

**SOME NEW $(H_p - L_p)$ TYPE INEQUALITIES FOR
WEIGHTED MAXIMAL OPERATORS OF FEJÉR MEANS
OF WALSH-FOURIER SERIES**

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ABSTRACT. In this paper we introduce some new weighted maximal operators of the Fejér means of the Walsh-Fourier series. We prove that for some "optimal" weights, $(H_p - L_p)$ type inequalities hold for these operators when $0 < p < 1/2$. We also prove sharpness of this result. As a consequence we obtain some new and well-known results.

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1. INTRODUCTION

All symbols used in this introduction can be found in Section 2.

The weak $(1, 1)$ -type inequality for the maximal operator σ^* of Fejér means $\{\sigma_n\}$ with respect to the Walsh system

$$\sigma^* f := \sup_{n \in \mathbb{N}} |\sigma_n f|$$

can be found in Schipp [20] and Pál, Simon [14] (see also [2] and [17]). Fujii [5] and Simon [22] proved that σ^* is bounded from H_1 to L_1 . Weisz [29] generalized this result and proved boundedness of σ^* from the martingale space H_p to the Lebesgue space L_p for $p > 1/2$. Simon [21] gave a counterexample, which shows that boundedness does not hold for $0 < p < 1/2$. A counterexample for $p = 1/2$ was given by Goginava [7]. Moreover, in [8] (see also [23]) he proved that there exists a martingale $F \in H_p$ ($0 < p \leq 1/2$), such that

$$\sup_{n \in \mathbb{N}} \|\sigma_n F\|_p = +\infty.$$

Weisz [32] proved that the maximal operator σ^* of the Fejér means is bounded from the Hardy space $H_{1/2}$ to the space *weak* $-L_{1/2}$.

Concerning convergence and summability of Fejér means of Walsh-Fourier series we refer to [4], [12], [13], [15], [26, 27], [29].

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For $0 < p < 1/2$ in [25] the weighted maximal operator $\tilde{\sigma}^{*,p}$, defined by

$$(1) \quad \tilde{\sigma}^{*,p} F := \sup_{n \in \mathbb{N}} \frac{|\sigma_n F|}{(n+1)^{1/p-2}},$$

was investigated and it was proved that the following inequality holds:

$$(2) \quad \left\| \tilde{\sigma}^* F \right\|_p \leq c_p \|F\|_{H_p},$$

where c_p is an absolute constant depending only on p . Moreover, it was proved that the rate of the sequence $(n+1)^{1/p-2}$ given in the denominator of (1) can not be improved. In the case $p = 1/2$ analogical results for the maximal operator $\tilde{\sigma}^*$, defined by

$$\tilde{\sigma}^* F := \sup_{n \in \mathbb{N}} \frac{|\sigma_n F|}{\log^2(n+1)},$$

was proved in [24].

To study convergence of subsequences of Fejér means and their restricted maximal operators on the martingale Hardy spaces H_p for $0 < p \leq 1/2$, the central role is played by the fact that any natural number $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{k=0}^{\infty} n_k 2^k$, $n_k \in Z_2$ ($k \in \mathbb{N}$), where only a finite numbers of n_k differ from zero and their important characters $[n]$, $|n|$, $\rho(n)$ and $V(n)$ are respectively defined by

$$[n] := \min\{j \in \mathbb{N}, n_j \neq 0\}, \quad |n| := \max\{j \in \mathbb{N}, n_j \neq 0\}, \quad \rho(n) := |n| - [n],$$

$$V(n) := n_0 + \sum_{k=1}^{\infty} |n_k - n_{k-1}|, \quad \text{for all } n \in \mathbb{N}.$$

Moreover, for any $\{n_{s_j}\}, j = 1, 2, \dots, r$, satisfying

$$2^s \leq n_{s_1} \leq n_{s_2} \leq \dots \leq n_{s_r} < 2^{s+1}, \quad s \in \mathbb{N},$$

we define numbers

$$(3) \quad s_- := \min\{[n_{s_j}]\}, \quad s_+ := \max\{|n_{s_j}|\} = s \text{ and } \rho_s(n_{s_j}) := s_+ - s_-.$$

