On the space of generalized theta-series for certain quadratic forms of six variables

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Abstract. The vector spaces of generalized theta-series with spherical polynomials of order ν, corresponding to some diagonal and non-diagonal quadratic forms in six variables are considered. The upper bound for the dimensions of this spaces is established.

1. Introduction

Let

$$Q(X) = Q(x_1, x_2, \ldots, x_r) = \sum_{1 \leq i \leq j \leq r} b_{ij} x_i x_j$$

be an integer positive definite quadratic form of r variables and let $A = (a_{ij})$ be the symmetric $r \times r$ matrix of the quadratic form $Q(X)$, where $a_{ii} = 2b_{ii}$ and $a_{ij} = a_{ji} = b_{ij}$, for $i < j$. If $X = (x_1 \ldots x_r)^T$ denotes a column matrix and X^T its transpose, then $Q(X) = \frac{1}{2} X^T A X$. Let A_{ij} denote the cofactor to the element a_{ij} in A and let $A^{-1} = (a_{ij}^*)_{ij=1}^r$ be the inverse matrix.

A homogeneous polynomial $P(X) = P(x_1, \cdots, x_r)$ of degree ν with complex coefficients, satisfying the condition

$$\sum_{1 \leq i,j \leq r} a_{ij}^* \left(\frac{\partial^2 P}{\partial x_i \partial x_j} \right) = 0 \tag{1}$$

is called a spherical polynomial of order ν with respect to $Q(X)$ (see [4]).

Let $\mathcal{P}(\nu, Q)$ denote the vector space over $\mathbb{C}$ of spherical polynomials $P(X)$ of even order ν with respect to $Q(X)$.

Hecke [6] calculated the dimension of the space $\mathcal{P}(\nu, Q)$ and showed that

$$\dim \mathcal{P}(\nu, Q) = \binom{\nu + r - 1}{r - 1} - \binom{\nu + r - 3}{r - 1}.$$

He formed a basis of the space of spherical polynomials of second order ($\nu = 2$) with respect to $Q(X)$.

Lomadze [7] constructed a basis of the space of spherical polynomials of fourth order ($\nu = 4$) with respect to $Q(X)$. In the next section a basis of the space $\mathcal{P}(\nu, Q)$ is constructed with a simpler way.

Let

$$\vartheta(\tau, P, Q) = \sum_{n \in \mathbb{Z}^r} P(n) z^{Q(n)}, \qquad z = e^{2\pi i \tau}, \qquad \tau \in \mathbb{C}, \qquad \operatorname{Im} \tau > 0$$

be the corresponding generalized r-fold theta-series. Schoeneberg [8] proved that the function $\vartheta(\tau, P, Q)$ is a modular form of weight $-(\frac{r}{2} + \nu)$ with respect to the congruence subgroup $\Gamma_0(N)$, where N is the least positive integer such that NA^{-1} is again an even integral symmetric matrix. The map which assigns to each P in $\mathcal{P}(\nu, Q)$ the modular form $\vartheta(\tau, P, Q)$ is a linear transformation.

Let $T(\nu, Q)$ denote the vector space over $\mathbb{C}$ of generalized multiple theta-series, i.e.

$$T(\nu, Q) = \{\vartheta(\tau, P, Q) \ : \ P \in \mathcal{P}(\nu, Q)\}.$$

Gooding [4, 5] calculated the dimension of the vector space $T(\nu, Q)$ for a reduced binary quadratic form Q and obtained an upper bound of the dimension of the space $T(\nu, Q)$ for some diagonal quadratic forms of r variables

$$\dim T(\nu, Q) \leq \binom{\frac{\nu}{2} + r - 2}{r - 2}. \tag{2}$$

In [9, 10, 11], the upper bounds for the dimensions of the spaces $T(\nu, Q)$ for certain quadratic forms in three, four, five and r variables were established. In several cases, the dimensions were calculated and bases for these spaces were constructed.

Gaigalas [1-3] obtained upper bounds for the dimensions of the spaces $T(4, Q)$ and $T(6, Q)$ for certain diagonal quadratic forms and presented upper bounds for the dimensions of the spaces $T(\nu, Q)$ for some diagonal quadratic forms in six variables.

In this paper, upper bounds for the dimensions of the spaces $T(\nu, Q)$ are obtained for certain diagonal and non-diagonal quadratic forms in six variables. The dimension of the space $T(2, Q)$ is calculated and a basis for this space is constructed.

In the sequel, we use the following definition and results:

An integral $r \times r$ matrix U is called an integral automorphism of the quadratic form $Q(X)$ in r variables if $U^T A U = A$.

