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ON A MIXED PROBLEM FOR A PSEUDO-PARABOLIC TYPE DIFFERENTIAL EQUATIONS WITH GERASIMOV–CAPUTO TYPE OPERATOR AND WITH DEGENERATION

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Abstract. In rectangle domain a linear mixed problem for a Gerasimov–Caputo type fractional pseudo-parabolic differential equation with degeneration is considered in the case of $0 < \alpha \leq 1$. The solution of this fractional order partial differential equation is studied in the class of regular functions. Corresponding spectral problem is studied ana eigenvalues, eigenfunctions and associated functions are defined. Adjoint spectral problem also is studied. Application of the Fourier series method is justified and by the aid of Mittag–Leffler function a scalar differential equation and three countable systems of ordinary differential equations are obtained. These countable systems are reduced to the countable systems of linear Volterra type integral equations. The unique solvability of these countable systems of integral equations are proved by the method of successive approximations in combination with the method of compressing mapping. The absolute and uniform convergence of the obtained Fourier series is proved by applying the Cauchy–Schwarz inequality and the Bessel inequality.

1. Introduction. Statement of the problem

Fractional calculus plays an important role in the mathematical modeling of many natural and engineering technology processes [1–7]. The construction of various models of theoretical physics by the aid of fractional calculus is described in [8, Vol. 4, 5], [9, 10]. A detailed review of the application of fractional calculus in solving problems of applied sciences is given in [11, Vol. 6–8], [12]. In [13], it is considered an inverse problem to determine right-hand side for a mixed type integro-differential equation with fractional order Gerasimov–Caputo operators. The problem of determining the source function for a degenerate parabolic equation with the Gerasimov–Caputo operator is investigated in [14]. In [15], the solvability of the nonlocal boundary problem for a mixed type differential equation with fractional order operator and degeneration is studied. In the direction of applications of fractional derivatives to solving partial differential equations were obtained interesting results also in [16–35].

In this paper, for the case of $0 < \alpha \leq 1$ we study the regular solvability of the mixed problem for the Gerasimov–Caputo type fractional differential equation with degeneration.

In the domain $\Omega = \left\{(t,x)|\, 0 < t < T,\ 0 < x < 1\right\}$ a pseudo-parabolic differential equation of the following form is considered

$$\left[{}_C D_{0t}^{\alpha} - {}_C D_{0t}^{\alpha} \frac{\partial^2}{\partial x^2} - t^{\beta} \frac{\partial^2}{\partial x^2}\right] U(t,x) = a(t)\, U(x,y) + f(t,x) \tag{1.1}$$

with initial value condition

$$U(0,x) = \varphi(x), \quad 0 \leq x \leq 1, \tag{1.2}$$

where $\beta,\ T$ are given positive real numbers, for $0 < \alpha \leq 1$ the function

$${}_C D_{0t}^{\alpha}\eta(t) = J_{0t}^{1-\alpha}\eta'(t) = \frac{1}{\Gamma(1-\alpha)} \int\limits_0^t \frac{\eta'(s)}{(t-s)^{\alpha}} ds, \ \ {}_C D_{0t}^{1}\eta(t) = J_{0t}^{0}\eta'(t) = \eta'(t), \ \ t \in (0,T)$$

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is Gerasimov–Caputo type fractional operator,

$$J_{0+}^{\alpha}\eta(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\eta(s)\,ds}{(t-s)^{1-\alpha}}, \quad t \in (0, T)$$

is Riemann–Liouville integral operator, $a(t) \in C[0,T]$, $\varphi(x) \in C^6[0,1]$ and $f(t,x) \in C_{t,x}^{0,6}(\overline{\Omega})$ are known functions, $\overline{\Omega} = \left\{(t,x) \mid 0 \le t \le T,\ 0 \le x \le 1\right\}$.

Problem Statement. We find an unknown function $U(t,x)$, which satisfies partial differential equation (1.1), initial value condition (1.2), boundary value conditions

$$U(t,0) = 0, \quad U_x(t,1) = U_x(t,x_0), \quad 0 \le t \le T, \quad 0 < x_0 < 1, \tag{1.3}$$

and class of functions

$$U(t,x) \in C(\overline{\Omega}), \quad {}_C D_{0+}^{\alpha} U(t,x), \ U_{xx}(t,x), \ {}_C D_{0+}^{\alpha} U_{xx}(t,x) \in C(\Omega), \tag{1.4}$$

where

$$\frac{d^i\varphi(x)}{dx^i}\,|_{x=0} = 0, \quad \frac{d^j\varphi(x)}{dx^j}\,|_{x=1} = \frac{d^j\varphi(x)}{dx^j}\,|_{x=x_0}, \quad i = 0,2,4,6, \quad j = 1,3,5,$$

$$\frac{d^i f(t,x)}{dx^i}\,|_{x=0} = 0, \quad \frac{d^j f(t,x)}{dx^j}\,|_{x=1} = \frac{d^j f(t,x)}{dx^j}\,|_{x=x_0}, \quad i = 0,2,4,6, \quad j = 1,3,5.$$

2. Spectral problem

For the problem (1.1) and (1.3) we consider the spectral problem

$$\vartheta''(x) + \lambda\vartheta(x) = 0, \quad \lambda \ge 0, \tag{2.1}$$

$$\vartheta(0) = 0, \quad \vartheta'(1) = \vartheta'(x_0), \quad 0 < x_0 < 1. \tag{2.2}$$

Solving the spectral problem (2.1), (2.2), we derive eigenvalues (see [36])

$$\lambda_0 = 0, \quad \lambda_{1,n} = \left(\frac{2n\pi}{1+x_0}\right)^2, \quad \lambda_{2,n} = \left(\frac{2n\pi}{1-x_0}\right)^2, \quad n \in \mathbb{N}. \tag{2.3}$$

Eigenfunctions, corresponding to the eigenvalues (2.3), have the forms

$$\vartheta_0(x) = x, \quad \vartheta_{i,n}(x) = \sin\sqrt{\lambda_{i,n}}x, \quad i = 1,2, \quad n \in \mathbb{N}. \tag{2.4}$$

Condition A. Let x_0 be a rational number from the interval $(0,1)$ such that $x_0 = \frac{p}{q}$, $p < q$, $q - p = 1$, p and q be positive integers. If condition A is satisfying, then solving problem (2.1), (2.2), instead (2.3) we obtain

$$\lambda_0 = 0, \quad \lambda_{1,n} = \left(\frac{2qn\pi}{p+q}\right)^2, \quad \lambda_{2,m} = (2qm\pi)^2, \quad n,m \in \mathbb{N}, \quad n \ne m(p+q). \tag{2.5}$$

These eigenvalues correspond to eigenfunctions in (2.4). For each value of $\lambda_{2,m}$, there also exist associated functions of the form

$$\tilde{\vartheta}_{2,m}(x) = x\cos\sqrt{\lambda_{2,m}}x. \tag{2.6}$$

Along with problem (2.1), (2.2), we also consider adjoint problem. It is not difficult to determine that the following spectral problem will be adjoint to problem (2.1), (2.2):

$$\omega''(x) + \lambda\,\omega(x) = 0, \quad \lambda \ge 0, \quad x \in (0,x_0) \cup (x_0,1), \tag{2.7}$$

$$\omega(0) = 0, \quad \omega'(1) = 0, \tag{2.8}$$

$$\omega'(x_0+0) = \omega'(x_0-0), \quad \omega(x_0+0) - \omega(x_0-0) = \omega(1). \tag{2.9}$$

Problem (2.7)–(2.9) also has eigenvalues of the form (2.5). Solving this problem, it is not difficult to see that the eigenfunctions have the form

$$\left\{\omega_0(x);\ \omega_{1,n}(x);\ \omega_{2,m}(x)\right\}, \quad n,m \in \mathbb{N}, \quad n \ne m(p+q). \tag{2.10}$$

where

$$\omega_0(x)=\begin{cases}0, & x\in[0,x_0),\\ \frac{2}{1-x_0^2}, & x\in(x_0,1],\end{cases}\quad \omega_{1,n}(x)=\begin{cases}\frac{4\sin\sqrt{\lambda_{1,n}}x}{1+x_0}, & x\in[0,x_0),\\ \frac{2\cos\sqrt{\lambda_{1,n}}(1-x)}{(1+x_0)\sin\sqrt{\lambda_{1,n}}}, & x\in(x_0,1],\end{cases}$$

$$\omega_{2,m}(x)=\begin{cases}0, & x\in[0,x_0),\\ \frac{4\cos\sqrt{\lambda_{2,m}}x}{1-x_0}, & x\in(x_0,1].\end{cases}$$

There also exist associated functions of the form

$$\tilde{\omega}_{2,m}(x)=\begin{cases}\frac{4\sin\sqrt{\lambda_{2,m}}x}{1+x_0}, & x\in[0,x_0),\\ \frac{4(1-x)\sin\sqrt{\lambda_{2,m}}x}{1-x_0^2}, & x\in(x_0,1].\end{cases}\tag{2.11}$$

