HARDY-LITTLEWOOD MAXIMAL OPERATOR ON ASSOCIATE SPACES

TENGIZ KOPALIANI AND ANI OZBETELASHVILI

Abstract. Let X be a Banach function space over a locally compact abelian group, that admits covering family. We show that if the Hardy-Littlewood maximal operator M is bounded on the space X, then its boundedness on the associate space X' is equivalent to a certain condition $\mathcal{A}_\infty$.

1. Intriduction

One of the central problems of Harmonic Analysis is the problem of the boundedness of the Hardy-Littlewood maximal operator M on the Banach function spaces. In 2005, Diening [4, Theorem 8.1] proved the following result: the Hardy-Littlewood maximal operator M (defined by Euclidean balls or cubes) is bounded on the reflexive variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ if and only if, it is bounded on its dual $L^{q(\cdot)}(\mathbb{R}^n)$ space. For Euclidean setting Lerner in [8] proved that, if the Hardy-Littlewood maximal operator M is bounded on a Banach function space X, then its boundedness on the associate space X' is equivalent to a certain condition $\mathcal{A}_\infty$. Analogous result for BFS on a space of homogeneous type was obtained in [7]. The boundedness of Hardy-littlewood maximal operator M on the Banach function space was considered in [1] and [9]. In [12] authors studied the weighted norm inequalities for maximal type operators such as Hardy-littlewood maximal operator associated with $\mathbb{E} := E_r(x)_{r\in\mathbb{I},x\in X}$ a family of open subsets of a topological space X endowed with a non-negative Borel measure μ satisfying certain basic conditions. However, there are many examples of important families of measurable sets arising in harmonic analysis and PDE which cannot be generated by a quasi-metric, and hence are not in the scope of spaces of homogeneous type (for examples please refer to [12]).

The aim of this paper is to present extension of Lerner's above mentioned theorem to the setting of Locally Compact Abelian (LCA) groups G having covering families.

Suppose that G is a locally compact abelian group equipped with a nontrivial and inner regular measure μ, such that $\mu(K) < \infty$ for all compact $K \subset G$. Notice that, μ does not need to be Haar measure because we do not assume μ to be translation invariant. The general assumption on the group will be that it admits

2020 *Mathematics Subject Classification.* 43A85, 46E30, 43A70.

Key words and phrases. Hardy-Littlewood maximal operator, locally compact abelian groups, Banach function space, associate space.

This work was supported by Shota Rustaveli National Science Foundation of Georgia FR-22 17770.

a sequence of neighborhoods of 0 with certain properties that we will describe in the next definition (see [5], Section 2.1).

Definition 1.1. A family $\{U_i\}_{i\in\mathbb{Z}}$ is a covering family for G if:
(1) $\{U_i\}_{i\in\mathbb{Z}}$ is an increasing base of relatively compact neighborhoods of 0, $\cup_{k\in\mathbb{Z}}U_k = G$, and $\cap_{k\in\mathbb{Z}}U_k = \{0\}$,
(2) there exist a positive constant c_D and a mapping $\theta : \mathbb{Z} \to \mathbb{Z}$ such that for all $k \in \mathbb{Z}$ and all $x \in G$,
a) $k < \theta(k)$,
b) $U_k - U_k \subset U_{\theta(k)}$,
c) $\mu(x + U_{\theta(k)}) \leq c_D\mu(x + U_k)$.

We will refer to the condition c) as the doubling property of measure μ with respect to θ and we will call c_D the doubling constant. Note that, $c_D \geq 1$ is required by the reason of $U_k \subset U_{\theta(k)}$. Any group G admiting a sequence $\{U_i\}_{i\in\mathbb{Z}}$ of neighbourhoods of 0 and satisfying the above postulates is said to have a covering family. This concept was introduced in [5] by *Edwards* and *Gaudry*.

Some basic properties of covering family that follow directly from Definition 1.1. for instance (1) for every $x \in G$ and $k \in \mathbb{Z}$ it holds $\mu(x + U_k) > 0$; (2) the interiors U'_k of the base sets U_k cover G, i.e., $\cup_{k\in\mathbb{Z}}U'_k = G$. In particular, for every compact $K \subset G$ there is $k \in \mathbb{Z}$ such that $K \subset U_k$ (see [11], Proposition 1).