Lemma 1. ([4], p. 37) *Let $Q(X) = Q(x_1, \ldots, x_r)$ be a positive definite quadratic form in r variables and $P(X) = P(x_1, \ldots, x_r) \in P(\nu, Q)$. Let G be the set of all integral automorphisms of Q. Suppose*

$$\sum_{i=1}^{t} P(U_i X) = 0 \qquad \text{for some} \qquad U_1, \ldots, U_t \subseteq G,$$

then $\vartheta(\tau, P, Q) = 0$.

Lemma 2. ([11], p. 92) *Let $Q_1(X)$ be the non-diagonal quadratic form of r variables, given by $Q_1(X) = b_{11}x_1^2 + b_{22}x_2^2 + b_{33}x_3^2 + \cdots + b_{rr}x_r^2 + b_{12}x_1x_2$ then*

$$\dim T(\nu, Q_1) \leq \frac{1}{r-2} \binom{\frac{\nu}{2} + r - 3}{r - 3} (\nu + r - 2). \tag{3}$$

2. The Basis of the Space $\mathcal{P}(\nu, Q)$

Let

$$P(X) = P(x_1, x_2, x_3, x_4, x_5, x_6) = \sum_{k=0}^{\nu}\sum_{i=0}^{k}\sum_{j=0}^{i}\sum_{l=0}^{j}\sum_{s=0}^{l} a_{kijls} x_1^{\nu-k} x_2^{k-i} x_3^{i-j} x_4^{j-l} x_5^{l-s} x_6^s$$

be a spherical function of order ν with respect to the positive quadratic form $Q(x_1, x_2, x_3, x_4, x_5, x_6)$ of six variables and

$$L = \left(a_{00000}, a_{10000}, a_{11000}, a_{11100}, a_{11110}, a_{11111}, a_{20000}, a_{21000}, \ldots, a_{\nu,\nu,\nu,\nu,\nu}\right)^T$$

be the column vector, where a_{kijls} ($\nu \geq k \geq i \geq j \geq l \geq s \geq 0$) are the coefficients of polynomial $P(X)$.

According to (1), the condition

$$\frac{1}{|A|} \sum_{1 \leq i,j \leq 6} A_{ij}\left(\frac{\partial^2 P}{\partial x_i \partial x_j}\right) = 0$$

is satisfied. Considering

$$\frac{\partial^2 P}{\partial x_1^2} = \sum_{k=0}^{\nu}\sum_{i=0}^{k}\sum_{j=0}^{i}\sum_{l=0}^{j}\sum_{s=0}^{l}(\nu - k)(\nu - k - 1)a_{kijls} x_1^{\nu-k-2} x_2^{k-i} x_3^{i-j} x_4^{j-l} x_5^{l-s} x_6^s$$

$$= \sum_{k=1}^{\nu-1}\sum_{i=0}^{k-1}\sum_{j=0}^{i}\sum_{l=0}^{j}\sum_{s=0}^{l}(\nu - k + 1)(\nu - k)a_{k-1ijls} x_1^{\nu-k-1} x_2^{k-i-1} x_3^{i-j} x_4^{j-l} x_5^{l-s} x_6^s$$

and also obtain similar formulas for other second partial derivatives, then condition (1) takes the form

$$\frac{1}{|A|} \sum_{k=1}^{\nu-1}\sum_{i=0}^{k-1}\sum_{j=0}^{i}\sum_{l=0}^{j}\sum_{s=0}^{l} \Big(A_{11}(\nu - k + 1)(\nu - k)a_{k-1ijls} + 2A_{12}(\nu - k)(k - i)a_{kijls} +$$

$$2A_{13}(\nu - k)(i - j + 1)a_{ki+1jls} + 2A_{14}(\nu - k)(j - l + 1)a_{ki+1j+1ls} +$$

$$+ 2A_{15}(\nu - k)(l - s + 1)a_{ki+1j+1l+1s} + 2A_{16}(\nu - k)(s + 1)a_{ki+1j+1l+1s+1} +$$

$$+ A_{22}(k - i + 1)(k - i)a_{k+1ijls} + 2A_{23}(k - i)(i - j + 1)a_{k+1i+1jls} + \ldots$$

$$+ A_{66}(s + 2)(s + 1)a_{k+1i+2j+2l+2s+2}\Big) x_1^{\nu-k-1} x_2^{k-i-1} x_3^{i-j} x_4^{j-l} x_5^{l-s} x_6^s = 0.$$