Weisz [31] (see also [30]) proved that for any $F \in H_p$ ($p > 0$), the maximal operator $\sup_{n \in \mathbb{N}} |\sigma_{2^n} F|$ is bounded from the Hardy space H_p to the Lebesgue space L_p . Persson and Tephnadze [16] generalized this result and proved that if $0 < p \leq 1/2$ and $\{n_k : k \geq 0\}$ is a sequence of positive integers, such that

$$(4) \quad \sup_{k \in \mathbb{N}} \rho(n_k) \leq c < \infty,$$

then the maximal operator $\tilde{\sigma}^{*,\nabla}$, defined by

$$(5) \quad \tilde{\sigma}^{*,\nabla} F := \sup_{k \in \mathbb{N}} |\sigma_{n_k} F|,$$

is bounded from the space H_p to the space L_p . Moreover, if $0 < p < 1/2$ and $\{n_k : k \geq 0\}$ is a sequence of positive numbers, such that $\sup_{k \in \mathbb{N}} \rho(n_k) = \infty$, then there exists a martingale $F \in H_p$ such that

$$\sup_{k \in \mathbb{N}} \|\sigma_{n_k} F\|_p = \infty.$$

Similar problem for $p = 1/2$ was studied in [1].

In [28] it was proved that if $F \in H_{1/2}$, then there exists an absolute constant c , such that the following inequality holds:

$$\|\sigma_n F\|_{H_{1/2}} \leq cV^2(n) \|F\|_{H_{1/2}}.$$

Moreover, the rate of sequence $V^2(n)$ can not be improved.

In [28] it was also proved that if $0 < p < 1/2$ and $F \in H_p$, then there exists an absolute constant c_p , depending only on p , such that the inequality

$$(6) \quad \|\sigma_n F\|_{H_p} \leq c_p 2^{\rho(n)(1/p-2)} \|F\|_{H_p}$$

holds. Moreover, the rate of sequence $2^{\rho(n)(1/p-2)}$ in the inequality (6) is sharp.

In [3] it was proved that the weighted maximal operator $\tilde{\sigma}^{*,*,p}$ defined by

$$(7) \quad \tilde{\sigma}^{*,*,p} F := \sup_{n \in \mathbb{N}} \frac{|\sigma_n F|}{2^{\rho(n)(1/p-2)}},$$

is bounded from the Hardy space H_p to the space *weak* L_p and is not bounded from the space H_p to the Lebesgue space L_p , for $0 < p < 1/2$.

One main aim of this paper is to generalize the estimate (2) for $f \in H_p$, $0 < p < 1/2$. Indeed, we will investigate the much more general maximal operators $\tilde{\sigma}^{*,\nabla}$, defined by

$$(8) \quad \tilde{\sigma}^{*,\nabla} F := \sup_{s \in \mathbb{N}} \sup_{2^s \leq n_{s_i} < 2^{s+1}} \frac{|\sigma_{n_{s_i}} F|}{2^{\rho_s(n_{s_i})(1/p-2)}}.$$

In particular, we prove that (8) is bounded from the Hardy space H_p to the Lebesgue space L_p (see Theorem 1). Moreover, we also prove sharpness of this result (see Theorem 2). As a consequence we obtain some new and well-known results.

This paper is organized as follows: In order not to disturb our discussions later on some preliminaries (definitions, notations and lemmas) are presented in Section 2. The main results and some of their consequences can be found in Section 3. Finally, the detailed proofs are given in Section 4.

2. PRELIMINARIES

Let \mathbb{N}_+ denote the set of the positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$. Denote by Z_2 the discrete cyclic group of order 2, that is $Z_2 := \{0, 1\}$, where the group operation is the modulo 2 addition and every subset is open. The Haar measure on Z_2 is given so that the measure of a singleton is $1/2$. Define the group G as the complete direct product of the group Z_2 , with the product of

the discrete topologies of Z_2 . The elements of G are represented by sequences $x := (x_0, x_1, \dots, x_j, \dots)$, where $x_k = 0 \vee 1$.