We note that systems of functions (2.4), (2.6) and (2.10), (2.11) are biorthonormal in $L_2[0,1]$, that is

$$\big(\vartheta_0(x),\,\omega_0(x)\big)=1,\ \big(\vartheta_0(x),\,\omega_{1,n}(x)\big)=\big(\vartheta_0(x),\,\omega_{2,m}(x)\big)=\big(\vartheta_0(x),\,\tilde{\omega}_{2,m}(x)\big)=0,$$

$$\big(\vartheta_{1,n}(x),\,\omega_{1,k}(x)\big)=\begin{cases}1, & n=k,\\ 0, & n\neq k,\end{cases}$$

$$\big(\vartheta_{1,n}(x),\,\omega_0(x)\big)=\big(\vartheta_{1,n}(x),\,\omega_{2,m}(x)\big)=\big(\vartheta_{1,n}(x),\,\tilde{\omega}_{2,m}(x)\big)=0,$$

$$\big(\vartheta_{2,m}(x),\,\tilde{\omega}_{2,k}(x)\big)=\begin{cases}1, & m=k,\\ 0, & m\neq k,\end{cases}$$

$$\big(\vartheta_{2,m}(x),\,\omega_0(x)\big)=\big(\vartheta_{2,m}(x),\,\omega_{1,n}(x)\big)=\big(\vartheta_{2,m}(x),\,\omega_{2,k}(x)\big)=0,$$

$$\big(\tilde{\vartheta}_{2,m}(x),\,\omega_{2,k}(x)\big)=\begin{cases}1, & m=k,\\ 0, & m\neq k,\end{cases}$$

$$\big(\tilde{\vartheta}_{2,m}(x),\,\omega_0(x)\big)=\big(\tilde{\vartheta}_{2,m}(x),\,\tilde{\omega}_{2,k}(x)\big)=0,$$

where by $(\cdot,\,\cdot)$ is denoted the inner product in $L_2[0,1]$.

We note that if the condition A is satisfying, then the systems of root functions of problems (2.1), (2.2) and (2.7)–(2.9) form a Riesz basis in $L_2[0,1]$.

3. Countable systems of differential equations

Taking into account the formulas (2.4), (2.6) and (2.10), (2.11) we look for a solution

$$U(t,x)=U_0(t,x)+U_1(t,x)+U_2(t,x)+\tilde{U}_2(t,x)\tag{3.1}$$

to the mixed problem (1.1)–(1.4) in the following Fourier series:

$$U(t,x)=u_0(t)\,\vartheta_0(x)+\sum_{n=1}^{\infty *}u_{1,n}(t)\,\vartheta_{1,n}(x)+\sum_{m=1}^{\infty}\big(u_{2,m}(t)\,\vartheta_{2,m}(x)+\tilde{u}_{2,m}(t)\,\tilde{\vartheta}_{2,m}(x)\big)\tag{3.2}$$

where

$$u_0(t)=\int_0^1 U_0(t,y)\omega_0(y)dy,\quad u_{1,n}(t)=\int_0^1 U_1(t,y)\omega_{1,n}(y)dy,$$

$$u_{2,m}(t)=\int_0^1 U_2(t,y)\tilde{\omega}_{2,m}(y)dy,\quad \tilde{u}_{2,m}(t)=\int_0^1 \tilde{U}_2(t,y)\omega_{2,m}(y)dy.$$

Here "*" means that the sum is taken over $n\in\mathbb{N}$, different from $k(q+p),\ k\in\mathbb{N}$.

Let the condition A be satisfied and a function $U(t,x)$ be a solution to the mixed problem (1.1)–(1.4). Then, applying representation (3.2) into equation (1.1) and taking (2.4) and (3.1) into account, we obtain

$${}_C D_{0t}^{\alpha}\Bigg\{x\,u_0(t)+\sum_{n=1}^{\infty *}u_{1,n}(t)\,\vartheta_{1,n}(x)+\sum_{m=1}^{\infty}\Big(u_{2,m}(t)\,\vartheta_{2,m}(x)+\tilde{u}_{2,m}(t)\,\tilde{\vartheta}_{2,m}(x)\Big)+$$

$$+\sum_{n=1}^{\infty *}\lambda_{1,n}u_{1,n}(t)\,\vartheta_{1,n}(x)+\sum_{m=1}^{\infty}\left[\lambda_{2,m}u_{2,m}(t)\,\vartheta_{2,m}(x)+\tilde{u}_{2,m}(t)\left(2\sqrt{\lambda_{2,m}}\,\vartheta_{2,m}(x)+\lambda_{2,m}\tilde{\vartheta}_{2,m}(x)\right)\right]\Bigg\}+$$

$$+\sum_{n=1}^{\infty *}\lambda_{1,n}t^{\beta}u_{1,n}(t)\,\vartheta_{1,n}(x)+\sum_{m=1}^{\infty}\left[\lambda_{2,m}t^{\beta}u_{2,m}(t)\,\vartheta_{2,m}(x)+t^{\beta}\tilde{u}_{2,m}(t)\left(2\sqrt{\lambda_{2,m}}\,\vartheta_{2,m}(x)+\lambda_{2,m}\tilde{\vartheta}_{2,m}(x)\right)\right]=$$

$$=a(t)\left[x\,u_0(t)+\sum_{n=1}^{\infty *}u_{1,n}(t)\,\vartheta_{1,n}(x)+\sum_{m=1}^{\infty}\left(u_{2,m}(t)\,\vartheta_{2,m}(x)+\tilde{u}_{2,m}(t)\,\tilde{\vartheta}_{2,m}(x)\right)\right]+$$

$$+x\,f_0(t)+\sum_{n=1}^{\infty *}f_{1,n}(t)\,\vartheta_{1,n}(x)+\sum_{m=1}^{\infty}\left(f_{2,m}(t)\,\vartheta_{2,m}(x)+\tilde{f}_{2,m}(t)\,\tilde{\vartheta}_{2,m}(x)\right),$$

where

$$f_0(t)=\int_0^1 f_0(t,y)\omega_0(y)dy,\quad f_{1,n}(t)=\int_0^1 f_1(t,y)\omega_{1,n}(y)dy,$$

$$f_{2,m}(t)=\int_0^1 f_2(t,y)\tilde{\omega}_{2,m}(y)dy,\quad \tilde{f}_{2,m}(t)=\int_0^1 \tilde{f}_2(t,y)\omega_{2,m}(y)dy.$$

Hence, taking (2.10) and (2.11) into account, we obtain

$${}_CD_{0t}^{\alpha}\,u_0(t)=a(t)\,u_0(t)+f_0(t),\tag{3.3}$$

$${}_CD_{0t}^{\alpha}\,u_{1,n}(t)+\mu_{1,n}t^{\beta}u_{1,n}(t)=\frac{1}{1+\lambda_{1,n}}\left[a(t)u_{1,n}(t)+f_{1,n}(t)\right],\tag{3.4}$$

$${}_CD_{0t}^{\alpha}\,u_{2,m}(t)+\mu_{2,m}t^{\beta}u_{2,m}(t)=$$

$$=-\frac{2\sqrt{\lambda_{2,m}}}{1+\lambda_{2,m}}\left({}_CD_{0t}^{\alpha}\tilde{u}_{2,m}(t)+\tilde{u}_{2,m}(t)\right)+\frac{1}{1+\lambda_{2,m}}\big[a(t)u_{2,m}(t)+f_{2,m}(t)\big],\tag{3.5}$$

$${}_CD_{0t}^{\alpha}\,\tilde{u}_{2,m}(t)+\mu_{2,m}t^{\beta}\tilde{u}_{2,m}(t)=\frac{1}{1+\lambda_{2,m}}\big[a(t)\tilde{u}_{2,m}(t)+\tilde{f}_{2,m}(t)\big],\tag{3.6}$$

where

$$\mu_{1,n}=\frac{\lambda_{1,n}}{1+\lambda_{1,n}},\quad \mu_{2,m}=\frac{\lambda_{2,m}}{1+\lambda_{2,m}},$$

$\lambda_{1,n}$ and $\lambda_{2,m}$ $\left(n,m\in\mathbb{N},\ n\neq m(p+q)\right)$ defined from (2.5).

The fractional differential equation (3.3) is scalar equation without degeneration. The others (3.4)–(3.6) are countable systems of fractional differential equations with degeneration. Taking into account the formulas (2.4), (2.6) and (2.10), (2.11) we consider the function $\varphi(x)$ as in the case (3.1) of function $U(t,x)$:

$$\varphi(x)=\varphi_0(x)+\varphi_1(x)+\varphi_2(x)+\tilde{\varphi}_2(x).$$

The solving method for the fractional differential equations (3.4)–(3.6) is the same. So, we show the scheme of solving only for the equations (3.3) and (3.4). In order to solve the differential equations (3.3)–(3.6), from condition (1.2) we determine the initial value condition

$$u_0(0)=\int_0^1 U_0(0,y)\omega_0(y)dy=\int_0^1\varphi_0(x)\omega_0(y)dy=\varphi_0,\tag{3.7}$$

$$u_{1,n}(0)=\int_0^1 U_1(0,y)\omega_{1,n}(y)dy=\int_0^1\varphi_1(x)\omega_{1,n}(y)dy=\varphi_{1,n},\tag{3.8}$$

$$u_{2,m}(0)=\int_0^1 U_2(0,y)\tilde{\omega}_{2,m}(y)dy=\int_0^1\varphi_2(x)\tilde{\omega}_{2,m}(y)dy=\varphi_{2,m},\tag{3.9}$$

$$\tilde{u}_{2,m}(0) = \int_0^1 \tilde{U}_2(0,y)\omega_{2,m}(y)dy = \int_0^1 \tilde{\varphi}_2(x)\omega_{2,m}(y)dy = \tilde{\varphi}_{2,m}. \tag{3.10}$$

