right\} \\
&= |\gamma| \inf \left\{ \delta > 0 : \frac{1}{2} \int_X \cdots \int_X \phi \left(\frac{1}{\delta} \left| \det \begin{pmatrix} f_1(x_1) & \cdots & f_1(x_n) \\ \vdots & \ddots & \vdots \\ f_n(x_1) & \cdots & f_n(x_n) \end{pmatrix} \right| \right) dx_1 \cdots dx_n \leq 1 \right\} \\
&= |\gamma|\, \|f_1, \ldots, f_n\|_{L_\phi(X)}.
\end{aligned}
$$

(4) Suppose that

$$
\begin{aligned}
&\|f_1, \ldots, f_{n-1}, f_n + f'\|_{L_\phi(X)} \\
&\quad = \inf \left\{ b > 0 : \frac{1}{n!} \int_X \cdots \int_X \phi \left(\frac{|\det[(f_i + f')(x_j)]|}{b} \right) dx_1 \cdots dx_n \leq 1 \right\} = \inf B
\end{aligned}
$$

where

$$
[(f_i + f')(x_j)] = \begin{pmatrix} f_1(x_1) & \cdots & f_1(x_n) \\ \vdots & \ddots & \vdots \\ f_{n-1}(x_1) & \cdots & f_{n-1}(x_{n-1}) \\ f_n(x_1) + f'(x_1) & \cdots & f_n(x_n) + f'(x_n) \end{pmatrix}.
$$

Let $c = \|f_1, \dots, f_n\|_{L_\Phi(X)}$ and $d = \|f_1, \dots, f_{n-1}, f'\|_{L_\Phi(X)}$. Using the properties of determinants, the properties of Young function (ϕ) and Lemma 1, we have

$$\begin{aligned}&\frac{1}{n!}\int\limits_X \cdots \int\limits_X \phi\left(\frac{1}{c+d}\left|\det[(f_i + f')(x_j)]\right|\right) dx_1 \cdots dx_n \\ &\leq \frac{c}{(c+d)n!}\int\limits_X \int\limits_X \phi\left(\frac{1}{c}\left|\det[(f_i(x_j))]\right|\right) dx_1 \cdots dx_n \\ &+\frac{d}{(c+d)n!}\int\limits_X \cdots \int\limits_X \phi\left(\frac{1}{d}\left|\det[(f'(x_j))]\right|\right) dx_1 \cdots dx_n \\ &\leq 1.\end{aligned}$$

In here, $\det[(f_i(x_j))] = \begin{pmatrix} f_1(x_1) & \cdots & f_1(x_n) \\ \vdots & \ddots & \vdots \\ f_n(x_1) & \cdots & f_n(x_n)\end{pmatrix}$ and $\det[(f'(x_j))] = \begin{pmatrix} f_1(x_1) & \cdots & f_1(x_n) \\ \vdots & \ddots & \vdots \\ f_{n-1}(x_1) & \cdots & f_{n-1}(x_n) \\ f'(x_1) & \cdots & f'(x_n)\end{pmatrix}$.

As consequence $c + d \in B$. Because $\inf(B) = \|f_1, ..., f_{n-1}, f_n + f'\|_{L_\phi(X)}$ then we conclude that

$$\|f_1, ..., f_{n-1}, f_n + f'\|_{L_\phi(X)} \leq \|f_1, ..., f_{n-1}, f_n\|_{L_\phi(X)} + \|f_1, ..., f_{n-1}, f'\|_{L_\phi(X)}.$$

□

Next, the relation between $L_\phi(X)$ that is equipped with the n-norm $\|\cdot, \dots, \cdot\|_{L_\phi(X)}$ and $L_p(X)$ that is equipped with the n-norm $\|\cdot, \dots, \cdot\|_{L_p(X)}$ in (1.1) as follows.

Fact: $\phi(y) = y^p$ for $1 \leq p < \infty$ implies $\|f_1, ..., f_n\|_{L_\phi(X)} = \|f_1, ..., f_n\|_{L_p(X)}$.