Among the well known groups satisfying Definition 1.1 are the groups $\mathbb{R}$, $\mathbb{Z}$, torus $\mathbb{T}$, p-adic group $\mathbb{Q}_p$ and finite products of these groups (for details see [5]). Moreover, if G is an LCA group equipped with nontrivial Haar measure and with an increasing sequence $\{U_k\}_{k\in\mathbb{Z}}$ of compact open subgroups, such that

$$\bigcup_{k\in\mathbb{Z}} U_k = G, \quad \bigcap_{k\in\mathbb{Z}} U_k = \{0\},$$

then Definition 1.1 satisfies this property if and only if $\sup_{k\in\mathbb{Z}} |U_k/U_{k-1}| < \infty$ ($|U_k/U_{k-1}|$ is the order of U_k/U_{k-1} factor groups), where and one may take $\theta(k) = k + 1$.

Let G be a LCA group with Haar measure μ and let H be a compact and open subgroup of G with $\mu(H) = 1$. Let A be an automorphism on G such that $H \subset AH$ and $\cap_{i<0}A^iH = \{0\}$. In addition, suppose that $G = \cup_{i\in\mathbb{Z}}A^iH$. Then $\{A^iH\}_{i\in\mathbb{Z}}$ satisfies the required properties to be a covering family (for details see [10]). A structure of this type is considered in [3] for constructing wavelets on LCA groups with open and compact subgroups.

We will call any set of the form $x + U_k$, where $x \in G, k \in \mathbb{Z}$ a base set and the collection of all base sets will be denoted by $\mathcal{B}$. For any natural number n we denote $\theta^n(k) = \theta(\theta^{n-1}(k)),\ n > 1,\ \theta^1(k) = \theta(k), k \in \mathbb{Z}$. For $V = x + U_k$ we denote $V^* = x + U_{\theta(k)}$ and $V^{**} = x + U_{\theta^2(k)}$. Observe that we can assume without loss of generality the base sets to be symmetric. This is not a restriction at all because one can always consider the new family of base sets formed by the difference sets $U_i - U_i$ which increases the doubling constant from c_D to c_D^2. We denote $2U_k := U_k - U_k = U_k + U_k$. We also may assume that the function θ be non-decreasing. Indeed if we replace θ by

$$\widetilde{\theta}(k) = \min\{l \in \mathbb{Z} :\ l > k \text{ with } U_k - U_k \subset U_l\},$$

we obtain that $\widetilde{\theta}$ function is non-decreasing, i.e. $\widetilde{\theta}(k) \le \theta(k)$, $k \in \mathbb{Z}$ and for all $x \in G$ and $k \in \mathbb{Z}$

$$\mu(U_{\widetilde{\theta}(k)}) \le \mu(U_{\theta(k)}) \le c_D \mu(U_k).$$

In the Euclidean setting the standard way to introduce maximal functions is by considering averages over cubes, balls or more general families of convex sets. However there are many LCA groups where we have possibility of consider a family of base sets satisfying the fundamental property of the the collection of cubes or balls: any point has family of decreasing base sets shrinking to it, in addition, the whole space can be covered by increasing union of such family. The notion of base sets allows to define a direct analogue of the Hardy-Littlewood maximal function (uncentered):

$$Mf(x) = \sup_{x \in V \in \mathcal{B}} \frac{1}{\mu(V)} \int_V |f| d\mu, \quad f \in L^1_{loc}(G).$$

Analogously we may define the centered maximal function for $f \in L^1_{loc}(G)$

$$\mathcal{M}f(x) = \sup_{k \in \mathbb{Z}} \frac{1}{\mu(x + U_k)} \int_{x+U_k} |f| d\mu.$$

Note that uncentered maximal operator is comparable to the centered maximal operator (see [11], Lemma 1). More precisely for $f \in L^1_{loc}(G)$ we have

$$\mathcal{M}f \le Mf \le c_D^2 \mathcal{M}f. \tag{1.1}$$

Let us recall the definition of a Banach function space (BFS) (see, e.g., [2, Chap.1, Definition 1.1]).