4. Volterra integral equation

To solve the equation (3.3), we rewrite it in the following form

$${}_C D_{0t}^{\alpha} u_0(t) = g_0(t), \tag{4.1}$$

where $g_0(t) = a(t)\, u_0(t) + f_0(t)$. Since ${}_C D_{0t}^{\alpha}\eta(t) = J_{0t}^{1-\alpha}\eta'(t)$, and the equation has not degeneration, we apply to the both side of fractional equation (4.1) the operator $I_{0\,t}^{\alpha}$ and on the interval $(0,T)$ we obtain:

$$\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g_0(s)\, ds = J_{0t}^{\alpha} J_{0t}^{1-\alpha} u_0'(t) = J_{0t}^{1} u_0'(t) =$$

$$= \int_0^t u_0'(s)\, ds = u_0(t) - u_0(0), \quad t \in (0,T).$$

Hence, taking into account the initial value condition (3.7), we have Volterra integral equation

$$u_0(t) = \varphi_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\,\alpha-1} \left[a(s)\, u_0(s) + f_0(s)\right] ds.$$

We rewrite this equation in the convenient form

$$u_0(t) = h_0(t) + \frac{1}{\Gamma(\alpha)} \int_0^t K_0(t,s)\, u_0(s)\, ds, \tag{4.2}$$

where

$$h_0(t) = \varphi_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\,\alpha-1} f_0(s)\, ds, \quad K_0(t,s) = (t-s)^{\,\alpha-1} a(s).$$

Smoothness conditions S_0**.** Let $\varphi_0(x) \in C[0,1], \;\; f_0(t,x) \in C(\overline{\Omega})$ be fulfilled.

Theorem 4.1. *Let the smoothness conditions* S_0 *and condition* A *are fulfilled,* $a(t) \in C[0,T]$ *and* $\rho_0 = \frac{\| a(t) \|_C}{\alpha\, \Gamma(\alpha)} T^{\,\alpha} < 1$.

Then Volterra type integral equation of second kind (4.2) *has a unique solution in the class of continuous functions* $C[0,T]$.

Proof. By virtue of the conditions of the theorem, $h_0(t) \in C[0,T], \;\; K_0(t,s) \in C([0,T] \times [0,T])$. However, according to the principle of contracting mapping, Volterra type integral equation of second kind (4.2) has a unique solution in $C[0,T]$. This solution can be found by the iteration process

$$\left[\begin{array}{l} u_0^{\tau+1}(t) = h_0(t) + \frac{1}{\Gamma(\alpha)} \int_0^t K_0(t,s)\, u_0^{\tau}(s)\, ds, \\ u_0^0(t) = h_0(t), \quad \tau = 0,1,2,... \end{array}\right.$$

Indeed, for this iteration process we have estimates:

$$\left\| u_0^0(t) \right\|_C = \| h_0(t) \|_C \le |\, \varphi_0 \,| + \frac{\| f_0(t) \|_C}{\alpha\, \Gamma(\alpha)} T^{\,\alpha} < \infty;$$

$$\left\| u_0^1(t) - u_0^0(t) \right\|_C \le$$

$$\le \frac{1}{\Gamma(\alpha)} \int_0^t K_0(t,s) \left\| u_0^0(s) \right\|_C ds \le \left[|\, \varphi_0 \,| + \frac{\| f_0(t) \|_C}{\alpha\, \Gamma(\alpha)} T^{\,\alpha} \right] \frac{\| a(t) \|_C}{\alpha\, \Gamma(\alpha)} T^{\,\alpha};$$

$$\left\| u_0^{\tau+1}(t) - u_0^{\tau}(t) \right\|_C \le$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^t K_0(t,s)\, ds \left\| u_0^{\tau}(t) - u_0^{\tau-1}(t) \right\|_C \leq \rho_0 \left\| u_0^{\tau}(t) - u_0^{\tau-1}(t) \right\|_C ,$$

where

$$\rho_0 = \frac{\| a(t) \|_C}{\alpha\, \Gamma(\alpha)} T^{\,\alpha} < 1.$$

The Theorem 4.1 is proved. □

From the representation (4.2) one can find that

$$t^{\,1-\gamma} U_0(t,x) = x\, h_0(t) + \frac{1}{\Gamma(\alpha)} \int_0^t K_0(t,s)\, U_0(s,x)\, ds. \tag{4.3}$$

5. CS of Volterra integral equations

It is well known, that the two-parametric Mittag–Leffler function defined as (see, [37])

$$E_{\alpha,\beta}(z) = \sum_{m=0}^{\infty} \frac{z^{\,m}}{\Gamma(\alpha\, m+\beta)}, \quad z, \alpha, \beta \in \mathbb{C}, \;\; \mathrm{Re}\,(\alpha) > 0.$$

The generalized Mittag–Lefler (Kilbas–Saigo) type function for real $\alpha, m, l \in \mathbb{R}$ and complex $l \in \mathbb{C}$ is defined by Kilbas and Saigo in the following form

$$E_{\alpha,m,l}(z) = \sum_{k=0}^{\infty} c_k z^k, \;\; c_0 < 1, \;\; c_k = \prod_{j=1}^{k-1} \frac{\Gamma(\alpha[jm+l]+1)}{\Gamma(\alpha[jm+l+1]+1)}, \;\; k = 1,2,...$$

These functions belong to the class of entire functions in the complex plane.

Let us consider the CS (3.4) of fractional order differential equation with initial value condition (3.8). We rewrite the equation (3.4) as

$${}_C D_{0t}^{\alpha}\, u_{1,n}(t) = -\mu_{1,n} t^{\beta} u_{1,n}(t) + g_{1,n}(t), \tag{5.1}$$

where

$$g_{1,n}(t) = \frac{1}{1+\lambda_{1,n}} \left[a(t) u_{1,n}(t) + f_{1,n}(t) \right], \quad \mu_{1,n} = \frac{\lambda_{1,n}}{1+\lambda_{1,n}}, \quad \lambda_{1,n} = \left(\frac{2qn\pi}{p+q} \right)^2 .$$

We consider the class of following functions [38, pp. 4, 205]:

$$C_{\gamma}[0;T] = \left\{ g_n(t) : \, t^{\gamma} g_n(t) \in C[0;T] \right\},$$

$$C_{\gamma}^{\alpha}[0;T] = \left\{ g_n(t) \in C[0;T] : {}_C D_{0t}^{\alpha}\, g_n(t) \in C_{\gamma}[0;T] \right\}.$$

Lemma 1. *Let $\gamma \in [0,\alpha]$, $\beta \geq 0$. Then for all $g_{1,n}(t) \in C_{\gamma}[0;T]$ there exists a unique solution $u_{1,n}(t) \in C_{\gamma}^{\alpha}[0;T]$ of the initial value problem (5.1), (3.8). This solution has the following form*

$$u_{1,n}(t) = \varphi_{1,n} E_{\alpha,\, 1+\frac{\beta}{\alpha},\, \frac{\beta}{\alpha}} \left(-\mu_{1,n} t^{\alpha+\beta} \right) + \int_0^t K_1(t,\tau) g_{1,n}(\tau) d\tau, \tag{5.2}$$

where

$$K_j(t,\tau) = \sum_{i=1}^{n} K_{j,i}(t,\tau), \tag{5.3}$$

$$K_{j,0}(t,\tau) = \frac{1}{\Gamma(\alpha)} (t-\tau)^{\alpha-1}, \;\; K_{j,i}(t,\tau) = -\frac{\mu_{j,n}}{\Gamma(\alpha)} \int_{\tau}^{t} s^{\beta} (t-s)^{\alpha-1} K_{j,i-1}(s,\tau) ds, \;\; i = 1,2,..., \tag{5.4}$$

$E_{\alpha,\, 1+\frac{\beta}{\alpha},\, \frac{\beta}{\alpha}} \left(-\mu_{j,n} t^{\alpha+\beta} \right)$ *is Kilbas–Saigo function, $j = 1,2,3$.*

Proof. The uniqueness of the solution $u_{1,n}(t) \in C_\gamma^\alpha[0,T]$ of the problem (5.1), (3.8) was proved in the work [38]. In this work [38], the existence of solution for the case $g_{1,n}(t)=0$ is proved also. So, we consider the inhomogeneous equation (5.1) and the solution of this problem we seek as the sum of two functions

$$u_{1,n}(t) = v_{1,n}(t) + w_{1,n}(t), \tag{5.5}$$

where the functions $v_{1,n}(t)$ and $w_{1,n}(t)$, respectively, are solutions of the following two problems:

$${}_C D_{0t}^\alpha v_{1,n}(t) = -\mu_{1,n} t^\beta v_{1,n}(t), \quad v_{1,n}(0) = \varphi_{1,n}, \tag{5.6}$$

$${}_C D_{0t}^\alpha w_{1,n}(t) = -\mu_{1,n} t^\beta w_{1,n}(t) + g_{1,n}(t), \quad w_{1,n}(0) = 0, \tag{5.7}$$