It is easy to give a base for the neighborhood of $x \in G$:

$$I_0(x) := G, \quad I_n(x) := \{y \in G : y_0 = x_0, \dots, y_{n-1} = x_{n-1}\} \quad (n \in \mathbb{N}).$$

Denote $I_n := I_n(0)$, $\overline{I_n} := G \setminus I_n$ and

$$e_n := (0, \dots, 0, x_n = 1, 0, \dots) \in G, \quad \text{for } n \in \mathbb{N}.$$

Then it is easy to prove that

$$(9) \quad \overline{I_M} = \left(\bigcup_{k=0}^{M-2} \bigcup_{l=k+1}^{M-1} I_{l+1}(e_k + e_l) \right) \cup \left(\bigcup_{k=0}^{M-1} I_M(e_k) \right).$$

The norms (or quasi-norms) of the spaces $L_p(G)$ and $weak - L_p(G)$, ($0 < p < \infty$) are, respectively, defined by

$$\|f\|_p^p := \int_G |f|^p d\mu \quad \text{and} \quad \|f\|_{weak-L_p(G)}^p := \sup_{\lambda > 0} \lambda^p \mu(f > \lambda) < +\infty.$$

The k -th Rademacher function $r_k(x)$ is defined by

$$r_k(x) := (-1)^{x_k} \quad (x \in G, k \in \mathbb{N}).$$

Now, define the Walsh system $w := (w_n : n \in \mathbb{N})$ on G as:

$$w_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) = r_{|n|}(x) (-1)^{\sum_{k=0}^{|n|-1} n_k x_k} \quad (n \in \mathbb{N}).$$

The Walsh system is orthonormal and complete in $L_2(G)$ (see [19]).

If $f \in L_1$, we can define the Fourier coefficients, partial sums, Dirichlet kernels, Fejér means and Fejér kernels as

$$\begin{aligned} \widehat{f}(n) &:= \int_G f w_n d\mu, \quad (n \in \mathbb{N}), \\ S_n f &:= \sum_{k=0}^{n-1} \widehat{f}(k) w_k, \quad D_n := \sum_{k=0}^{n-1} w_k, \quad (n \in \mathbb{N}_+, S_0 f := 0), \\ \sigma_n f &:= \frac{1}{n} \sum_{k=1}^n S_k f, \quad K_n := \frac{1}{n} \sum_{k=1}^n D_k, \quad (n \in \mathbb{N}_+). \end{aligned}$$

Recall that (see [6], [10] and [19]) for any $t, n \in \mathbb{N}$,

$$(10) \quad D_{2^n}(x) = \begin{cases} 2^n & \text{if } x \in I_n \\ 0 & \text{if } x \notin I_n. \end{cases}$$

and

$$(11) \quad K_{2^n}(x) = \begin{cases} 2^{t-1}, & \text{if } x \in I_n(e_t), n > t, x \in I_t \setminus I_{t+1}, \\ (2^n + 1)/2, & \text{if } x \in I_n, \\ 0, & \text{otherwise.} \end{cases}$$

Let $n = \sum_{i=1}^r 2^{n^i}$, $n^1 > n^2 > \dots > n^r \geq 0$ and $n^{(k)} := 2^{n^{k+1}} + 2^{n^{k+2}} + \dots + 2^{n^r}$. Then (see [10] and [19]), for any $n \in \mathbb{N}$,

$$(12) \quad nK_n = \sum_{A=1}^r \left(\prod_{j=1}^{A-1} w_{2^{n^j}} \right) \left(2^{n^A} K_{2^{n^A}} + n^{(A)} D_{2^{n^A}} \right).$$

The proof of the next lemma can be found in [16]:

Lemma 1. *Let $n \in \mathbb{N}$, $[n] \neq |n|$ and $x \in I_{[n]+1}(e_{[n]-1} + e_{[n]})$. Then, the inequality*

$$|nK_n(x)| = \left| (n - 2^{|n|}) K_{n-2^{|n|}}(x) \right| \geq \frac{2^{2[n]}}{4}.$$

holds. We note that if $[n] = 0$, we have the set $I_2(e_0)$.