Definition 1.2. Let $L^0 := L^0(G, \mu)$ denote as the set of all real-valued measurable functions on G and let $L^0_+ := L^0_+(G, \mu)$ be the set of all non-negative measurable functions on G. The characteristic function of a set $E \subset G$ is denoted by χ_E. A mapping $\rho : L^0_+ \to [0, \infty]$ is called a Banach function norm if, for all functions $f, g, f_n \in L^0_+$ with $n \in \mathbb{N}$, for all constant $a \ge 0$, and for all measurable subset E of G, the following properties hold:

$(A1)$ $\rho(f) = 0 \Leftrightarrow f = 0$ a.e., $\rho(af) = a\rho(f)$, $\rho(f + g) \le \rho(f) + \rho(g)$,

$(A2)$ $0 \le g \le f$ a.e. $\Rightarrow \rho(g) \le \rho(f)$ (the lattice property),

$(A3)$ $0 \le f_n \uparrow f$ a.e $\Rightarrow \rho(f_n) \uparrow \rho(f)$ (the Fatou property),

$(A4)$ $\mu(E) < \infty \Rightarrow \rho(\chi_E) < \infty$,

$(A5)$ $\displaystyle\int_E f d\mu \le C_E \rho(f)$

with a constant $C_E \in (0, \infty)$ that depends on E and ρ, but is independent of f.

When functions differing only on a set of measure zero are identifed, the set X of all functions $f \in L^0$ for which $\rho(|f|) < \infty$ is called a Banach function space. For each $f \in X$, the norm of f is defined by

$$\|f\|_X = \rho(|f|).$$

The set X under the natural linear space operations and under this norm becomes a Banach space (see [2, Chapt.1, Theorems 1.4 and 1.6]). If ρ is a Banach function norm, its associate norm ρ' is defined on L^0_+ by

$$\rho'(g) = \sup \left\{ \int_G fg d\mu : \ f \in L^0_+, \ \rho(f) \le 1 \right\}.$$

It is known that the associated norm ρ' is a Banach function norm itself [2, Chapt.1,Theorem 2.2]. The Banach function space X' determined by the Banach function norm ρ' is called the associate space of X. By Lorentz-Luxemburg theorem, $\|f\|_X = \|f\|_{X''}$, where $X'' = (X')'$.

The definition of $\|f\|_{X'}$ implies that

$$\int_G |fg| d\mu \leq \|f\|_X \|g\|_{X'} \tag{1.2}$$

and

$$\|g\|_{X'} = \sup_{f\in X,\, \|f\|_X \leq 1} \int_G |fg| d\mu. \tag{1.3}$$

Definition 1.3. We say that a collection $\mathcal{S} \subset \mathcal{B}$ is sparse if for every base set $V \in \mathcal{S}$, there is a a measurable subset $E(V) \subset V$ such that $\mu(V) \leq 2\mu(E(V))$ and the sets $\{E(V)\}_{V\in\mathcal{S}}$ are pairwise disjoint.

Sparse domination is a recent technique allowing one to estimate many operators in harmonic analysis by simple expressions of the form

$$\sum_{Q\in\mathcal{S}} f_{p,Q}\chi_Q$$

where $f_{p,Q} = \left(\frac{1}{|Q|}\int_Q |f|^p\right)^{1/p}$ for $p \in (0,\infty)$ and $\mathcal{S}$ is a sparse family of cubes in $\mathbb{R}^n$.

Definition 1.4. (The condition $\mathcal{A}_\infty$). We say that a Banach function space X over a LCA group G satisfies the condition $\mathcal{A}_\infty$ if there exist constants $C = C_{[\infty]}$ and $\gamma > 0$ such that for every finite sparse collection $\mathcal{S} \subset \mathcal{B}$, every family of non-negative numbers $\{\alpha_V\}_{V\in\mathcal{S}}$, and every family of pairwise disjoint measurable sets $\{G_V\}_{V\in\mathcal{S}}$ such that $G_V \subset V, V \in \mathcal{S}$ one has

$$\left\|\sum_{V\in\mathcal{S}} \alpha_V \chi_{G_V}\right\|_X \leq C_{[\infty]} \left(\max_{V\in\mathcal{S}} \frac{\mu(G_V)}{\mu(V)}\right)^\gamma \left\|\sum_{V\in\mathcal{S}} \alpha_V \chi_V\right\|_X. \tag{1.4}$$

The question about conditions on BFS X such that for Hardy-Littlewood maximal operator M (with respect cubes or balls) boundedness on X causes, that M is bounded on X' solved by Lerner in terms of sparse families and $\mathcal{A}_\infty$-type conditions in Euclidean setting. Analogous problem for a BFS over a space of homogeneous type investigated by Karlovich in [7].

The aim of this paper is to prove following theorem.

Theorem 1.5. *Let X be a BFS over a LCA group G with a covering family such that corresponding Hardy-Littlewood maximal operator M is bounded on X. The following conditions are equivalent:*

(i) M is bounded on X';

(ii) X satisfies the condition $\mathcal{A}_\infty$.