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Thus, for $0 \le s \le l \le j \le i < k \le \nu - 1$, we obtain

$$A_{11}(\nu - k + 1)(\nu - k)a_{k-1ijls} + 2A_{12}(\nu - k)(k - i)a_{kijls}+$$

$$2A_{13}(\nu - k)(i - j + 1)a_{ki+1jls} + 2A_{14}(\nu - k)(j - l + 1)a_{ki+1j+1ls}+$$

$$+2A_{15}(\nu - k)(l - s + 1)a_{ki+1j+1l+1s} + 2A_{16}(\nu - k)(s + 1)a_{ki+1j+1l+1s+1}+$$

$$+A_{22}(k - i + 1)(k - i)a_{k+1ijls} + 2A_{23}(k - i)(i - j + 1)a_{k+1i+1jls} + \ldots$$

$$+A_{66}(s + 2)(s + 1)a_{k+1i+2j+2l+2s+2} = 0.$$

It follows that condition (1), in matrix notation has the following form

$$S \cdot L = 0,$$

where the matrix S has the form

$$S = \begin{pmatrix} A_{11}\nu(\nu-1) & 2A_{12}(\nu-1) & 2A_{13}(\nu-1) & 2A_{14}(\nu-1) & \ldots & \ldots & \ldots & 0 \\ 0 & A_{11}(\nu-1)(\nu-2) & 0 & 0 & \ldots & \ldots & \ldots & 0 \\ 0 & 0 & A_{11}(\nu-1)(\nu-2) & 0 & \ldots & \ldots & \ldots & 0 \\ 0 & 0 & 0 & A_{11}(\nu-1)(\nu-2) & \ldots & \ldots & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & 2A_{11} & \ldots & A_{66}(\nu-1)\nu \end{pmatrix}$$

and is $\binom{\nu+3}{5} \times \binom{\nu+5}{5}$ matrix (the number of rows of the matrix S is equal to the number of (k, i, j, l, s) with $0 \le s \le l \le j \le i < k \le \nu - 1$, the number of columns is equal to the number of coefficients a_{kijls}, i.e. to the number of (k, i, j, l, s) with $0 \le s \le l \le j \le i \le k \le \nu$).

We partition the matrix S into two matrices S_1 and S_2, where S_1 is the left square nondegenerate $\binom{\nu+3}{5} \times \binom{\nu+3}{5}$ matrix. It consists of the first $\binom{\nu+3}{5}$ columns of the matrix S; the matrix S_2 consists of the last $\binom{\nu+5}{5} - \binom{\nu+3}{5}$ columns of the matrix S.

Similarly, we partition the matrix L into two matrices L_1 and L_2, where L_1 is the $\binom{\nu+3}{5} \times 1$ matrix consisting of the upper $\binom{\nu+3}{5}$ elements of L; the matrix L_2 consists of the lower $\binom{\nu+5}{5} - \binom{\nu+3}{5}$ elements of the matrix L.

According to the new notation, the matrix equality takes the form

$$S_1 L_1 + S_2 L_2 = 0,$$

i.e.

$$L_1 = -S_1^{-1} S_2 L_2.$$

It follows from this equality that the matrix L_1 is expressed through the matrix L_2. Consequently, the first $\binom{\nu+3}{5}$ elements of the matrix L can be expressed in terms of its remaining elements. Since the matrix L consists of the coefficients of the spherical polynomial $P(X)$, its first $\binom{\nu+3}{5}$ coefficients can be expressed through the last $\binom{\nu+5}{5} - \binom{\nu+3}{5}$ coefficients.

Hence, the polynomials

$$P_{\nu-1,0000}(a^{(1)}_{00000}, a^{(1)}_{10000}, \ldots, a^{(1)}_{\nu-2,\nu-2,\nu-2,\nu-2,\nu-2}, 1, 0, 0, \ldots, 0),$$

$$P_{\nu-1,1000}(a^{(2)}_{00000}, a^{(2)}_{10000}, \ldots, a^{(2)}_{\nu-2,\nu-2,\nu-2,\nu-2,\nu-2}, 0, 1, 0, \ldots, 0),$$

(4)

$$\cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots$$

$$P_{\nu,\nu,\nu,\nu,\nu}(a^{(t)}_{00000}, a^{(t)}_{10000}, \ldots, a^{(t)}_{\nu-2,\nu-2,\nu-2,\nu-2,\nu-2}, 0, 0, 0, \ldots, 1),$$

form the basis of the space $\mathcal{P}(\nu, Q)$, where the first $\binom{\nu+3}{5}$ coefficients, from a_{00000} to $a_{\nu-2,\nu-2,\nu-2,\nu-2,\nu-2}$ are calculated in terms of the remaining $\binom{\nu+5}{5} - \binom{\nu+3}{5}$ coefficients. The coefficients of the basis polynomial P_{kijls} are given in the brackets, where $k = \nu - 1$ or $k = \nu$. Moreover, the polynomial P_{kijls} is such that its coefficient a_{kijls} is equal to 1 and all other coefficients for which k is $\nu - 1$ or ν are equal to 0 .