It is proved in [38, theorem 4.4] that, the problem (5.6) has a unique solution $v_{1,n}(t) \in C_\gamma^\alpha[0,T]$ of the form

$$v_{1,n}(t) = \varphi_{1,n} E_{\alpha,\, 1+\frac{\beta}{\alpha},\, \frac{\beta}{\alpha}} \left(-\mu_{1,n} t^{\alpha+\beta}\right). \tag{5.8}$$

We study the problem (5.7). According to the [38, theorem 3.24, p. 202], the problem is equivalent to the unique solvability of CS of Volterra integral equation of the second kind

$$w_{1,n}(t) = \mu_{1,n} J_{0t}^\alpha \left(t^\beta w_{1,n}(t)\right) + J_{0t}^\alpha g_{1,n}(t), \quad t \in (0,T). \tag{5.9}$$

We apply the method of successive approximations to solve the integral equation (5.9). In this order we consider the following iteration process

$$\begin{cases} w_{1,n}^0(t) = J_{0t}^\alpha g_{1,n}(t), \\ w_{1,n}^k(t) = w_{1,n}^0(t) - \mu_{1,n} J_{0t}^\alpha \left(t^\beta w_{1,n}^{k-1}(t)\right), \ \ k = 1,2,3,... \end{cases}$$

Hence, it is not difficult to obtain that the desired solution of (5.9) has the form

$$w_{1,n}(t) = \int\limits_0^t K_{1,i}(t,\tau) g_{1,n}(\tau) d\tau, \tag{5.10}$$

where the kernel $K_{1,i}(t,\tau)$ defines by the formulas (5.3) and (5.4). According to the formula (5.5), from the representations (5.8) and (5.10) we arrive at (5.2). Lemma 1 is proved. □

Lemma 2. *([39, p. 9)]) For every $\alpha \in [0,1]$, $m > 0$ and $\theta \geq 0$ the following estimate is true:*

$$\frac{1}{1+\Gamma(1-\alpha)\theta} \leq E_{\alpha,m,m-1}(-\theta) \leq \frac{1}{1+\frac{\Gamma(1+\alpha(m-1))}{\Gamma(1+\alpha m)}\theta}.$$

Lemma 3. *([40, p. 136]) Let $\alpha < 2$, $z \in \mathbb{C}$, δ be real constant and σ be fixed number from the interval $\left(\frac{\pi\alpha}{2}, \min\left\{\pi, \pi\alpha\right\}\right)$. Then, for $|\arg z| \leq \sigma$ and $|z| \geq 0$, the following estimate is true*

$$|E_{\alpha,\delta}(z)| \leq M_1 (1+|z|)^{\frac{1-\delta}{\alpha}} e^{\mathrm{Re} z^{\frac{1}{\alpha}}} + \frac{M_2}{1+|z|},$$

where $M_{j,1}$ and $M_{j,2}$ are constants, not depending from z.

Lemma 4. *For the case of $\gamma \in [0,\alpha]$, $\beta \geq 0$ the following estimate*

$$|K_j(t,\tau)| \leq (t-\tau)^{\alpha-1} E_{\alpha,\,\alpha}\left(\mu_{j,n} t^\beta (t-\tau)^\alpha\right) \leq (t-\tau)^{\alpha-1} M_{j,3} \tag{5.11}$$

holds, where $M_{j,3} = \mathrm{const}$, $j = 1,2,3$.

Proof. From the presentations (5.3) and (5.4) we obtain that there hold the following estimates

$$|K_j(t,\tau)| \leq |K_{j,0}(t,\tau)| + |K_{j,1}(t,\tau)| + |K_{j,2}(t,\tau)| + ... + |K_{j,k}(t,\tau)| + ..., \tag{5.12}$$

$$|\, K_{j,0}(t,\tau)\,| \leq \frac{1}{\Gamma(\alpha)} (t-\tau)^{\alpha-1}, \tag{5.13}$$

$$|\, K_{j,1}(t,\tau)\,| \leq \frac{\mu_{j,n}}{\Gamma(\alpha)} \int\limits_\tau^t s^\beta (t-s)^{\alpha-1} \,|\, K_{j,0}(s,\tau)\,|\, ds \leq$$

$$\leq \frac{\mu_{j,n}}{\Gamma^2(\alpha)} \int\limits_{\tau}^{t} s^{\beta} (t-s)^{\alpha-1} (s-\tau)^{\alpha-1} ds \leq \frac{\mu_{j,n} t^{\beta}}{\Gamma^2(\alpha)} \int\limits_{\tau}^{t} (t-s)^{\alpha-1} (s-\tau)^{\alpha-1} ds. \tag{5.14}$$

Using the following known formula

$$\int\limits_{\tau}^{t} (t-s)^{\rho-1} (s-\tau)^{\sigma-1} ds = \frac{\Gamma(\rho)\Gamma(\sigma)}{\Gamma(\rho+\sigma)} (t-\tau)^{\rho+\sigma-1}, \tag{5.15}$$

for $\rho = \sigma = \alpha$ from the estimate (5.14) we derive

$$|\, K_{j,1}(t,\tau) \,| \leq \frac{\mu_{j,n} t^{\beta}}{\Gamma^2(\alpha)} \frac{\Gamma^2(\alpha)}{\Gamma(2\alpha)} (t-\tau)^{2\alpha-1} = \frac{\mu_{j,n}}{\Gamma(2\alpha)} t^{\beta} (t-\tau)^{2\alpha-1}. \tag{5.16}$$

Similarly, we estimate the next kernel $K_{j,2}(t,\tau)$:

$$|\, K_{j,2}(t,\tau) \,| \leq \frac{\mu_{j,n}}{\Gamma(\alpha)} \int\limits_{\tau}^{t} s^{\beta} (t-s)^{\alpha-1} \,|\, K_{j,1}(s,\tau) \,|\, ds \leq$$

$$\leq \frac{\mu_{j,n}^2}{\Gamma^2(\alpha)\Gamma(2\alpha)} \int\limits_{\tau}^{t} s^{2\beta} (t-s)^{\alpha-1} (s-\tau)^{2\alpha-1} ds \leq \frac{\mu_{j,n}^2 t^{2\beta}}{\Gamma(\alpha)\,\Gamma(2\alpha)} \int\limits_{\tau}^{t} (t-s)^{\alpha-1} (s-\tau)^{2\alpha-1} ds.$$

Hence, taking the estimate (5.16) into account, by the aid of formula (5.15) for $\rho = \alpha$, $\sigma = 2\alpha$ we derive

$$|\, K_{j,2}(t,\tau) \,| \leq \frac{\mu_{j,n}^2}{\Gamma(3\alpha)} t^{2\beta} (t-\tau)^{3\alpha-1}, \tag{5.17}$$

and continuing this process, by the induction method one can obtain that

$$|\, K_{j,k}(t,\tau) \,| \leq \frac{\mu_{j,n}^k}{\Gamma((k+1)\alpha)} t^{k\beta} (t-\tau)^{(k+1)\alpha-1}. \tag{5.18}$$

Taking estimates (5.13)–(5.18) into account, for (5.12) we have

$$|\, K_j(t,\tau) \,| \leq \frac{1}{\Gamma(\alpha)} (t-\tau)^{\alpha-1} + \frac{\mu_{j,n}}{\Gamma(2\alpha)} t^{\beta} (t-\tau)^{2\alpha-1} + \frac{\mu_{j,n}^2}{\Gamma(3\alpha)} t^{2\beta} (t-\tau)^{3\alpha-1} +$$

$$+ ... + \frac{\mu_{j,n}^k}{\Gamma((k\alpha+\alpha)} t^{k\beta} (t-\tau)^{(k+1)\alpha-1} + ... \leq (t-\tau)^{\alpha-1} E_{\alpha,\alpha} \left(\mu_{j,n} t^{\beta} (t-\tau)^{\alpha} \right).$$

We put $z = \mu_{j,n} t^{\beta} (t-\tau)^{\alpha}$, $\delta = \alpha$. Since $0 < \mu_{j,n} < 1$, $j = 1, 2, 3$, then we have

$$\left|\, E_{\alpha,\alpha} \left(\mu_{j,n} t^{\beta} (t-\tau)^{\alpha} \right) \right| \leq M_{j,1} \left[1 + \mu_{j,n} t^{\beta} (t-\tau)^{\alpha} \right]^{\frac{1-\alpha}{\alpha}} e^{\left(\mu_{j,n} t^{\beta} (t-\tau)^{\alpha} \right)^{\frac{1}{\alpha}}} +$$

$$+ \frac{M_{j,2}}{1 + \mu_{j,n} t^{\beta} (t-\tau)^{\alpha}} \leq M_{j,3}.$$

From the last two estimates, we obtain the estimate (5.11). The Lemma 4 is proved. □

Similarly to the Lemmas 1–4, as a solution of CS of ordinary differential equations (5.1) with initial value condition (3.8), we rewrite (5.2) as a CS of linear integral equations (CSLIE)

$$u_{1,n}(t) = I_1(t; u_{1,n}(t)) \equiv h_{1,n}(t) + \frac{1}{1+\lambda_{1,n}} \int\limits_{0}^{t} K_1(t,s) a(s)\, u_{1,n}(s)\, ds, \tag{5.19}$$

where

$$h_{1,n}(t) = \varphi_{1,n} E_{\alpha, 1+\frac{\beta}{\alpha}, \, \frac{\beta}{\alpha}} \left(-\mu_{1,n}\, t^{\alpha+\beta} \right) + \frac{1}{1+\lambda_{1,n}} \int\limits_{0}^{t} K_1(t,s) f_{1,n}(s)\, ds.$$

By similar way, as a solution of CS of ordinary differential equations (3.6) with initial value condition (3.10), we study the following CS of linear integral equations (CSLIE)

$$\tilde{u}_{2,m}(t) = \tilde{I}_2(t; \tilde{u}_{2,m}(t)) \equiv \tilde{h}_{2,m}(t) + \frac{1}{1+\lambda_{2,m}} \int\limits_0^t \tilde{K}_2(t,s) a(s)\, \tilde{u}_{2,m}(s)\, ds, \tag{5.20}$$

where

$$\tilde{h}_{2,m}(t) = \tilde{\varphi}_{2,m} E_{\alpha,\, 1+\frac{\beta}{\alpha},\, \frac{\beta}{\alpha}} \left(-\mu_{2,m}\, t^{\,\alpha+\beta}\right) + \frac{1}{1+\lambda_{2,m}} \int\limits_0^t \tilde{K}_2(t,s) \tilde{f}_{2,m}(s)\, ds.$$

CS of ordinary differential equations (3.5) has two unknown variables $u_{2,m}(t)$ and $\tilde{u}_{2,m}(t)$. So, we consider it after solving CS of ordinary differential equations (3.6), i.e. after solving CS of integral equations (5.20).