The paper is organized as it follows. In Section 2 we give some preliminary results. We investigate some properties of weak reverse Hölder inequality defined by the cover family. In Section 3 we will prove the main theorem. Throughout, we use C and c to stand for an absolute positive constant, which may have different values in different occurrences.

2. Preliminaries

In [6], Theorem 44.18 it is proven a version of the Lebesgue differentiation theorem with respect to Haar measure for LCA groups having a D' sequence (see [6], Definition 44.10). Note that the result is still true with the obvious changes for measures which are not translation invariant. Thus, since a covering family is in particular D' sequence, we have that the Lebesgue Differentiation theorem holds in our context (see for details [10], Remak 2.6).

For a given $V \in \mathcal{B}$, we will denote by $j(V)$ the maximum integer such that $V = x + U_{j(V)}$ for some $x \in G$. Such a number exists (see for details [10], Remark 2.3).

Let $V \in \mathcal{B}$ be a fixed base set and $k = j(V)$. The local base $\mathcal{B}_V$ is defined as

$$\mathcal{B}_V = \{y + U_j : y \in V, j \leq k\}.$$

We also defined the enlarged set $\widehat{V}$ by the formula

$$\widehat{V} = \bigcup_{U \in \mathcal{B}_V} U.$$

We have the following geometric properties of $\widehat{V}$ (see [10]): For any $z \in V$, $\widehat{V} \subset z + U_{\theta^2(k)}$ and as a consequence of this property, we have

$$\mu(\widehat{V}) \leq \mu(z + U_{\theta^2(k)}) \leq c_D^2 \mu(z + U_k). \tag{2.1}$$

Any covering family has the so-called engulfing property:

Lemma 2.1. *([10], Lemma 2.2) Let U, V be two base sets such that $U = x + U_i$ and $V = y + U_j$ with $i \leq j$ and $x, y \in G$. If $U \cap V \neq \emptyset$, then $x + U_i \subset y + U_{\theta^2(j)}$.*

We need following covering lemma. Note that Lemma 2.2 has been proved in [5] in the case of an translation invariant measure μ. For regular measure μ with doubling property with respect to θ Lemma 2.2 is proved in [11].

Lemma 2.2. *(see [11], Lemma 2) Let E be a subset of G and $k : E \to \mathbb{Z}$ a mapping bounded from above such that for every $k_0 \in \mathbb{Z}$ the set $\{x \in E : k(x) \geq k_0\}$ is relatively compact in G. Then there is a sequence (x_n) of elements of E, finite or infinite, such that*
(i) the sequence $(k_n) := (k(x_n))$ is non-increasing,
(ii) the sets $x_n + U_{k_n}$ are pairwise disjoint, and
(iii) $E \subset \bigcup(x_n + 2U_{k_n})$.

We now define the local maximal function as follows

$$M_V f(x) = \sup_{x \in U \in \mathcal{B}_V} \frac{1}{\mu(U)} \int_U |f(y)| d\mu$$

for any $x \in \widehat{V}$ and $M_V f(x) = 0$ otherwise.

A nonnegative, locally integrable function is called a weight. Below we will use the standard notation: for any measurable set E we denote

$$w(E) = \int_E w d\mu, \text{ and } f_E = \frac{1}{\mu(E)} \int_E f d\mu.$$

Consider for a fixed $V \in \mathcal{B}$, the level set for the local maximal function acting on a function w at level $\lambda > 0$:

$$\Omega_\lambda = \{x \in \widehat{V} : \ M_V w(x) > \lambda\}.$$

The principal method for investigateting level sets is the following Calderón-Zygmund decomposition of the set Ω_λ.

Lemma 2.3. *([10], Lemma 2.5) Let $V \in \mathcal{B}$ be a fixed base set in G and w be a nonnegative and integrable function supported on $\widehat{V}$ and $\lambda > w_{\widehat{V}}$. If Ω_λ is nonempty, then there exists a finite or countable index set $Q \subset \mathbb{Z}$ and family $\{y_i + U_{\alpha_i}\}_{i\in Q}$ of pairwise disjoint base sets from $\mathcal{B}_V$ such that*

(a) The sequence $\{\alpha_i\}_{i\in Q}$ is decreasing.