Proof. Assume that $\phi(y) = y^p$ for $1 \leq p < \infty$. We have

$$\begin{aligned}&\|f_1, \dots, f_n\|_{L_\phi(X)} \\ &= \inf\left\{ b > 0 : \frac{1}{n!}\int\limits_X \cdots \int\limits_X \frac{1}{b^p}\left|\det\begin{pmatrix} f_1(x_1) & \cdots & f_1(x_n) \\ \vdots & \ddots & \vdots \\ f_n(x_1) & \cdots & f_n(x_n)\end{pmatrix}\right|^p dx_1 \cdots dx_n \leq 1\right\} \\ &= \inf\left\{ b > 0 : \frac{1}{n!}\int\limits_X \cdots \int\limits_X \left|\det\begin{pmatrix} f_1(x_1) & \cdots & f_1(x_n) \\ \vdots & \ddots & \vdots \\ f_n(x_1) & \cdots & f_n(x_n)\end{pmatrix}\right|^p dx_1 \cdots dx_n \leq b^p\right\} \\ &= \inf Y.\end{aligned}$$

Because

$$\|f_1, \dots, f_n\|^p_{L_p(X)} = \frac{1}{n!}\int\limits_X \cdots \int\limits_X \left|\det\begin{pmatrix} f_1(x_1) & \cdots & f_1(x_n) \\ \vdots & \ddots & \vdots \\ f_n(x_1) & \cdots & f_n(x_n)\end{pmatrix}\right|^p dx_1 \cdots dx_n$$

then $\|f_1, \dots, f_n\|_{L_p(X)} \leq b$ for any $b \in Y$. As consequence, $\|f_1, \dots, f_n\|_{L_p(X)}$ is lower bound Y. Hence, $\|f_1, \dots, f_n\|_{L_p(X)} \leq \|f_1, \dots, f_n\|_{L_\phi(X)}$. Conversely, we observe that

$$\frac{1}{n!}\int\limits_X \cdots \int\limits_X \frac{1}{\|f_1, ..., f_n\|^p_{L_p(X)}}\left|\det\begin{pmatrix} f_1(x_1) & \cdots & f_1(x_n) \\ \vdots & \ddots & \vdots \\ f_n(x_1) & \cdots & f_n(x_n)\end{pmatrix}\right|^p dx_1 \cdots dx_n = 1.$$

Therefore, $\|f_1, \dots, f_n\|_{L_p(X)} \in Y$. Because $\inf Y = \|f_1, \dots, f_n\|_{L_\phi(X)}$ then we can conclude that $\|f_1, \dots, f_n\|_{L_p(X)} \geq \|f_1, \dots, f_n\|_{L_\phi(X)}$. Hence,

$$\|f_1, \dots, f_n\|_{L_p(X)} = \|f_1, \dots, f_n\|_{L_\phi(X)}.$$

□

This fact show that Orlicz spaces that is equipped the n-norm can be seen as a generalization of Lebesgue spaces that is equipped the n-norm.

3. A new norm derived from the n-norm

In this Section of the manuscript, we introduce the norm derived from the n-norm $(\|\cdot,\dots,\cdot\|_{L_p(X)})$. Let $\{a_1,\dots,a_n\}$ be a linearly independent set in $L_\phi(X)$. One may observe that

$$\|g\|^*_{L_\phi(X)} = \sum_{\{j_2,\dots,j_n\}\subseteq\{1,\dots,n\}} \|g, a_{j_2}, \dots, a_{j_n}\|_{L_\phi(X)} \tag{3.1}$$

defines a norm on $L_\phi(X)$. We can observe that (1) $\|g\|^*_{L_\phi(X)} \geq 0$. If $g=0$ then $\|g\|^*_{L_\phi(X)} = 0$ by the properties of n-norm. Conversely if $\|g\|^*_{L_\phi(X)} = 0$, then $\|g, a_{j_2}, \dots, a_{j_n}\|_{L_\phi(X)} = 0$ for every $\{j_2,\dots,j_n\} \subseteq \{1,\dots,n\}$. As consequence, g is a linear combination from $a_{j_2},\dots,a_{j_n}$. Let $g = \alpha_2 a_2 + \dots + \alpha_n a_n$. Substituting g in $\|a_1, g, a_3, \dots, a_n\|_{L_\phi(X)} = 0$, we have

$$\|a_1, \alpha_2 a_2 + \dots + \alpha_n a_n, a_3, \dots, a_n\|_{L_\phi(X)} = |\alpha_2| \|a_1, a_2, a_3, \dots, a_n\|_{L_\phi(X)} = 0.$$