3. On the dimension of $T(\nu, Q)$ for certain quadratic forms of six variables

Let

$$Q(X) = b_{11}x_1^2 + b_{22}x_2^2 + b_{33}x_3^2 + \cdots + b_{rr}x_r^2$$

with $0 < b_{11} < b_{22} < \cdots < b_{rr}$ be a diagonal quadratic form of r variables. We have shown [11 p.91] that a basis of the space $T(\nu, Q)$ is among the theta-series $\vartheta(\tau, P, Q)$ with spherical polynomials $P = P_{kij\ldots s}$, where all indices $k = \nu, i, j, \ldots s$ are even. Furthermore, the dimension of the space $T(\nu, Q)$ satisfies the estimate (2).

Now consider the quadratic form of six variables

$$Q_2 = b_{11}x_1^2 + b_{22}x_2^2 + b_{33}x_3^2 + b_{44}x_4^2 + b_{55}(x_5^2 + x_6^2),$$

where $0 < b_{11} < b_{22} < b_{33} < b_{44} < b_{55} = b_{66}$. Using (2), we have

$$\dim T(\nu, Q_2) \leq \binom{\frac{\nu}{2} + r - 2}{r - 2} = \binom{\frac{\nu}{2} + 4}{4}.$$

(5)

We improve this estimation.

We construct the integral automorphisms U of the quadratic form Q_2. Since

$$b_{11} = Q_2(\pm 1, 0, 0, 0, 0, 0), \quad b_{22} = Q_2(0, \pm 1, 0, 0, 0, 0), \quad b_{33} = Q_2(0, 0, \pm 1, 0, 0, 0),$$
$$b_{44} = Q_2(0, 0, 0, \pm 1, 0, 0), \quad b_{55} = b_{66} = Q_2(0, 0, 0, 0, \pm 1, 0) = Q_2(0, 0, 0, 0, 0, \pm 1),$$

it is easy to verify that the integral automorphisms of the quadratic form Q_2 are

$$\begin{pmatrix} e_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & e_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & e_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & e_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & e_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & e_6 \end{pmatrix}, \quad \begin{pmatrix} e_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & e_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & e_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & e_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e_5 \\ 0 & 0 & 0 & 0 & e_6 & 0 \end{pmatrix}, (e_i = \pm 1, \quad i = 1, 2, 3, 4, 5, 6).$$

Consider all possible polynomials $P = P_{kijls}(UX)$, where $P_{kijls}(X) \in P(\nu, Q_2)$ are spherical basis polynomials (4) of order ν with respect to $Q_2(X)$ and $U \in G$ is an integral automorphism of the quadratic form $Q_2(X)$.

We have to find which polynomials P satisfy equality

$$\sum_{U_h} P_{kijls}(U_h X) = 0, \qquad \text{for some} \qquad U_h \in G.$$

For such polynomials, according to Lemma 1, we have $\vartheta(\tau, P, Q) = 0$.

For example, if $U_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$ then

$$P_{kijls}(X) + P_{kijls}(U_1 X) = \sum_{k=0}^{\nu} \sum_{i=0}^{k} \sum_{j=0}^{i} \sum_{l=0}^{j} \sum_{s=0}^{l} (1 + (-1)^s) a_{kijls} x_1^{\nu-k} x_2^{k-i} x_3^{i-j} x_4^{j-l} x_5^{l-s} x_6^{s}.$$

The equality

$$P_{kijls}(X) + P_{kijls}(U_1 X) = 0$$

takes place if and only if the condition

$$(1 + (-1)^s) a_{kijls} = 0$$

is satisfied. This means that index s of the coefficient equal to one must be odd. Similarly, it follows that if among the last coefficients of P, at least one of the indices k, i, j, l, s of the coefficient (which is equal to one) is odd, then, by Lemma 1, for the spherical polynomial $P = P_{kijls}$, the theta-series satisfies $\vartheta(\tau, P, Q) = 0$. Hence, if theta-series are linearly independent, then the indices k, i, j, l, s of the corresponding spherical polynomial P must all be even. According to (5), such k, i, j, l, s indices are $\binom{\frac{\nu}{2}+4}{4}$.