We also need the following lemma (see [9]):

Lemma 2. *Let $n \geq 2^M$ and $x \in I_M^{k,l}$, $k = 0, \dots, M-1$ and $l = k+1, \dots, M$. Then, the following inequality holds:*

$$\int_{I_M} |K_n(x+t)| d\mu(t) \leq \frac{c2^{k+l}}{2^{2M}}.$$

The σ -algebra generated by the intervals $\{I_n(x) : x \in G\}$ will be denoted by ζ_n ($n \in \mathbb{N}$). Denote by $F = (F_n, n \in \mathbb{N})$ a martingale with respect to ζ_n ($n \in \mathbb{N}$) (see e.g. [18] and [30]). The maximal function F^* of a martingale F is defined by

$$F^* := \sup_{n \in \mathbb{N}} |F_n|.$$

In the case $f \in L_1(G)$, the maximal function f^* is given by

$$f^*(x) := \sup_{n \in \mathbb{N}} \left(\frac{1}{\mu(I_n(x))} \left| \int_{I_n(x)} f(u) d\mu(u) \right| \right).$$

For $0 < p < \infty$ the Hardy martingale spaces $H_p(G)$ consists of all martingales for which

$$\|F\|_{H_p} := \|F^*\|_p < \infty.$$

It is easy to check that for every martingale $F = (F_n, n \in \mathbb{N})$ and every $k \in \mathbb{N}$ the limit

$$\widehat{F}(k) := \lim_{n \rightarrow \infty} \int_G F_n(x) w_k(x) d\mu(x)$$

exists and it is called the k -th Walsh-Fourier coefficients of F .

If $F := (S_{2^n} f : n \in \mathbb{N})$ is a regular martingale, generated by $f \in L_1(G)$, then $\widehat{F}(k) = \widehat{f}(k)$, $k \in \mathbb{N}$.

A bounded measurable function a is called p -atom, if there exists a dyadic interval I , such that

$$\text{supp}(a) \subset I, \quad \int_I a d\mu = 0, \quad \|a\|_\infty \leq \mu(I)^{-1/p}.$$

The dyadic Hardy martingale spaces H_p for $0 < p \leq 1$ have an atomic characterization. Namely, the following holds (see [18], [30, 31]):

Lemma 3. *A martingale $F = (F_n, n \in \mathbb{N})$ belongs to H_p ($0 < p \leq 1$) if and only if there exist a sequence $(a_k, k \in \mathbb{N})$ of p -atoms and a sequence $(\mu_k, k \in \mathbb{N})$ of a real numbers, such that for every $n \in \mathbb{N}$*

$$(13) \quad \sum_{k=0}^{\infty} \mu_k S_{2^n} a_k = F_n, \quad \sum_{k=0}^{\infty} |\mu_k|^p < \infty.$$

Moreover, $\|F\|_{H_p} \sim \inf (\sum_{k=0}^{\infty} |\mu_k|^p)^{1/p}$, where the infimum is taken over all decomposition of F of the form (13).

From Lemma 3 follows the following important lemma (see Weisz [30]):

Lemma 4. *Suppose that an operator T is σ -linear and*

$$\int_I |Ta|^p d\mu \leq c_p < \infty, \quad (0 < p \leq 1)$$

for every p -atom a , where I denotes the support of the atom. If T is bounded from L_{∞} to L_{∞} , then

$$\|TF\|_p \leq c_p \|F\|_{H_p}.$$

3. THE MAIN RESULTS

Our first main result reads:

Theorem 1. *Let $0 < p < 1/2$, $f \in H_p$ and $\{n_k : k \geq 0\}$ be a sequence of positive numbers and let $\{n_{s_i} : 1 \leq i \leq r\} \subset \{n_k : k \geq 0\}$ be numbers such that $2^s \leq n_{s_1} \leq n_{s_2} \leq \dots \leq n_{s_r} \leq 2^{s+1}$, $s \in \mathbb{N}$. Then the weighted maximal operator $\tilde{\sigma}^{*, \nabla}$, defined by (8), where $\rho_s(n_{s_i})$ are defined by (3), is bounded from the martingale Hardy space H_p to the Lebesgue space L_p .*

We also prove sharpness of this result:

Theorem 2. *Let $0 < p < 1/2$, $f \in H_p(G)$, $\{n_k : k \geq 0\}$ be a sequence of positive numbers and let $\{n_{s_i} : 1 \leq i \leq r\} \subset \{n_k : k \geq 0\}$ be numbers such that*

$$2^s \leq n_{s_1} \leq n_{s_2} \leq \dots \leq n_{s_r} \leq 2^{s+1}, \quad s \in \mathbb{N}.$$

Then, for any non-negative and non-decreasing function $\varphi : \mathbb{N}_+ \rightarrow \mathbb{R}$ satisfying the condition

$$(14) \quad \sup_{s \in \mathbb{N}} \sup_{2^s \leq n_{s_i} < 2^{s+1}} \frac{2^{\rho_s(n_{s_i})(1/p-2)}}{\varphi(n_{s_i})} = \infty,$$

there exist p -atoms f_s , such that

$$\frac{\left\| \sup_{s \in \mathbb{N}} \sup_{2^s \leq n_{s_i} < 2^{s+1}} \frac{|\sigma_{n_{s_i}} f_s|}{\varphi(n_{s_i})} \right\|_p}{\|f_s\|_{H_p}} \rightarrow \infty, \quad \text{as } s \rightarrow \infty.$$

From Theorems 1 and 2 follow immediately the following result (see [25]):

Corollary 1. *a) Let $0 < p < 1/2$ and $f \in H_p$. Then the weighted maximal operator $\tilde{\sigma}^*$, defined by (1), is bounded from the martingale Hardy space H_p to the Lebesgue space L_p .*

b) Let $\{\varphi_n\}$ is any nondecreasing sequence satisfying the condition

$$\lim_{n \rightarrow \infty} \frac{(n+1)^{1/p-2}}{\varphi_n} = \infty.$$

Then, the maximal operator

$$\sup_{n \in \mathbb{N}} \frac{|\sigma_n f|}{\varphi_n}$$

is not bounded from the martingale Hardy space H_p to the Lebesgue space L_p , for any $0 < p < 1/2$.

We also have the following new consequences of Theorem 1, which all are sharp in a special sense (see Theorem 2):

Corollary 2. *Let $p > 0$ and $f \in H_p$. Then the restricted maximal operators $\tilde{\sigma}_i^{*, \nabla}$, $i = 1, 2$, defined by*

$$\tilde{\sigma}_1^{*, \nabla} F := \sup_{k \in \mathbb{N}} |\sigma_{2^k} F| \quad \text{and} \quad \tilde{\sigma}_2^{*, \nabla} F := \sup_{k \in \mathbb{N}} |\sigma_{2^k + 2^{k-1}} F|,$$

are both bounded from the Hardy space H_p to the Lebesgue space L_p .

Corollary 3. *If $0 < p < 1/2$, $f \in H_p$ and $\{n_k : k \geq 0\}$ is any sequence of positive numbers, then the maximal operator defined by (5) is bounded from the Hardy space H_p to the Lebesgue space L_p if and only if the condition (4) is fulfilled.*

Corollary 4. *Let $0 < p < 1/2$ and $f \in H_p$. Then the restricted maximal operator $\tilde{\sigma}_i^{*, \nabla}$, $i = 3, 4$, defined by*

$$\tilde{\sigma}_3^{*, \nabla} F := \sup_{k \in \mathbb{N}} \frac{|\sigma_{2^{2k} + 2^k} F|}{2^{k(1/p-2)}} \quad \text{and} \quad \tilde{\sigma}_4^{*, \nabla} F := \sup_{k \in \mathbb{N}} \frac{|\sigma_{2^{3k} + 2^k} F|}{2^{2k(1/p-2)}},$$

are both bounded from the Hardy space H_p to the Lebesgue space L_p .