6. Solvability of CS of integral equations

The CS of linear integral equations (5.19) and (5.20) we consider in the class $B_2(0,T)$ of Banach spaces with norm

$$\left\| \vec{\psi}(t) \right\|_{B_2(0,T)} = \sqrt{\sum_{n=1}^{\infty} \left(\max_{t\in[0,T]} |\, \psi_n(t)\, | \right)^2} < \infty$$

for arbitrary sequences of continuous functions $\{\psi_n(t)\}_{n=1}^{\infty}$. We consider also coordinate Hilbert space ℓ_2 of number sequences $\left\{\zeta_n\right\}_{n=1}^{\infty}$ with a norm

$$\left\| \vec{\zeta} \right\|_{\ell_2} = \sqrt{\sum_{n=1}^{\infty} |\, \zeta_n\, |^2} < \infty.$$

We use also the space $L_2[0,1]$ of square summable functions on an interval $[0,1]$ with a norm

$$\|\, \eta(x)\, \|_{L_2[0,1]} = \sqrt{\int\limits_0^1 |\, \eta(y)\, |^2\, dy} < \infty.$$

Smoothness conditions S_1. Let the functions $\varphi_1(x) \in C^3[0,1]$ and $f_1(t,x) \in C^{0,1}_{t,x}(\Omega)$ have piecewise continuous derivatives with respect to x up to the fourth and second orders, respectively. Then, we integrate by parts $\varphi_{1,n} = \int_0^1 \varphi_1(y)\, \omega_{1,n}(y) dy$ four times and $\int_0^1 f_1(t,x)\, \omega_{1,n}(y)\, dy$ – two times. Then we obtain the following estimates

$$|\, \varphi_{1,n}\, | \le \left(\frac{p+q}{2q\pi}\right)^4 \frac{\left|\, \varphi_{1,n}^{(IV)}\, \right|}{n^4}, \quad |\, f_{1,n}(t)\, | \le \left(\frac{p+q}{2q\pi}\right)^2 \frac{\left|\, f''_{1,n}(t)\, \right|}{n^2}, \tag{6.1}$$

where

$$\varphi_{1,n}^{(IV)} = \int\limits_0^1 \frac{\partial^4 \varphi_1(y)}{\partial\, y^4}\, \omega_{1,n}(y)\, dy, \quad f''_{1,n}(t) = \int\limits_0^1 \frac{\partial^2 f_1(t,y)}{\partial\, y^2}\, \omega_{1,n}(y)\, dy.$$

Theorem 6.1. *Let the condition A and smoothness condition S_1 be fulfilled and*

$$\rho_1 = M_{1,5}\, \|\, a(t)\, \|_C < 1, \quad M_{1,5} = \frac{T^\alpha}{\alpha} \left(\frac{p+q}{2q\pi}\right)^2 \sqrt{\sum_{n=1}^{\infty *} \frac{1}{n^4}}.$$

Then CSLIE (5.19) *has a unique solution in the space $B_2(0,T)$.*

Proof. We use the method of successive approximations as follows:

$$\begin{cases} u_{1,n}^{0}(t) = h_{1,n}(t), \\ u_{1,n}^{\tau+1}(t) = I_1(t; u_{1,n}^{\tau}(t)) \quad \tau = 0,\, 1,\, 2,\, 3,\, \ldots \end{cases} \tag{6.2}$$

Let us estimate the zero approximation $u_{1,n}^{0}(t)$. In the Lemma 2 we put

$$x = -\mu_{1,n}\, t^{\alpha+\beta}, \quad m = 1 + \frac{\beta}{\alpha}, \quad \Gamma(1+\alpha(m-1)) = \Gamma(1+\beta), \quad \Gamma(1+\alpha m) = \Gamma(1+\alpha+\beta).$$

Then we have

$$E_{\alpha,\, 1+\frac{\beta}{\alpha},\, \frac{\beta}{\alpha}}\left(-\mu_{1,n}\, t^{\alpha+\beta}\right) \le \frac{1}{1 + \frac{\Gamma(1+\beta)}{\Gamma(1+\alpha+\beta)}\, \mu_{1,n}\, t^{\alpha+\beta}} \le 1.$$

We estimate the zero approximation. By virtue of formulas in (6.1) and the fact that $\mu_{1,n} = \frac{\lambda_{1,n}}{1+\lambda_{1,n}} < 1$, $\lambda_{1,n} = \left(\frac{2qn\pi}{p+q}\right)^2$, applying the Cauchy–Shwartz inequality and Bessel inequality, from approximation (6.2) we obtain

$$\| \vec{u}_1^{0}(t) \|_{B_2(0,T)} \le \sqrt{\sum_{n=1}^{\infty *} \max_{0\le t\le T} | u_{1,n}^{0}(t) |^2} \le \sum_{n=1}^{\infty *} \max_{0\le t\le T} | h_{1,n}(t) | \le$$

$$\le \sum_{n=1}^{\infty *} \max_{0\le t\le T} \left| \varphi_{1,n} E_{\alpha,\, 1+\frac{\beta}{\alpha},\, \frac{\beta}{\alpha}}\left(-\mu_{1,n}\, t^{\alpha+\beta}\right) \right| + \left(\frac{p+q}{2q\pi}\right)^2 \sum_{n=1}^{\infty *} \frac{1}{n^2} \max_{0\le t\le T} \int_0^t K_{1,n}(t,s)\, | f_{1,n}(s) |\, ds \le$$

$$\le \left(\frac{p+q}{2q\pi}\right)^4 \sum_{n=1}^{\infty *} \frac{1}{n^4} \left| \varphi_{1,n}^{(IV)} \right| + M_{1,3} \left(\frac{p+q}{2q\pi}\right)^2 \sum_{n=1}^{\infty *} \frac{1}{n^2} \max_{0\le t\le T} | f_{1,n}(t) | \int_0^t (t-s)^{\alpha-1} ds \le$$

$$\le M_{1,4} \left\| \vec{\varphi}_1^{(IV)} \right\|_{\ell_2} + M_{1,5} \left\| \vec{f}_1(t) \right\|_{B_2(0,T)} \le$$

$$\le M_{1,4} \left\| \frac{\partial^4 \varphi_1(x)}{\partial x^4} \right\|_{L_2[0,1]} + M_{1,5} \max_{0\le t\le T} \| f_1(t,x) \|_{L_2[0,1]} = \delta_1 < \infty, \tag{6.3}$$

where

$$M_{1,4} = \left(\frac{p+q}{2q\pi}\right)^4 \sqrt{\sum_{n=1}^{\infty *} \frac{1}{n^8}}, \quad M_{1,5} = \frac{T^\alpha}{\alpha} \left(\frac{p+q}{2q\pi}\right)^2 \sqrt{\sum_{n=1}^{\infty *} \frac{1}{n^4}}.$$

Due to the conditions of the Theorem 6.1, estimate (6.3) and applying the Cauchy–Shwartz inequality, Bessel's inequality, for the first difference $u_{1,n}^{1}(t) - u_{1,n}^{0}(t)$ we obtain

$$\| \vec{u}_1^{1}(t) - \vec{u}_1^{0}(t) \|_{B_2(0,T)} \le \left(\frac{p+q}{2q\pi}\right)^2 \sum_{n=1}^{\infty *} \frac{1}{n^2} \max_{0\le t\le T} \int_0^t K_{1,n}(t,s) \left| a(s)\, u_{1,n}^{0}(s) \right| ds \le$$

$$\le \|a(t)\|_C \frac{T^\alpha}{\alpha} \left(\frac{p+q}{2q\pi}\right)^2 \sum_{n=1}^{\infty} \frac{1}{n^2} \max_{0\le t\le T} | u_{1,n}^{0}(t) | \le M_{1,6} \left\| u_1^{0}(t) \right\|_{B_2(0,T)} \le M_{1,6}\delta_1, \tag{6.4}$$

where $M_{1,6} = M_{1,5} \| a(t) \|_C$.