For automorphism $U_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$ we have

$$P(U_2 X) = \sum_{k=0}^{\nu} \sum_{i=0}^{k} \sum_{j=0}^{i} \sum_{l=0}^{j} \sum_{s=0}^{l} a_{kijls}^{(h)} x_1^{\nu-k} x_2^{k-i} x_3^{i-j} x_4^{j-l} x_6^{l-s} x_5^{s}$$

$$= \sum_{k=0}^{\nu} \sum_{i=0}^{k} \sum_{j=0}^{i} \sum_{l=0}^{j} \sum_{s=0}^{l} a_{kijls}^{(h)} x_1^{\nu-k} x_2^{k-i} x_3^{i-j} x_4^{j-l} x_5^{l-(l-s)} x_6^{l-s}.$$

From here, it follows that if all the last coefficients of the basis polynomial $P(X)$ are equal to zero except one $a_{kijls}^{(h)} = 1$, then all the last coefficients of the polynomial $P(U_2 X)$

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are equal to zero except one $a^{(t)}_{k,i,j,l,l-s} = 1$. Hence, $P_{kijls}(U_2 X) = P_{kijll-s}(X)$ is a basis polynomial of the space $\mathcal{P}(\nu, Q_2)$. Further, it is known ([4], p. 38) that

$$\vartheta(\tau, P(X), Q_2) = \vartheta(\tau, P(U_2 X), Q_2).$$

Thus, the theta-series $\vartheta(\tau, P(X), Q_2)$ and $\vartheta(\tau, P(U_2 X), Q_2)$, corresponding to different basis polynomials $P(X) = P_{kijls}(X)$ and $P(U_2 X) = P_{kijll-s}(X)$, are linearly dependent.

Calculate how many such linearly dependent theta-series we have. Let k, i, j, l and s be even (otherwise, it can be shown that $\vartheta(\tau, P, Q_2) = 0$), i.e. $k = \nu, 2 \mid i, 2 \mid j, 2 \mid l, 2 \mid s$ and s takes

$$\sum_{2 \mid s, s=0}^{l} 1 = \frac{l}{2} + 1$$

even values for each even l. Hence we have

$$\left[\frac{1}{2}\left(\frac{l}{2} + 1\right) \right] = \begin{cases} \frac{l}{4} & \text{if } l \equiv 0 (\mathrm{mod}4), \\ \frac{l+2}{4} & \text{if } l \equiv 2 (\mathrm{mod}4) \end{cases}$$

linearly dependent theta-series for each even l. Similarly, for each even j, we have

$$\sum_{\substack{l = 0 \\ l \equiv 0 (\mathrm{mod}4)}}^{j} \frac{l}{4} + \sum_{\substack{l = 0 \\ l \equiv 2 (\mathrm{mod}4)}}^{j} \frac{l+2}{4} = \begin{cases} (1 + \frac{j}{4})\frac{j}{4} & \text{if } j \equiv 0 (\mathrm{mod}4), \\ \left(\frac{j+2}{4}\right)^2 & \text{if } j \equiv 2 (\mathrm{mod}4) \end{cases}$$

linearly dependent theta-series. Also, for each even i, we have

$$\sum_{\substack{j = 0 \\ j \equiv 0 (\mathrm{mod}4)}}^{i} (1 + \frac{j}{4})\frac{j}{4} + \sum_{\substack{j = 0 \\ j \equiv 2 (\mathrm{mod}4)}}^{i} \left(\frac{j+2}{4}\right)^2 =$$

$$= \begin{cases} \frac{1}{24}(\frac{i}{4} + 1)i(i + 5) & \text{if } i \equiv 0 (\mathrm{mod}4), \\ \frac{1}{24}(\frac{i}{2} + 1)(\frac{i}{2} + 3)(i + 1) & \text{if } i \equiv 2 (\mathrm{mod}4) \end{cases}$$

linearly dependent theta-series. The number of linearly dependent theta-series for even ν is

$$\sum_{\substack{i = 0 \\ i \equiv 0 (\mathrm{mod}4)}}^{\nu} \frac{1}{24}(\frac{i}{4} + 1)i(i + 5) + \sum_{\substack{i = 0 \\ i \equiv 2 (\mathrm{mod}4)}}^{\nu} \frac{1}{24}(\frac{i}{2} + 1)(\frac{i}{2} + 3)(i + 1)$$

$$= \begin{cases} \frac{1}{3 \cdot 2^8}\nu(\nu + 4)^2(\nu + 8) & \text{if } \nu \equiv 0 (\mathrm{mod}4), \\ \frac{1}{3 \cdot 2^8}(\nu + 2)(\nu + 6)(\nu^2 + 8\nu + 4) & \text{if } \nu \equiv 2 (\mathrm{mod}4). \end{cases} \quad (6)$$

Hence, from (5), for the maximal number of linearly independent theta-series, we obtain (a similar result is given in [3])

$$\dim T(\nu, Q_2) \le \begin{cases} \binom{\frac{\nu}{2}+4}{4} - \frac{1}{3 \cdot 2^8}\nu(\nu + 4)^2(\nu + 8) & \text{if } \nu \equiv 0 (\mathrm{mod}4), \\ \binom{\frac{\nu}{2}+4}{4} - \frac{1}{3 \cdot 2^8}(\nu + 2)(\nu + 6)(\nu^2 + 8\nu + 4) & \text{if } \nu \equiv 2 (\mathrm{mod}4). \end{cases}$$