4. PROOFS

Proof of Theorem 1. Since σ_n is bounded from L_∞ to L_∞ , by Lemma 4, the proof of Theorem 1 will be complete, if we prove that

$$(15) \quad \int_{I_M} \left(\sup_{s \in \mathbb{N}} \sup_{2^s \leq n_{s_i} < 2^{s+1}} \left| \frac{\sigma_{n_{s_k}} a}{2^{\rho_s(n_{s_k})(1/p-2)}} \right| \right)^p d\mu \leq c < \infty,$$

for every p -atom a . We may assume that a is an arbitrary p -atom, with support I , $\mu(I) = 2^{-M}$ and $I = I_M$. It is easy to see that $\sigma_{n_{s_k}} a = 0$, when

$n_{s_k} < 2^M$. Therefore, we can suppose that $n_{s_k} \geq 2^M$. Let $2^s \leq n_{s_i} < 2^{s+1}$, for some $s \geq M$. Since $\|a\|_\infty \leq 2^{M/p}$, we find that

$$\begin{aligned} \frac{|\sigma_{n_{s_i}} a|}{2^{\rho_s(n_{s_i})(1/p-2)}} &\leq \frac{1}{2^{\rho_s(n_{s_i})(1/p-2)}} \|a\|_\infty \int_{I_M} |K_{n_{s_i}}(x+t)| d\mu(t) \\ &\leq \frac{1}{2^{\rho_s(n_{s_i})(1/p-2)}} 2^{M/p} \int_{I_M} |K_{n_{s_i}}(x+t)| d\mu(t). \end{aligned}$$

Let $x \in I_{l+1}(e_k + e_l)$, $0 \leq k < l \leq [n_{s_i}] \leq M$. Then $x+t \in I_{l+1}(e_k + e_l)$, $0 \leq k < l \leq [n_{s_i}] \leq M$ and by applying (10), (11) and (12) we get that

$$K_{n_{s_i}}(x+t) = 0, \quad \text{for } t \in I_M$$

and $|\sigma_{n_{s_i}} a| = 0$, for any $2^s \leq n_{s_i} < 2^{s+1}$. Since $[n_{s_i}] \geq s_-$, we obtain that

$$(16) \quad \sup_{2^s \leq n_{s_i} < 2^{s+1}} \frac{|\sigma_{n_{s_i}} a|}{2^{\rho_s(n_{s_i})(1/p-2)}} = 0, \quad \text{for } x \in I_{l+1}(e_k + e_l), \quad 0 \leq k < l < s_-.$$

Next, we suppose that $x \in I_{l+1}(e_k + e_l)$, where either $[n_{s_i}] \leq k < l \leq M$ or $k \leq [n_{s_i}] < l \leq M$. Since $n_{s_i} \geq 2^M$ and $|n_{s_i}| = s$ if we apply Lemma 2 we find that

$$(17) \quad \frac{|\sigma_{n_{s_i}} a|}{2^{\rho_s(n_{s_i})(1/p-2)}} \leq \frac{2^{M(1/p-2)+k+l}}{2^{\rho_s(n_{s_i})(1/p-2)}} \leq \frac{2^{M(1/p-2)}}{2^{s(1/p-2)}} 2^{k+l+s_-(1/p-2)}.$$

By applying (17) for any $x \in I_{l+1}(e_k + e_l)$, $s_- \leq k < l \leq M$ or $k \leq s_- < l \leq M$ we obtain that

$$(18) \quad \sup_{2^s \leq n_{s_i} < 2^{s+1}} \frac{|\sigma_{n_{s_i}} a|}{2^{\rho_s(n_{s_i})(1/p-2)}} \leq \frac{2^{M(1/p-2)}}{2^{s(1/p-2)}} 2^{k+l+s_-(1/p-2)}.$$

If we now define $\tilde{\sigma}_s^*$ by

$$\tilde{\sigma}_s^* f := \sup_{2^s \leq n_{s_i} < 2^{s+1}} \frac{|\sigma_{n_{s_i}} f|}{2^{\rho_s(n_{s_i})(1/p-2)}},$$

then we can conclude that

$$\left(\sup_{s \in \mathbb{N}} \sup_{k \in \mathbb{N}} \left| \frac{\sigma_{n_{s_k}} a}{2^{\rho_s(n_{s_k})(1/p-2)}} \right| \right)^p \leq \sum_{s=M}^{\infty} |\tilde{\sigma}_s^* a|^p.$$