Now we consider the second difference $u_{1,n}^{2}(t) - u_{1,n}^{1}(t)$ and, taking the estimate (6.4) into account, we obtain

$$\| \vec{u}_1^{2}(t) - \vec{u}_1^{1}(t) \|_{B_2(0,T)} \le \left(\frac{p+q}{2q\pi}\right)^2 \sum_{n=1}^{\infty *} \frac{1}{n^2} \max_{0\le t\le T} \int_0^t K_{1,n}(t,s)\, | a(s) | \cdot \left| u_{1,n}^{1}(s) - u_{1,n}^{0}(s) \right| ds \le$$

$$\le \|a(t)\|_C \frac{T^\alpha}{\alpha} \left(\frac{p+q}{2q\pi}\right)^2 \sum_{n=1}^{\infty} \frac{1}{n^2} \max_{0\le t\le T} \left| u_{1,n}^{1}(t) - u_{1,n}^{0}(t) \right| \le$$

$$\le M_{1,6} \left\| \vec{u}_1^{1}(t) - \vec{u}_1^{0}(t) \right\|_{B_2(0,T)} \le \left[M_{1,6}\right]^2 \delta_1. \tag{6.5}$$

For an arbitrary natural number τ, similarly to (6.5) we obtain

$$\left\| \vec{u}_1^{\tau+1}(t) - \vec{u}_1^{\tau}(t) \right\|_{B_2(0,T)} \leq \left[\rho_1\right]^{\tau+1} \delta_1, \tag{6.6}$$

where $\rho_1 = M_{1,6} < 1$. Then, passing to the limit as $\tau \to \infty$ in inequality (6.6), we obtain

$$\lim_{\tau \to \infty} \left\| \vec{u}_1^{\tau+1}(t) - \vec{u}_1^{\tau}(t) \right\|_{B_2(0,T)} = 0. \tag{6.7}$$

From estimates (6.3), (6.6) and limit (6.7) it follows the existence and uniqueness of the solution $\vec{u}_1(t) \in B_2(0,T)$ to CSLIE (5.19). $\square$

Smoothness conditions $\tilde{S}_2$. Let the functions $\tilde{\varphi}_2(x) \in C^3[0,1]$ and $\tilde{f}_2(t,x) \in C^{0,1}_{t,x}(\Omega)$ have piecewise continuous derivatives with respect to x up to the fourth and second orders, respectively. Then, we integrate by parts $\tilde{\varphi}_{2,n} = \int_0^1 \tilde{\varphi}_2(y)\,\omega_{2,n}(y)dy$ four times and $\int_0^1 \tilde{f}_2(t,x)\,\omega_{2,n}(y)\,dy$ – two times. Then we obtain the following estimates

$$|\,\tilde{\varphi}_{2,m}\,| \leq \left(\frac{1}{2q\pi}\right)^4 \frac{\left|\tilde{\varphi}_{2,m}^{(IV)}\right|}{m^4}, \quad \left|\tilde{f}_{2,m}(t)\right| \leq \left(\frac{1}{2q\pi}\right)^2 \frac{\left|\tilde{f}''_{2,m}(t)\right|}{m^2},$$

where

$$\tilde{\varphi}_{2,m}^{(IV)} = \int_0^1 \frac{\partial^4 \tilde{\varphi}_2(y)}{\partial y^4}\,\omega_{2,m}(y)\,dy, \quad \tilde{f}''_{2,m}(t) = \int_0^1 \frac{\partial^2 \tilde{f}_2(t,y)}{\partial y^2}\,\omega_{2,m}(y)\,dy.$$

Theorem 6.2. *Let the condition A and smoothness condition $\tilde{S}_2$ be fulfilled and*

$$\tilde{\rho}_2 = \tilde{M}_{2,1} \left\| a(t) \right\|_C < 1, \quad \tilde{M}_{2,1} = \frac{T^\alpha}{\alpha} \left(\frac{1}{2q\pi}\right)^2 \sqrt{\sum_{m=1}^{\infty} \frac{1}{m^4}}.$$

Then CSLIE (5.20) has a unique solution in the space $B_2(0,T)$.

The existence and uniqueness theorem for CS of linear integral equations (5.20) is proved by similar way as in the case of the Theorem 6.1.

The solution of the CSLIE (5.20) we denote by $\tilde{F}_{2,m}(t)$ and substitute it into equation (3.5):

$$D_{0t}^\alpha u_{2,m}(t) + \mu_{2,m} u_{2,m}(t) = \frac{1}{1+\lambda_{2,m}} \left[a(t)u_{2,m}(t) + f_{2,m}(t)\right] -$$

$$-\frac{2\sqrt{\lambda_{2,m}}}{1+\lambda_{2,m}} \left(D_{0t}^\alpha \tilde{F}_{2,m}(t) + \tilde{F}_{2,m}(t)\right). \tag{6.8}$$

The equation (6.8) with initial value condition (3.9) is equivalent to the following CSLIE

$$u_{2,m}(t) = I_2(t; u_{2,m}(t)) \equiv h_{2,m}(t) + \frac{1}{1+\lambda_{2,m}} \int_0^t K_2(t,s) a(s)\, u_{2,m}(s)\, ds, \tag{6.9}$$

where

$$h_{2,m}(t) = \varphi_{2,m} E_{\alpha, 1+\frac{\beta}{\alpha}, \frac{\beta}{\alpha}} \left(-\mu_{2,m}\, t^{\alpha+\beta}\right) + \frac{1}{1+\lambda_{2,m}} \int_0^t K_2(t,s) f_{2,m}(s)\, ds -$$

$$-\frac{2\sqrt{\lambda_{2,m}}}{1+\lambda_{2,m}} \int_0^t K_2(t,s) \left({}_C D_{0s}^\alpha \tilde{F}_{2,m}(s) + \tilde{F}_{2,m}(s)\right) ds,$$

$$\mu_{2,m} = \frac{\lambda_{2,m}}{1+\lambda_{2,m}}, \quad \lambda_{2,m} = (2qm\pi)^2, \ \ m = 1,\, 2,\, \ldots$$

Smoothness condition S_2. Let in the domain $[0,1]$ the functions $\varphi_2(x) \in C^4[0,1]$, $f_2(t,x) \in C^{0,2}_{t,x}(\Omega)$ have the continuous derivative with respect to x up to the fourth order and second order, respectively. Then, by integrations by parts

$$\varphi_{2,m} = \int\limits_0^1 \varphi_2(y)\,\tilde{\omega}_{2,m}(y)dy, \quad f_{2,m}(t) = \int\limits_0^1 f_2(t,y)\,\tilde{\omega}_{2,m}(y)\,dy$$

we obtain the results

$$|\,\varphi_{2,m}\,| \le \left(\frac{1}{2q\pi}\right)^4 \frac{\left|\,\varphi_{2,m}^{(IV)}\,\right|}{m^4} + 4\left(\frac{1}{2q\pi}\right)^5 \frac{\left|\,\tilde{\varphi}_{2,m}^{(V)}\,\right|}{m^5}, \tag{6.10}$$

$$|\,f_{2,m}(t)\,| \le \left(\frac{1}{2q\pi}\right)^2 \frac{\left|\,f''_{2,m}(t)\,\right|}{m^2} + 4\left(\frac{1}{2q\pi}\right)^3 \frac{\left|\,\tilde{f}''_{2,m}(t)\right|}{m^3}, \tag{6.11}$$

where

$$\varphi_{2,m}^{(IV)} = \int\limits_0^1 \frac{\partial^4\,\varphi_2(y)}{\partial\,y^4}\,\tilde{\omega}_{2,m}(y)\,dy, \quad f''_{2,m}(t) = \int\limits_0^1 \frac{\partial^2 f_2(t,y)}{\partial\,y^2}\,\tilde{\omega}_{2,m}(y)\,dy.$$

Theorem 6.3. *Let the condition A and smoothness condition S_2 be fulfilled and*

$$\rho_2 = M_{2,8} < 1, \quad M_{2,8} = \frac{T^\alpha}{\alpha}\left(\frac{1}{2q\pi}\right)^2 \sqrt{\sum_{m=1}^{\infty}\frac{1}{m^4}}\,\|\,a(t)\,\|_C\,.$$

Then CSLIE (6.9) has a unique solution in the space $B_2(0,T)$.