Thus. we have the following:

Theorem 1. *Let $Q_2(X)$ be the diagonal quadratic form of six variables, given by* $Q_2(X) = b_{11}x_1^2 + b_{22}x_2^2 + b_{33}x_3^2 + b_{44}x_4^2 + b_{55}(x_5^2 + x_6^2)$, *then*

$$\dim T(\nu, Q_2) \leq \begin{cases} \frac{1}{3\cdot2^8}(\nu+4)(\nu+8)(\nu^2+12\nu+24) & \text{if } \nu \equiv 0(\text{mod}4), \\ \frac{1}{3\cdot2^8}(\nu+2)(\nu+6)^2(\nu+10) & \text{if } \nu \equiv 2(\text{mod}4). \end{cases}$$

Similarly, consider the non-diagonal quadratic form of six variables

$$Q_3 = b_{11}x_1^2 + b_{22}x_2^2 + b_{33}x_3^2 + b_{44}x_4^2 + b_{55}(x_5^2 + x_6^2) + b_{12}x_1x_2,$$

where $0 < |b_{12}| < b_{11} < b_{22} < b_{33} < b_{44} < b_{55} = b_{66}$.

We construct the integral automorphisms U of the quadratic form Q_3. It is easy to verify that the integral automorphisms of the quadratic form Q_3 are

$$\begin{pmatrix} e_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & e_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & e_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & e_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & e_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & e_5 \end{pmatrix}, \begin{pmatrix} e_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & e_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & e_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & e_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e_4 \\ 0 & 0 & 0 & 0 & e_5 & 0 \end{pmatrix}, (e_i = \pm 1, \quad i = 1,2,3,4,5).$$

Consider all possible polynomials $P = P_{kijls}(UX)$, where $P_{kijls}(X) \in P(\nu, Q_3)$ are spherical basis polynomials (4) of order ν with respect to $Q_3(X)$, and $U \in G$ is an integral automorphism of the quadratic form $Q_3(X)$.

We have to find which polynomials P satisfy the equality

$$\sum_{U_h} P_{kijls}(U_h X) = 0 \qquad \text{for some} \qquad U_h \in G.$$

As in the case of the quadratic form Q_2, here too, P_{kijls} satisfies this condition if, among the last coefficients of P, for which k is $\nu - 1$ or ν, at least one of the indices i, j, l, s of the coefficient, equaled to one, is odd. For such spherical a polynomial $P = P_{kijls}$, by Lemma 1, $\vartheta(\tau, P, Q) = 0$. Hence, if theta-series are linearly independent, then the indices i, j, l, s of the corresponding spherical polynomial P must be even. According to (3), such k, i, j, l, s indices are

$$\frac{1}{r-2}\binom{\frac{\nu}{2}+r-3}{r-3}(\nu+r-2) = \frac{\nu+4}{4}\binom{\frac{\nu}{2}+3}{3} \tag{7}$$

For the automorphism U_2, we have

$$P(U_2 X) = \sum_{k=0}^{\nu}\sum_{i=0}^{k}\sum_{j=0}^{i}\sum_{l=0}^{j}\sum_{s=0}^{l} a_{kijls}^{(h)} x_1^{\nu-k} x_2^{k-i} x_3^{i-j} x_4^{j-l} x_6^{l-s} x_5^{s}$$

$$= \sum_{k=0}^{\nu}\sum_{i=0}^{k}\sum_{j=0}^{i}\sum_{l=0}^{j} a_{kijls}^{(h)} x_1^{\nu-k} x_2^{k-i} x_3^{i-j} x_4^{j-l} x_5^{l-(l-s)} x_6^{l-s}.$$

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From here, $P(U_2X)$ is a basis polynomial of the space $\mathcal{P}(\nu, Q_3)$. Furthermore, it is known ([4], p. 38) that
$$\vartheta(\tau, P(X), Q_3) = \vartheta(\tau, P(U_2X), Q_3).$$
Thus, the theta-series $\vartheta(\tau, P(X), Q_3)$ and $\vartheta(\tau, P(U_2X), Q_3)$, corresponding to the different basis polynomials $P(X) = P_{kijls}(X)$ and $P(U_2X) = P_{kijll-s}(X)$, are linearly dependent.

Calculate how many such linearly dependent theta-series we have. Let i, j, l, s be even (otherwise, it can be shown that $\vartheta(\tau, P, Q_3) = 0$), i.e., $2 \mid i$, $2 \mid j$, $2 \mid l$, $2 \mid s$.