Hence, by combining (9), (16) and (18) we obtain that

$$(19) \quad \int_{I_M} \left(\sup_{s \in \mathbb{N}} \sup_{k \in \mathbb{N}} \left| \frac{\sigma_{n_{s_k}} a}{2^{\rho_s(n_{s_k})(1/p-2)}} \right| \right)^p d\mu \leq \sum_{s=M}^{\infty} \int_{I_M} |\tilde{\sigma}_s^* a|^p d\mu$$

$$\begin{aligned}
&= \sum_{s=M}^{\infty} \left(\sum_{k=0}^{s-2} \sum_{l=k+1}^{s-1} + \sum_{k=0}^{s-1} \sum_{l=s-}^{M-1} + \sum_{k=s-}^{M-2} \sum_{l=k+1}^{M-1} \right) \int_{I_{l+1}(e_k+e_l)} \left| \widetilde{\sigma}_s^* a \right|^p d\mu \\
&+ \sum_{s=M}^{\infty} \sum_{k=0}^{M-1} \int_{I_M(e_k)} \left| \widetilde{\sigma}_s^* a \right|^p d\mu \\
&\leq c_p \sum_{s=M}^{\infty} \left(\frac{2^{M(1/p-2)}}{2^{s(1/p-2)}} \right)^p \sum_{k=0}^{s-1} \sum_{l=s-}^{M-1} \frac{1}{2^l} 2^{s-(1-2p)} 2^{p(k+l)} \\
&+ c_p \sum_{s=M}^{\infty} \left(\frac{2^{M(1/p-2)}}{2^{s(1/p-2)}} \right)^p \sum_{k=s-}^{M-2} \sum_{l=k+1}^{M-1} \frac{1}{2^l} 2^{s-(1-2p)} 2^{p(k+l)} \\
&+ c_p \sum_{s=M}^{\infty} \left(\frac{2^{M(1/p-2)}}{2^{s(1/p-2)}} \right)^p \frac{2^{s-(1-2p)}}{2^M} \sum_{k=0}^{s-} 2^{p(k+M)} \\
&:= I + II + III.
\end{aligned}$$

Since $\sum_{s=M}^{\infty} \left(\frac{2^{M(1/p-2)}}{2^{s(1/p-2)}} \right)^p < c < \infty$ we find that

$$(20) \quad I \leq c_p \sum_{k=0}^{s-1} \sum_{l=s-}^{M-1} \frac{1}{2^l} 2^{s-(1-2p)} 2^{p(k+l)} < c_p < \infty,$$

$$(21) \quad II \leq c_p \sum_{k=s-}^{M-2} \sum_{l=k+1}^{M-1} \frac{1}{2^l} 2^{s-(1-2p)} 2^{p(k+l)} < c_p < \infty$$

and

$$(22) \quad III \leq \frac{c_p 2^{s-(1-2p)}}{2^M} \sum_{k=0}^{s-} 2^{p(k+M)} < c_p < \infty.$$

By combining (19)-(22) we can conclude that (15) holds so the proof is complete. \square

Proof of Theorem 2. Let $n_{s_k} := q_{n_k}^s \in \mathbb{N}$ is such that $2^{n_k} \leq q_{n_k}^s \leq 2^{n_k+1}$, where $0 \leq s < 2^{n_k}$. If s_0 is such that $q_{n_k}^{s_0} = s_-$, then we get that $\rho_{n_k}(q_{n_k}^s) = n_k - s_- = \rho(q_{n_k}^{s_0})$. In view of (14) we have that

$$(23) \quad \frac{2^{\rho(q_{n_k}^{s_0})(1/p-2)}}{\varphi(q_{n_k}^{s_0})} \rightarrow \infty, \quad \text{as } k \rightarrow \infty.$$

Set

$$f_{n_k} = D_{2^{n_k+1}} - D_{2^{n_k}}, \quad n_k \geq 3.$$

It is evident that

$$\widehat{f_{n_k}}(i) = \begin{cases} 1, & \text{if } i = 2^{n_k}, \dots, 2^{n_k+1} - 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then we easily can derive that

$$(24) \quad S_i f_{n_k} = \begin{cases} D_i - D_{2^{n_k}}, & \text{if } i = 2^{n_k}, \dots, 2^{n_k+1} - 1, \\ f_{n_k}, & \text{if } i \geq 2^{n_k+1}, \\ 0, & \text{otherwise.} \end{cases}$$