(b) $\bigcup_{i\in Q} y_i + U_{\alpha_i} \subset \Omega_\lambda \subset \bigcup_{i\in Q} y_i + U_{\theta^2(\alpha_i)}$,

(c) For any $i \in Q$, we have that

$$\lambda < \frac{1}{\mu(y_i + U_{\alpha_i})} \int_{y_i+U_{\alpha_i}} w d\mu, \tag{2.2}$$

(d) Given $r > \alpha_i$ for some $i \in Q$, then

$$\frac{1}{\mu(y_i + U_r)} \int_{y_i+U_r} w d\mu \le c_D^2 \lambda. \tag{2.3}$$

From part (b) of Lemma 2.3 and Lebesgue differentiation theorem we obtain that

$$w(x) \le \lambda \ \text{ for a.e } \ x \in \widehat{V} \setminus \bigcup_{i\in Q} y_i + U_{\theta^2(\alpha_i)}. \tag{2.4}$$

A weight w is an $A_1 = A_1(G, d\mu)$ weight if

$$[w]_{A_1} := \sup_{V\in\mathcal{B}} \left(\frac{1}{\mu(V)} \int_V w d\mu \right) \operatorname{ess\,sup}_V (w^{-1}) < +\infty,$$

which is equivalent to w having the property

$$Mw(x) \le [w]_{A_1} w(x) \ \ \mu - \text{a.e.} \ \ x \in G.$$

The Muckenhoupt A_1 weights can be characterized as those weight functions, such that the maximal operator is weakly bounded in the weighted function space $L_w^{(1)}(G)$ (see Theorem 4 in [11]).

We prove following version of reverse Hölder inequality.

Theorem 2.4. *(Weak reverse Hölder inequality). Let $w \in A_1(G)$. Then there are positive constants c and δ such that for every base set V*

$$\left(\frac{1}{\mu(\widehat{V})} \int_{\widehat{V}} w^{1+\delta} d\mu \right)^{\frac{1}{1+\delta}} \le c \frac{1}{\mu(\widehat{V})} \int_{\widehat{V}} w d\mu. \tag{2.5}$$

We need following kind of "reverse" weak $(1,1)$ inequality for the local maximal function M_V.

Lemma 2.5. *Let $w \in L^1(G)$ be a nonnegative function, V be a base set, and $\lambda > w_{\widehat{V}}$. Then there exists a constant $C > 0$ such that*

$$\int_{\{x\in\widehat{V} : \, w(x) > \lambda\}} w d\mu \le C \lambda \mu(\{x \in \widehat{V} : \ M_V w > \lambda\}). \tag{2.6}$$

Proof. Let $\{y_i + U_{\alpha_i}\}_{i\in Q}$ be the Calderón-Zygmund decomposition of set Ω_λ according to Lemma 2.3 . Using (2.2), (2.3) and (2.4) we obtain

$$\int_{\{x\in\widehat{V}:\, w(x)>\lambda\}} w d\mu \le c_D^2 \lambda \sum \mu(y_i + U_{\theta^2(\alpha_i)})$$

$$\le c_D^4 \lambda \sum \mu(y_i + U_{\alpha_i}) \le C\lambda\mu(\{x\in\widehat{V} :\, M_V w > \lambda\}).$$

□

Proof of Theorem 2.4. We use inequality (2.6) and the fact that $w \in A_1$ to write

$$\begin{aligned} w(\{x\in\widehat{V} :\, w(x) > \lambda\}) &\le C\lambda\mu(\{x\in\widehat{V} :\, M_V w(x) > \lambda\}) \\ &\le C\lambda\mu(\{x\in\widehat{V} :\, w(x) > [w]_{A_1}^{-1}\lambda\}) \end{aligned}$$

for $\lambda > w_{\widehat{V}}$.

Using (2.6) we obtain

$$\begin{aligned} \int_{\widehat{V}} w^r d\mu &= (r-1)\int_0^\infty \lambda^{r-2} w(\{x\in\widehat{V};\, w(x)>\lambda\})d\lambda \\ &= (r-1)\int_0^{w_{\widehat{V}}} \lambda^{r-2} w(\{x\in\widehat{V};\, w(x)>\lambda\})d\lambda \\ &+ (r-1)\int_{w_{\widehat{V}}}^\infty \lambda^{r-2} w(\{x\in\widehat{V};\, w(x)>\lambda\})d\lambda \\ &\le w_{\widehat{V}}^{r-1} w(\widehat{V}) + C(r-1)\int_{w_{\widehat{V}}}^\infty \lambda^{r-1}\mu(\{x\in\widehat{V};\, w(x) > [w]_{A_1}^{-1}\lambda\})d\lambda \\ &\le \frac{w(\widehat{V})^r}{\mu(\widehat{V})^{r-1}} + C\frac{r-1}{r}[w]_{A_1}^r \int_{\widehat{V}} w^r d\mu. \end{aligned}$$