For $k = \nu - 1$, the number of linearly dependent theta-series is

$$\sum_{\substack{i = 0 \\ i \equiv 0 (\mathrm{mod}4)}}^{\nu-2} \frac{1}{24}(\frac{i}{4} + 1)i(i + 5) + \sum_{\substack{i = 0 \\ i \equiv 2 (\mathrm{mod}4)}}^{\nu-2} \frac{1}{24}(\frac{i}{2} + 1)(\frac{i}{2} + 3)(i + 1)$$

$$= \begin{cases} \frac{1}{3 \cdot 2^8}\nu(\nu + 4)(\nu^2 + 4\nu - 8) & \text{if } \nu \equiv 0 (\mathrm{mod}4), \\ \frac{1}{3 \cdot 2^8}(\nu - 2)(\nu + 2)^2(\nu + 6) & \text{if } \nu \equiv 2 (\mathrm{mod}4). \end{cases}$$

For $k = \nu$, the number of linearly dependent theta-series is given by estimation (6). Thus, the total number of linearly dependent theta-series for $k = \nu - 1$ and $k = \nu$ is

$$\frac{1}{3 \cdot 2^7}\nu(\nu + 2)(\nu + 4)(\nu + 6).$$

For the maximal number of linearly independent theta-series, from (7) we obtain

$$\dim T(\nu, Q_3) \leq \frac{\nu + 4}{4}\binom{\frac{\nu}{2} + 3}{3} - \frac{1}{3 \cdot 2^7}\nu(\nu + 2)(\nu + 4)(\nu + 6) = \binom{\frac{\nu}{2} + 4}{4}.$$

Thus, we have the following:

Theorem 2. *Let $Q_3(X)$ be the non-diagonal quadratic form of six variables, given by $Q_3(X) = b_{11}x_1^2 + b_{22}x_2^2 + b_{33}x_3^2 + b_{44}x_4^2 + b_{55}(x_5^2 + x_6^2) + b_{12}x_1x_2$, then*

$$\dim T(\nu, Q_3) \leq \binom{\frac{\nu}{2} + 4}{4}.$$

We now construct the basis of the space $T(\nu, Q_3)$ when $\nu = 2$. For the quadratic form $Q_3(X)$, we have:

$$|A| = \det A = 2^4(4b_{11}b_{22} - b_{12}^2)b_{33}b_{44}b_{55}^2, \qquad a_{11}^* = \frac{2b_{22}}{4b_{11}b_{22} - b_{12}^2},$$

$$a_{12}^* = a_{21}^* = -\frac{b_{12}}{4b_{11}b_{22} - b_{12}^2}, \qquad a_{22}^* = \frac{2b_{11}}{4b_{11}b_{22} - b_{12}^2}, \qquad a_{33}^* = \frac{1}{2b_{33}},$$

$$a_{44}^* = \frac{1}{2b_{44}}, \qquad a_{55}^* = a_{66}^* = \frac{1}{2b_{55}}, \qquad \text{and other} \quad a_{ij}^* = 0 \quad \text{for} \quad i \neq j.$$

9

It is easy to verify that the spherical polynomials (4) of second order:

$$P_{10000} = \frac{b_{12}}{4b_{22}}x_1^2 + x_1 x_2, \qquad P_{11000} = x_1 x_3,$$

$$P_{11100} = x_1 x_4, \qquad \ldots, \qquad P_{20000} = -\frac{b_{11}}{b_{22}}x_1^2 + x_2^2,$$

$$P_{21000} = x_2 x_3, \qquad P_{22000} = -\frac{4b_{11}b_{22} - b_{12}^2}{4b_{22}b_{33}}x_1^2 + x_3^2, \qquad P_{22100} = x_3 x_4,$$

$$P_{22200} = -\frac{4b_{11}b_{22} - b_{12}^2}{4b_{22}b_{44}}x_1^2 + x_4^2, \qquad \ldots, \qquad P_{22220} = P_{22222} = -\frac{4b_{11}b_{22} - b_{12}^2}{4b_{22}b_{55}}x_1^2 + x_5^2$$

form the basis of the space of spherical polynomials of second order with respect to $Q_3(x)$.