Proof. We use the method of successive approximations as:

$$\begin{cases} u^0_{2,m}(t) = h_{2,m}(t), \\ u^{\tau+1}_{2,m}(t) = I_2(t;u^\tau_{2,m}(t)) \quad \tau = 0,\,1,\,2,\,3,\,\ldots \end{cases} \tag{6.12}$$

We estimate the zero approximation. By virtue of formulas (6.10) and applying the Cauchy–Shwartz inequality and Bessel inequality, from approximation (6.12) we

$$\|\,\vec{u}^0_2(t)\,\|_{B_2(0,T)} \le \sum_{m=1}^{\infty}\max_{0\le t\le T}|\,h_{2,m}(t)\,| \le$$

$$\le \left(\frac{1}{2q\pi}\right)^4 \sum_{m=1}^{\infty}\frac{1}{m^4}\left|\,\varphi_{2,m}^{(IV)}\,\right| + 4\left(\frac{1}{2q\pi}\right)^5 \sum_{m=1}^{\infty}\frac{1}{m^5}\left|\,\tilde{\varphi}_{2,m}^{(V)}\,\right| +$$

$$+\max_{0\le t\le T}\left[\sum_{m=1}^{\infty}\frac{2}{\sqrt{\lambda_{2,m}}}\left|\,{}_C D^\alpha_{0t}\tilde{F}_{2,m}(t)\,\right| + \sum_{m=1}^{\infty}\frac{2}{\sqrt{\lambda_{2,m}}}\left|\,\tilde{F}_{2,m}(t)\,\right|\right]\int\limits_0^t K_2(t,s)ds +$$

$$+M_{2,3}\left(\frac{1}{2q\pi}\right)^2 \sum_{m=1}^{\infty}\frac{1}{m^2}\max_{0\le t\le T}|\,f_{2,m}(t)\,|\int\limits_0^t (t-s)^{\alpha-1}ds \le$$

$$\le M_{2,4}\left\|\,\vec{\varphi}_2^{(IV)}\,\right\|_{\ell_2} + M_{2,5}\left\|\,\vec{\tilde{\varphi}}_2^{(V)}\,\right\|_{\ell_2} + M_{2,6}\left[\left\|\,{}_C D^\alpha_{0t}\vec{\tilde{F}}_2(t)\,\right\|_{B_2(0,T)} + \left\|\,\vec{\tilde{F}}_2(t)\,\right\|_{B_2(0,T)}\right] +$$

$$+M_{2,6}\left\|\,\vec{f}_2(t)\,\right\|_{B_2(0,T)} \le M_{2,4}\left\|\,\frac{\partial^4\,\varphi_2(x)}{\partial\,x^4}\,\right\|_{L_2[0,1]} + M_{2,5}\left\|\,\frac{\partial^5\,\tilde{\varphi}_2(x)}{\partial\,x^5}\,\right\|_{L_2[0,1]} +$$

$$+M_{2,6}\max_{0\le t\le T}\left[\left\|\,{}_C D^\alpha_{0t}\tilde{F}_2(t,x)\,\right\|_{L_2[0,1]} + \left\|\,\tilde{F}_2(t,x)\,\right\|_{L_2[0,1]}\right] +$$

$$+M_{2,7}\max_{0\le t\le T}\|\,f_2(t,x)\,\|_{L_2[0,1]} = \delta_2 < \infty, \tag{6.13}$$

where

$$M_{2,4}=\left(\frac{1}{2q\pi}\right)^4\sqrt{\sum_{m=1}^{\infty}\frac{1}{m^8}},\quad M_{2,5}=4\left(\frac{1}{2q\pi}\right)^5\sqrt{\sum_{m=1}^{\infty}\frac{1}{m^{10}}},$$

$$M_{2,6}=\frac{T^\alpha}{\alpha q\pi}\sqrt{\sum_{m=1}^{\infty}\frac{1}{m^2}},\quad M_{2,7}=\frac{T^\alpha}{\alpha}\left(\frac{1}{2q\pi}\right)^2\sqrt{\sum_{m=1}^{\infty}\frac{1}{m^4}}.$$

For the first difference $u_{2,m}^1(t)-u_{2,m}^0(t)$ we obtain

$$\left\|\,\vec{u}_2^1(t)-\vec{u}_2^0(t)\,\right\|_{B_2(0,T)}\leq$$

$$\leq\|a(t)\|_C\,\frac{T^\alpha}{\alpha}\left(\frac{1}{2q\pi}\right)^2\sum_{m=1}^{\infty}\frac{1}{m^2}\max_{0\leq t\leq T}\left|\,u_{2,n}^0(t)\,\right|\leq M_{2,8}\left\|\,u_2^0(t)\,\right\|_{B_2(0,T)}\leq M_{2,8}\delta_2,$$

where

$$M_{2,8}=\|a(t)\|_C\,\frac{T^\alpha}{\alpha}\left(\frac{1}{2q\pi}\right)^2.$$

Similarly, for an arbitrary natural number τ we obtain

$$\left\|\,\vec{u}_2^{\tau+1}(t)-\vec{u}_2^{\tau}(t)\,\right\|_{B_2(0,T)}\leq\left[\rho_2\right]^{\tau+1}\delta_2, \tag{6.14}$$

where $\rho_2=M_{2,8}<1$. Then, passing to the limit as $\tau\to\infty$ in inequality (6.14), we obtain

$$\lim_{\tau\to\infty}\left\|\,\vec{u}_2^{\tau+1}(t)-\vec{u}_2^{\tau}(t)\,\right\|_{B_2(0,T)}=0. \tag{6.15}$$

From estimates (6.13), (6.14) and limit (6.15) follows the existence and uniqueness of the solution $\vec{u}_2(t)\in B_2(0,T)$ to CSLIE (6.9). □

7. Mixed problem

Since the solution of the mixed problem (1.1)–(1.4) we look at (3.1), then for the function (3.2) from (4.3), (5.19), (5.20) and (6.9) we have

$$U(t,x)=h_0(t)\,\vartheta_0(x)+\frac{1}{\Gamma(\gamma)}\,\vartheta_0(x)\int_0^t K_0(t,s)\,u_0(s)\,ds+$$

$$+\sum_{n=1}^{\infty *}\vartheta_{1,n}(x)\left[h_{1,n}(t)+\frac{1}{1+\lambda_{1,n}}\int_0^t K_1(t,s)\,a(s)\,u_{1,n}(s)\,ds\right]+$$

$$+\sum_{m=1}^{\infty}\vartheta_{2,m}(x)\left[h_{2,m}(t)+\frac{1}{1+\lambda_{2,m}}\int_0^t K_2(t,s)\,a(s)\,u_{2,m}(s)\,ds\right]+$$

$$+\sum_{m=1}^{\infty}\tilde{\vartheta}_{2,m}(x)\tilde{F}_{2,m}(t). \tag{7.1}$$

The Fourier series (7.1) we consider as a formal solution of the problem (1.1)–(1.4).

We show the absolute and uniform convergence of the series (7.1) as a solution to the mixed problem (1.1)–(1.4).

Theorem 7.1. *Let the conditions of Theorems 4.1–6.3 be satisfied. Then function* (7.1) *will be the unique solution to the mixed problem* (1.1)–(1.3) *and this solution belongs to the class* (1.4).

Proof. From the theorems 4.1–6.3 we have that $u_0(t) \in C_2[0,T]$ and $\vec{u}_1(t)$, $\vec{u}_2(t)$, $\vec{\tilde{u}}_2(t) \in B_2(0,T)$. We will prove the convergence of the function (7.1). Using the proofs of the theorems 4.1–6.3, we obtain

$$|\, U(t,x)\,| \leq |\, \varphi_0\,| + \frac{\|\, f_0(t)\,\|_C}{\alpha\,\Gamma(\alpha)} T^\alpha + \|\, u_0(t)\,\|_C \,\frac{\|\, a(t)\,\|_C}{\alpha\,\Gamma(\alpha)} T^\alpha + \delta_1 + \rho_1 \,\|\, \vec{u}_1(t)\,\|_{B_2(0,T)} +$$

$$+\delta_2 + \rho_2 \,\|\, \vec{u}_2(t)\,\|_{B_2(0,T)} + \left\| \vec{\tilde{F}}_2(t) \right\|_{B_2(0,T)}.$$

Taking into account the smoothness conditions, as well as condition (6.11) in particular, we see that the solution to the mixed problem (1.1)–(1.3) belongs to class (1.4). The theorem is proved. □

8. Conclusion

In rectangle $\Omega = \{(t,x)|\, 0 < t < T,\ 0 < x < l\}$ domain the questions of unique solvability to the mixed problem (1.1)–(1.4) for a partial differential equation with degeneration is considered in the case of $0 < \alpha \leq 1$ order Gerasimov–Caputo type fractional operator. The solution of partial differential equation is studied in the class of regular functions. The equation (1.1) depends from two independent arguments. First argument is time argument and with respect to this argument the equation (1.1) is fractional Gerasimov–Caputo type ordinary differential equation with degeneration. Second argument is spatial and the equation (1.1) with respect to one is differential equation of second order. With respect to spatial argument is obtained spectral problem (2.1), (2.2) to define eigenvalues, eigenfunctions and associated functions. This spectral problem is solved when the condition A is fulfilled:
Let x_0 be a rational number from the interval $(0,1)$ such that $x_0 = \frac{p}{q},\ p < q,\ q - p = 1,\ p$ and q be positive integers.
Adjoint spectral problem (2.7)–(2.9) also is solved. Associated functions are found. From the spectral problems (2.1), (2.2) and (2.7)–(2.9) we obtain that the systems of functions (2.4), (2.6) and (2.10), (2.11) are biorthonormal in $L_2[0,1]$. Moreover, if the condition A is satisfying, then the systems of root functions of problems (2.1), (2.2) and (2.7)–(2.9) form a Riesz basis in $L_2[0,1]$. So, the Fourier series method is applied and some countable systems of linear differential equations are obtained. When conditions of smoothness are fulfilled, then by the aid of the Cauchy–Schwarz inequality and the Bessel inequality is proved the absolute and uniform convergence of the obtained Fourier series.

This work is of a theoretical nature and develops the theory of differential equations with different spectral conditions and fractional operators. In the future, we intend to continue our research in the direction of superposition of several fractional order operators.