Choosing $r = 1+\delta$ close enough to 1, then the last term can be absorbed by the left-hand side and (2.5) follows. □

Corollary 2.6. *Let $w \in A_1(G)$. Then there are positive constants c and δ such that for every base set V*

$$\left(\frac{1}{\mu(V)}\int_V w^{1+\delta} d\mu\right)^{\frac{1}{1+\delta}} \le c\frac{1}{\mu(\widehat{V})}\int_{\widehat{V}} w d\mu. \tag{2.7}$$

Proof. After using (2.1), then from (2.5) we obtain (2.7).□

Corollary 2.7. *Let $w \in A_1(G)$. Then there are positive constants c and γ such that for every base set V and any measurable subset $E \subset V$*

$$\frac{w(E)}{w(V)} \le c\left(\frac{\mu(E)}{\mu(V)}\right)^\gamma . \tag{2.8}$$

Proof. Using $w \in A_1$ condition and (2.1) we obtain

$$\begin{aligned} w(\widehat{V}) &\le w(V^{**}) \le c\mu(V^{**})\text{ess inf}_{x\in V^{**}} w(x) \\ &\le c\mu(V)\text{ ess inf}_{x\in V} w(x) \le cw(V). \end{aligned}$$

Combining Hölder's inequality (1.2) and (2.7) we get, for any base set V and measurable subset $E \subset V$

$$w(E) = \int_E w d\mu \leq \left(\int_V w^{1+\delta} d\mu \right)^{1/(1+\delta)} \mu(E)^{\delta/(1+\delta)}$$
$$\leq c \left(\frac{\mu(E)}{\mu(V)} \right)^{\delta/(1+\delta)} w(\widehat{V}) \leq c \left(\frac{\mu(E)}{\mu(V)} \right)^{\delta/(1+\delta)} w(V).$$

□

3. Proof of Theorem 1.5

First, lets prove $(i) \Rightarrow (ii)$. Let $g \in L^0_+$ with $\|g\|_{X'} \leq 1$. Using an idea of Rubio de Francia , put

$$\mathcal{R}g(x) = \sum_{k=0}^{\infty} \frac{M^k g(x)}{(2\|M\|_{X'})^k}, \quad x \in G,$$

where M^k denotes the k-th iteration of M and $M^0 g = g$, and $|M\|_{X'}$ stands for the norm of the operator M on the X'. Then $g \leq \mathcal{R}g$ and $\|\mathcal{R}g\|_{X'} \leq 2$. Since M is sublinear, we have

$$M\mathcal{R}g(x) \leq \sum_{k=0}^{\infty} \frac{M^{k+1} g(x)}{(2\|M\|_{X'})^k} \leq 2\|M\|_{X'} \sum_{k=0}^{\infty} \frac{M^{k+1} g(x)}{(2\|M\|_{X'})^{k+1}}$$
$$\leq 2\|M\|_{X'} \sum_{k=0}^{\infty} \frac{M^k g(x)}{(2\|M\|_{X'})^k} \leq 2\|M\|_{X'} \mathcal{R}g(x),$$

whence $\mathcal{R}g \in A_1$ with $[\mathcal{R}g]_{A_1} \leq 2\|M\|_{X'}$.

For every sparse collection $\mathcal{S} \subset \mathcal{B}$, every family of pairwise disjoint measurable subsets $G_V \subset V$, $V \in \mathcal{S}$ and every family of non-negative numbers α_V, $V \in \mathcal{S}$ and every $g \in L^0_+$, using the properties of $\mathcal{R}g$ along with (2.8) one has

$$\int_G \left(\sum_{V \in \mathcal{S}} \alpha_V \chi_{G_V} \right) g d\mu \leq \sum_{V \in \mathcal{S}} \alpha_V \int_{G_V} g d\mu$$
$$\leq \sum_{V \in \mathcal{S}} \alpha_V \int_{G_V} \mathcal{R}g d\mu \leq C \sum_{V \in \mathcal{S}} \alpha_V \left(\frac{\mu(G_V)}{\mu(V)} \right)^{\gamma} \int_V \mathcal{R}g d\mu$$
$$\leq C \left(\max_{V \in \mathcal{S}} \frac{\mu(G_V)}{\mu(V)} \right)^{\gamma} \int_G \left(\sum_{V \in \mathcal{S}} \alpha_V \chi_V \right) \mathcal{R}g d\mu$$
$$\leq C \left(\max_{V \in \mathcal{S}} \frac{\mu(G_V)}{\mu(V)} \right)^{\gamma} \left\| \sum_{V \in \mathcal{S}} \alpha_V \chi_V \right\|_X \|\mathcal{R}g\|_{X'}$$
$$\leq C \left(\max_{V \in \mathcal{S}} \frac{\mu(G_V)}{\mu(V)} \right)^{\gamma} \left\| \sum_{V \in \mathcal{S}} \alpha_V \chi_V \right\|_X .$$