Now, we construct the corresponding generalized theta-series. Suppose the quadratic form $Q_3(x)$ is such that $b_{22} \neq b_{11}x_1^2$, $b_{mm} \neq \sum_{i=1}^{m-1} b_{ii}x_i^2 + b_{12}x_1 x_2$, $m = 3, 4, 5,$. Consider all possible polynomials P_{kijls}, with even indices i, j, l, s and $k = \nu - 1, \nu$; their number is 5.

$$\vartheta(\tau, P_{10000}, Q_3) = \sum_{n=1}^{\infty} \Big(\sum_{Q_3(x)=n} P_{10000}(x) \Big) z^n = \sum_{n=1}^{\infty} \Big(\sum_{Q_3(x)=n} \big(\frac{b_{12}}{4b_{22}}x_1^2 + x_1 x_2 \big) \Big) z^n$$

$$= \frac{b_{12}}{2b_{22}} z^{b_{11}} + \cdots + 0z^{b_{22}} + \cdots + 0z^{b_{33}} + \cdots + 0z^{b_{44}} + \cdots + 0z^{b_{55}} + \cdots,$$

$$\vartheta(\tau, P_{20000}, Q_3) = \sum_{n=1}^{\infty} \Big(\sum_{Q_3(x)=n} P_{20000}(x) \Big) z^n = \sum_{n=1}^{\infty} \Big(\sum_{Q_3(x)=n} \big(-\frac{b_{11}}{b_{22}}x_1^2 + x_2^2 \big) \Big) z^n$$

$$= -\frac{2b_{11}}{b_{22}} z^{b_{11}} + \cdots + 2z^{b_{22}} + \cdots + 0z^{b_{33}} + \cdots + 0z^{b_{44}} + \cdots + 0z^{b_{55}} + \cdots,$$

$$\vartheta(\tau, P_{22000}, Q_3) = \sum_{n=1}^{\infty} \Big(\sum_{Q_3(x)=n} P_{22000}(x) \Big) z^n = \sum_{n=1}^{\infty} \Big(\sum_{Q_3(x)=n} \big(-\frac{4b_{11}b_{22} - b_{12}^2}{4b_{22}b_{33}}x_1^2 + x_3^2 \big) \Big) z^n$$

$$= -\frac{4b_{11}b_{22} - b_{12}^2}{2b_{22}b_{33}} z^{b_{11}} + \cdots + 0z^{b_{22}} + \cdots + 2z^{b_{33}} + \cdots + 0z^{b_{44}} + \cdots + 0z^{b_{55}} + \cdots,$$

$$\vartheta(\tau, P_{22200}, Q_3) = \sum_{n=1}^{\infty} \Big(\sum_{Q_3(x)=n} P_{22200}(x) \Big) z^n = \sum_{n=1}^{\infty} \Big(\sum_{Q_3(x)=n} \big(-\frac{4b_{11}b_{22} - b_{12}^2}{4b_{22}b_{44}}x_1^2 + x_4^2 \big) \Big) z^n$$

$$= -\frac{4b_{11}b_{22} - b_{12}^2}{2b_{22}b_{44}} z^{b_{11}} + \cdots + 0z^{b_{22}} + \cdots + 0z^{b_{33}} + \cdots + 2z^{b_{44}} + \cdots + 0z^{b_{55}} + \cdots.$$

$$\vartheta(\tau, P_{22220}, Q_3) = \sum_{n=1}^{\infty} \Big(\sum_{Q_3(x)=n} P_{22220}(x) \Big) z^n = \sum_{n=1}^{\infty} \Big(\sum_{Q_3(x)=n} \big(-\frac{4b_{11}b_{22} - b_{12}^2}{4b_{22}b_{55}}x_1^2 + x_5^2 \big) \Big) z^n$$

$$= -\frac{4b_{11}b_{22} - b_{12}^2}{2b_{22}b_{55}} z^{b_{11}} + \cdots + 0z^{b_{22}} + \cdots + 0z^{b_{33}} + \cdots + 0z^{b_{44}} + \cdots + 2z^{b_{55}} + \cdots.$$

These generalized theta-series are linearly independent since the determinant constructed from the coefficients of these theta-series is not equal to zero. By Theorem 2, we have $\dim T(2, Q_3) \leq 5$. Hence, these theta-series form the basis of the space $T(2, Q_3)$. We have the following:

Theorem 3. *Let $Q_3(X)$ be the non-diagonal quadratic form of six variables, given by $Q_3(X) = b_{11}x_1^2 + b_{22}x_2^2 + b_{33}x_3^2 + b_{44}x_4^2 + b_{55}(x_5^2 + x_6^2) + b_{12}x_1x_2$, and let $b_{22} \neq b_{11}x_1^2$, $b_{mm} \neq \sum_{i=1}^{m-1} b_{ii}x_i^2 + b_{12}x_1x_2$, $m = 3, 4, 5$, then $\dim T(2, Q_3) = 5$ and the generalized theta-series:*

$$\vartheta(\tau, P_{10000}, Q_3); \vartheta(\tau, P_{20000}, Q_3); \vartheta(\tau, P_{22000}, Q_3); \vartheta(\tau, P_{22200}, Q_3), \vartheta(\tau, P_{22220}, Q_3)$$

form the basis of the space $T(2, Q_3)$.

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