Since

$$(25) \quad D_{j+2^{n_k}} - D_{2^{n_k}} = w_{2^{n_k}} D_j, \quad j = 1, 2, \dots, 2^{n_k},$$

from (10) it follows that

$$(26) \quad \|f_{n_k}\|_{H_p} = \left\| \sup_{i \in \mathbb{N}} |S_{2^i} f_{n_k}| \right\|_p = \|D_{2^{n_k+1}} - D_{2^{n_k}}\|_p \leq 2^{n_k(1-1/p)}.$$

By combining (24) and (25) we can conclude that

$$\begin{aligned} |\sigma_{q_{n_k}^s} f_{n_k}| &= \frac{1}{q_{n_k}^s} \left| \sum_{j=2^{n_k}}^{q_{n_k}^s-1} S_j f_{n_k} \right| = \frac{1}{q_{n_k}^s} \left| \sum_{j=2^{n_k}}^{q_{n_k}^s-1} (D_j - D_{2^{n_k}}) \right| \\ &= \frac{1}{q_{n_k}^s} \left| \sum_{j=0}^{q_{n_k}^s-2^{n_k}-1} (D_{j+2^{n_k}} - D_{2^{n_k}}) \right| \\ &= \frac{1}{q_{n_k}^s} \left| \sum_{j=0}^{q_{n_k}^s-2^{n_k}-1} D_j \right| = \frac{q_{n_k}^s - 2^{n_k} - 1}{q_{n_k}^s} |K_{q_{n_k}^s-2^{n_k}-1}|. \end{aligned}$$

Let $q_{n_k}^{s_0}$ is n_{s_j} so that $[q_{n_k}^{s_0}] = s_-$, that is $s_0 = s_-$ and $x \in I_{s_0+1}(e_{s_0-1} + e_{s_0})$. By using Lemma 1 we have that

$$\left| \sigma_{q_{n_k}^{s_0}} f_{n_k} \right| \geq \frac{c2^{2s_0}}{2^{n_k}} \quad \text{and} \quad \frac{|\sigma_{q_{n_k}^{s_0}} f_{n_k}|}{\varphi(q_{n_k}^{s_0})} \geq \frac{c2^{2s_0}}{2^{n_k} \varphi(q_{n_k}^{s_0})}.$$

Hence,

$$\begin{aligned} \int_G \left(\sup_{k \in \mathbb{N}} \frac{|\sigma_{q_{n_k}^s} f_{n_k}|}{\varphi(q_{n_k}^s)} \right)^p d\mu &\geq \int_{I_{s_0+1}(e_{s_0-1} + e_{s_0})} \left(\frac{|\sigma_{q_{n_k}^{s_0}} f_{n_k}|}{\varphi(q_{n_k}^{s_0})} \right)^p d\mu \\ &\geq c_p \frac{1}{2^{s_0}} \frac{2^{2ps_0}}{2^{pn_k} \varphi^p(q_{n_k}^{s_0})} \geq \frac{C_p 2^{(2p-1)s_0}}{2^{pn_k} \varphi^p(q_{n_k}^{s_0})}. \end{aligned}$$

Finally, by combining (23) and (26) we find that

$$\begin{aligned} &\frac{\left(\int_G \left(\sup_{k \in \mathbb{N}} \sup_{0 \leq s < n_k} \frac{|\sigma_{q_{n_k}^s} f_{n_k}|}{\varphi(q_{n_k}^s)} \right)^p d\mu \right)^{1/p}}{\|f_{n_k}\|_{H_p}} \\ &\geq \frac{C_p 2^{(2-1/p)s_0}}{2^{n_k} \varphi(q_{n_k}^{s_0})} \frac{1}{2^{n_k(1-1/p)}} \geq \frac{C_p 2^{\rho(q_{n_k}^{s_0})(1/p-2)}}{\varphi(q_{n_k}^{s_0})} \rightarrow \infty, \quad \text{as } k \rightarrow \infty. \end{aligned}$$

The proof is complete. □

Final remark: The research in this paper is partly inspired of the research in [11]. The authors plan to further investigate the maximal operators studied in this paper and their possibilities to find concrete geometric constructions of more general means (than power means) and their related inequalities.

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