It remains to take here the supremum over all $g \in L^0_+$ with $\|g\|_{X'} \leq 1$ and use (1.3).

Turn to proof of $(ii) \Rightarrow (i)$. Let us show that there exists $C > 0$ such that for every $f \in L^1(G) \cap X'$, $f \geq 0$

$$\|Mf\|_{X'} \leq C\|f\|_{X'}. \tag{3.1}$$

Notice that (3.1) implies the boundedness of M on X'. Indeed, having (3.1) established, for arbitrary nonnegative $f \in X'$ we apply to $f_n = f\chi_{U_n}$ (clearly $f_n \in L^1(G) \cap X'$). Letting then $n \to \infty$ and using the Fatou property $(A3)$ of Definition 1.2, we obtain (3.1) holds for any $f \in X'$.

We consider only the case $\mu(G) = \infty$, the case $\mu(G) < \infty$ need minor modification. Suppose $f \in L^1(G) \cap X', f \geq 0$ and $||f||_{X'} \leq 1$. It is sufficient to prove that (see (1.3)) there exists a positive constant C (independent of f) such that for any non-negative function $g \in X$, with $\|g\|_X \leq 1$

$$\int_G Mf(x)g(x)d\mu \leq C. \tag{3.2}$$

Let A be an any number not less than $c_D^2 + 4$. For each integer k set

$$\Omega_k = \{x \in G \ : \ Mf(x) > A^k\}.$$

Denote $D_k = \Omega_k \backslash \Omega_{k+1}$. Let F_k be an arbitrary compact subset of D_k. We will prove that

$$\int_{\cup F_k} Mf(x)g(x)d\mu \leq C. \tag{3.3}$$

We recall that the group G is σ-compact since $G = \cup_{k \in \mathbb{Z}} \overline{U}_k$. Using the fact that μ is inner regular measure, by simple limiting argument from (3.3) we obtain (3.2).

Without loss of generality we may assume that $\mu(F_k) > 0$ for all $k \in \mathbb{Z}$. For $x \in F_k$ there exists a base set $V\alpha_x = y(x) + U_{\alpha_x}$, such that $|f|_{V_{\alpha_x}} > A^k$. Note that $\bigcup V\alpha_x \subset \Omega_k$ and $\alpha_x : F_k \to \mathbb{Z}$ bounded from above mapping (see [11], Theorem 3). We may select from collection $\{V_{\alpha_x}, x \in F_k\}$ a sequence finite or infinite of pairwise disjoint base sets $V_{_{k_j}}$ such that $F_k \subset \bigcup_j V^{**}_{k_j}$,(see Lemma 2.2).

Define the sets

$$E_1^k = V^{**}_{k_1} \cap F_k, \ E_j^k = (V^{**}_{k_j} \backslash \bigcup_{s<j} V^{**}_{k_s}) \cap F_k, \ j > 1.$$

Note that the sets E_j^k are pairwise disjoint and $\cup_j E_j^k = F_k$.

We give the following estimate

$$\mu(V_{k_j} \cap D_{k+l}) \leq \frac{\widetilde{C}}{A^{l-1}} \mu(V_{k_j}) \quad l \geq 3. \tag{3.4}$$

Let $x \in V_{k_j}$ and V be an arbitrary base set such that $x \in V$. Observe that either $V \subset V^{**}_{k_j}$ or $V_{k_j} \subset V^{**}$ (see Lemma 2.1). If the second inclusion holds, then $V^{**} \cap D_k \neq \emptyset$ and hence

$$|f|_V \leq c_D^2 |f|_{V^{**}} \leq c_D^2 \cdot A^{k+1} \leq A^{k+l} \ (l \geq 3).$$