References

1. A. N. Gerasimov, Generalization of laws of the linear deformation and their application to problems of the internal friction. *Prikl. Mat. Mekh.* **12** (1948), 251–260.
2. E. J. Ibrahimov, G. A. Dadashova, S. Ar. Jafarova. Boundedness of higher order commutators of G-fractional integral and G-fractional maximal operators with G-BMO functions. *Transactions of A. Razmadze Mathematical Institute* **174(3)** (2020), 325–341.
3. F. Mainardi, *Fractional calculus: some basic problems in continuum and statistical mechanics*. In: Carpinteri, A., Mainardi, F. (eds.) Fractals and Fractional Calculus in Continuum Mechanics. Springer, Wien, 1997.
4. A. Meskhi, Boundness weighted criteria for multilinear Riemann–Liouville integral operators. *Transactions of A. Razmadze Mathematical Institute* **177(1)** (2023), 147–148.
5. O. G. Novozhenova, Life and science of Alexey Gerasimov, one of the pioneers of fractional calculus in Soviet Union. *Fract. Calc. Appl. Anal.* **20** (2017), 790–809.
6. Y. A. Rossikhin, Reflections on two parallel ways in the progress of fractional calculus in mechanics of solids. *Appl.Mech. Rev.* **63(010701)** (2010).
7. S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional Integrals and Derivatives.* Theory and Applications. Gordon and Breach, Yverdon, 1993.
8. Handbook of Fractional Calculus with Applications. Vols. 1-8. Tenreiro Machado J. A. (ed.). Walter de Gruyter GmbH, Berlin, Boston, 2019.
9. D. Kumar, D. Baleanu, Editorial: fractional calculus and its applications in physics. *Front. Phys.*, **7(6)** (2019) https://doi.org/10.3389/fphy.2019.00081

10. H. Sun, A. Chang, Y. Zhang, W. Chen, A review on variable-order fractional differential equations: mathematical foundations, physical models, numerical methods and applications. *Fract. Calc. Appl. Anal.* **22(1)** (2019), 27–59. DOI: 10.1515/fca-2019-0003
11. R. Hilfer, *Application of Fractional Calculus in Physics*. World Scientific Publishing Company, Singapore, 2000.
12. S. Patnaik, J. P. Hollkamp, F. Semperlotti, Applications of variable-order fractional operators: a review. *Proc. A.* **476 (2234)** (2020), 1–32. https://doi.org/10.1098/rspa.2019.0498
13. T. K. Yuldashev, E. T. Karimov, Inverse problem for a mixed type integro-differential equation with fractional order Caputo operators and spectral parameters. *Axioms* **9(4)** (2020) 1–24. https://doi.org/10.3390/axioms9040121
14. M. Kirane, M. A. Sadybekov, A. A. Sarsenbi, On an inverse problem of reconstructing a subdiffusion process from nonlocal data. *Mathematical Methods in the Applied Sciences* **42(6)** (2019), 2043–2052.
15. B. Kh. Turmetov, B. J. Kadirkulov, On a problem for nonlocal mixed-type fractional order equation with degeneration. *Chaos, Solitons and Fractals* **146** (2021), 1–5.
16. A. S. Berdyshev, J. B. Kadirkulov, On a nonlocal problem for a fourth-order parabolic equation with the fractional Dzhrbashyan–Nersesyan operator. *Differ. Equations* **52(1)** (2016), 122–127. https://doi.org/10.1134/S0012266116010109
17. M. Kh. Beshtokov, To boundary-value problems for degenerating pseudoparabolic equations with Gerasimov–Caputo fractional derivative. *Russian Mathematics (Izv. VUZ)* **62(10)** (2018), 3–16. https://doi.org/10.3103/S1066369X18100018
18. A. Danelia, On some local properties of the congugate function and the modulus of smoothness of fractional order. *Transactions of A. Razmadze Mathematical Institute* **175(3)** (2021), 337–343.
19. P. N. Duc, H. D. Binh, L. D. Long, H. T. Kim Van, Reconstructing the right-hand side of the Rayleigh–Stokes problem with nonlocal in time condition. *Advances in Difference Equations* **470** (2021), 1–18. https://doi.org/10.1186/s13662-021-03626-z
20. V. E. Fedorov, A. V. Nagumanova, Inverse linear problems for a certain class of degenerate fractional evolution equations. Itogi nauki i tekhniki. (in Russian) Seriya Sovremennaya matematika i ikh prilozheniya. Tematicheskiye obzory **167** (2019), 97–111. https://doi.org/10.36535/0233-6723-2019-167-97-111
21. A. Guezane-Lakoud, R. Khaldi, Positive solution to a higher order fractional boundary value problem with fractional integral condition. *Romanian Journal of Mathematics and Computer Science* textbf2 (2012), 41–54.
22. S. Kerbal, B. J. Kadirkulov, M. Kirane, Direct and inverse problems for a Samarskii–Ionkin type problem for a two dimensional fractional parabolic equation. *Progr. Fract. Differ. Appl.* **4(3)** (2018), 1–14. http://dx.doi.org/10.18576/pfda/01010
23. Y. Liu, Zh. Li, M. Yamamoto, Inverse problems of determining sources of the fractional partial differential equations. *In Handbook of Fractional Calculus with Applications* **2** (2019), J.A.T. Marchado Ed. DeGruyter, Berlin, 411–429.
24. S. M. Ege and E. Misirli, Traveling wave solutions of the some fractional differential equations. *Romanian Journal of Mathematics and Computer Science* textbf6(1) (2016), 106–115.
25. S. A. Malik, S. Aziz, An inverse source problem for a two parameter anomalous diffusion equation with nonlocal boundary conditions. *Comput. Math. Appl.* **73(12)** (2017), 2548–2560. https://doi.org/10.1016/j.camwa.2017.03.019
26. D. Serikbaev, N. Tokmagambetov, A source inverse problem for the pseudo-parabolic equation with the fractional Sturm-Liouville operator. *Bulletin of the Karaganda university. Mathematics series* **4(99)** (2020), 143–151. https://doi.org/10.31489/2020M4/143-151
27. N. K. Ochilova, T. K. Yuldashev, On a nonlocal boundary value problem for a degenerate parabolic-hyperbolic equation with fractional derivative. *Lobachevskii Journal of Mathematics* **43(1)** (2022), 229–236. https://doi.org/10.1134/S1995080222040175
28. A. V. Pskhu, *Fractional Partial Differential Equations*. Nauka, Moscow, 2005. https://www.elibrary.ru/item.asp?id=19448655 [in Russian].
29. M. Ruzhansky, N. Tokmagambetov, B. T. Torebek, Inverse source problems for positive operators. I: Hypoelliptic diffusion and subdiffusion equations. *J. Inverse Ill-Possed Probl.* **27** (2019), 891–911.
30. T. K. Yuldashev, O. Kh. Abdullaev, Unique solvability of a boundary value problem for a loaded fractional parabolic-hyperbolic equation with nonlinear terms. *Lobachevskii Journal of Mathematics* **42(5)** (2021), 1113–1123 , https://doi.org/10.1134/S1995080221050218
31. T. K. Yuldashev, B. J. Kadirkulov, Inverse boundary value problem for a fractional differential equations of mixed type with integral redefinition conditions. *Lobachevskii Journal of Mathematics* **42(3)** (2021), 649–662. https://doi.org/10.1134/S1995080221030227
32. T. K. Yuldashev, B. J. Kadirkulov, Nonlocal problem for a mixed type fourth-order differential equation with Hilfer fractional operator. *Ural Math. J.* **6(1)** (2020), 153–167.
33. T. K. Yuldashev, B. J. Kadirkulov, Inverse problem for a partial differential equation with Gerasimov–Caputo-type operator and degeneration. *Fractal Fract* **5(2)** (2021), ID 58, 1–13. https://doi.org/10.3390/fractalfract5020058
34. T. K. Yuldashev and B. J. Kadirkulov, On a boundary value problem for a mixed type fractional differential equations with parameters. *Proceedings of the Institute of Mathematics and Mechanics, National Academy of Sciences of Azerbaijan* **47(1)** (2021), 112–123.

35. T. K. Yuldashev, B. J. Kadirkulov, R. A. Bandaliyev, On a Mixed Problem for Hilfer type fractional differential equation with degeneration. *Lobachevskii Journal of Mathematics* **43(1)** (2022), 263–274. https://doi.org/10.1134/S1995080222040229
36. R. Ashurov, B. Kadirkulov and M. Jalilov, On an inverse problem of the Bitsadze–Samarskii type for a parabolic equation of fractional order. *Boletín de la Sociedad Matemática Mexicana* **29(3)** (2023) 1–21. https://doi.org/10.1007/s40590-023-00542-y
37. R. Gorenflo, A. A. Kilbas, F. Mainardi, S. V. Rogosin, *Mittag–Leffler Functions, Related Topics and Applications*. Springer Monographs in Mathematics. Springer-Verlag, Berlin Heidelberg, 2014. https://doi.org/10.1007/978-3-662-43930-2
38. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*. North-Holland Mathematics Studies, 204. Elsevier Science B. V., Amsterdam, 2006. xvi+523 pp.
39. L. Boudabsa, T. Simon, Some properties of the Kilbas–Saigo Function. *Mathematics* **9(217)** (2021). https://doi.org/10.3390/math9030217
40. M. M. Dzhrbashian, *Integral Transformation and Representation of Functions in Complex Domain*. Moscow, 1966, p. 672.

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