Because $\{a_1,\dots,a_n\}$ is a set of linearly independent vectors then

$$\|a_1, a_2, \dots, a_n\|_{L_\phi(X)} \neq 0.$$

Hence $\alpha_2 = 0$. In the same way, we obtain $\alpha_3 = \alpha_4 = \dots = \alpha_n = 0$ so $g = 0$. Next, by properties of the n-norm, we have (2) $\|\gamma g\|^*_{L_\phi(X)} = |\gamma| \|g\|^*_{L_\phi(X)}$ and (3) $\|g+h\|^*_{L_\phi(X)} \leq \|g\|^*_{L_\phi(X)} + \|h\|^*_{L_\phi(X)}$.

Using the definition $\|\cdot\|^*_{L_\phi(X)}$, we have the following theorem about the convergence of a sequence in the n-norm $\|\cdot,\dots,\cdot\|_{L_\phi(X)}$ implies convergence in the norm $\|\cdot\|^*_{L_\phi(X)}$.

Theorem 3.1. *Let $\{a_1,\dots,a_n\}$ be set of linearly independent vectors on $L_\phi(X)$. If a sequence $(g_k) \in L_\phi(X)$ converges to limit $g \in L_\phi(X)$ according to the n-norm $\|\cdot,\dots,\cdot\|_{L_\phi(X)}$, then it also converges to g according to the derived norm $\|\cdot\|^*_{L_\phi(X)}$. Also, if (g_k) is a Cauchy sequence according to the n-norm $\|\cdot,\dots,\cdot\|_{L_\phi(X)}$, then it is a Cauchy sequence according to the derived norm $\|\cdot\|^*_{L_\phi(X)}$.*

Proof. Assume that $(g_k) \in L_\phi(X)$ converges to some $g \in L_\phi(X)$ according to the n-norm $\|\cdot,\dots,\cdot\|_{L_\phi(X)}$ that is

$$\|g_k - g, f_2, \dots, f_n\|_{L_\phi(X)} \to 0.$$

for every $f_2,\dots,f_n \in L_\phi(X)$ and $k \to \infty$. Because $\{a_1,\dots,a_2\}$ is set of linearly independent vectors on $L_\phi(X)$ then for every $\{j_2,\dots,j_n\} \subseteq \{1,\dots,n\}$, we have $\|g_k - f, a_{j_2}, \dots, a_{j_n}\|_{L_\phi(X)} \to 0$ as $k \to \infty$. Consequently, we obtain

$$\|g_k - g\|^*_{L_\phi(X)} = \sum_{\{j_2,\dots,j_n\}\subseteq\{1,\dots,n\}} \|g_k - g, a_{j_2}, \dots, a_{j_n}\|_{L_\phi(X)} \to 0.$$

Hence, (g_k) converges to $g \in L_\phi(X)$ with respect to the norm $\|\cdot\|^*_{L_\phi(X)}$. The proof of the second part is established in a similar manner.

□

Unfortunately, up to now, we have not been able to prove the converse of the above theorem.

4. Conclusion

In this manuscript, we have presented the n-norm in Orlicz space. We also have proved the Orlicz space equipped the n-norm is a generalization the Lebesgue space equipped with its n-norm. Furthermore, we have defined a norm that is derived from the n-norm in the Orlicz space and have showed the connection between the convergence in the n-norm and the norm. We plan to investigate the equivalence between the norm derived from the n-norm and the standard norm on Orlicz space.

Declaration of competing interes

The authors declare that they have no conflicts of interest or personal relationships that could have influenced the work presented in this manuscript.

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