Therefore, if $|f|_V > A^{k+l}$, $(l \geq 3)$ then $V \subset V^{**}_{k_j}$. From this and from weak type property of M, (see [11, Theorem 3]) we get (3.4). Indeed

$$\mu(V_{k_j} \cap D_{k+l}) \leq \mu\{x \in V_{k_j} \ : \ M(f\chi_{V^{**}_{k_j}})(x) > A^{k+l}\}$$

$$\leq \frac{C}{A^{k+l}} \int_{V^{**}_{k_j}} |f| d\mu \leq C \frac{\mu(V_{k_j})}{A^{k+l}} |f|_{V^{**}_{k_j}} \leq C \frac{A^{k+1}}{A^{k+l}} \mu(V_{k_j}) = \frac{\widetilde{C}}{A^{l-1}} \mu(V_{k_j}).$$

Using the fact that $\Omega_{k+l} = \bigcup_{i\geq k+l} D_i$, from (3.4) we deduce that there exists constant $\widetilde{C}$ such that

$$\mu(V_{k_j}\cap\Omega_{k+l}) \leq \frac{\widetilde{C}}{A^{l-1}}\mu(V_{k_j}),\quad l\geq 3. \tag{3.5}$$

From (3.5) we deduce that if we fix $A=\widetilde{A} > c_D^2+4$ such that $\widetilde{C}/\widetilde{A} < 1/2$, we obtain

$$\mu(V_{k_j}\backslash\Omega_{k+l}) \geq \frac{1}{2}\mu(V_{k_j})\ (l\geq 3) \tag{3.6}$$

for any pair of indexes (k,j).

Define

$$Tg(x) = \sum_{k\in\mathbb{Z}}\sum_{j}\left(\frac{1}{\mu(V_{k_j})}\int_{E_j^k} gd\mu\right)\chi_{V_{k_j}}(x).$$

Using the above definition, we get

$$\int_{\cup_k F_k} Mf(x)g(x)d\mu \leq A^{k+1}\sum_{k\in\mathbb{Z}}\sum_{j}\int_{E_j^k} gd\mu \leq A\sum_{k\in\mathbb{Z}}\sum_{j} f_{V_{k_j}}\int_{E_j^k} gd\mu$$
$$= A\int_G fTg \leq 2A\|f\|_{X'}\|Tg\|_X,$$

and consequently for proving (3.3), it is sufficient to show that $\|Tg\|_X \leq C$.

Let $\mathcal{K}$ be a finite index set of pairs (k,j) of integer numbers from definition of function Tg. Fix a natural number $\nu > 2$. Let $\mathbb{Z}_i$, $i=0,1,...,\nu-1$ be the equivalent classes of $\mathbb{Z}/\nu\mathbb{Z}$. Define the index sets $\mathcal{K}_i = \{(k,j):\ (k,j)\in\mathcal{K},\ k\in Z_i\}$ $(i=0,1,...,\nu-1)$. From (3.5) we deduce that if the collection $\{V_{k_j} : (k,j)\in\mathcal{K}_i\}$ $(i=0,1,...,\nu-1)$ is non-empty then it is sparse set.

Define

$$T_{\mathcal{K}}g(x) = \sum_{(k,j)\in\mathcal{K}}\left(\frac{1}{\mu(V_{k_j})}\int_{E_j^k} gd\mu\right)\chi_{V_{k_j}}(x).$$

Using the Fatou property for obtain the estimate $\|Tg\|_X \leq C$ it is sufficient to prove validity of the estimate $\|T_{\mathcal{K}}g\|_X \leq C$ for any finite subfamily $\mathcal{K}$ of indexes.

Take $\nu\in\mathbb{N}, \nu > 2$ such that

$$\nu C_{[\infty]}\sum_{l=\nu}^{\infty}\frac{\widetilde{C}^\gamma}{\widetilde{A}^{(l-1)\gamma}} \leq \frac{1}{2} \tag{3.7}$$

where the constants $C_{[\infty]}$ and γ are from (1.4).

Denote $\alpha_{k,j} = \frac{1}{\mu(V_{k_j})}\int_{E_j^k} fd\mu$. Then for all $x\in G$,

$$T_{\mathcal{K}}g(x) = \sum_{(k,j)\in\mathcal{K}}\left(\frac{1}{\mu(V_{k_j})}\int_{E_j^k} gd\mu\right)\chi_{V_{k_j}\backslash\Omega_{k+\nu}}(x) \tag{3.8}$$
$$+ \sum_{(k,j)\in\mathcal{K}}\left(\frac{1}{\mu(V_{k_j})}\int_{E_j^k} gd\mu\right)\chi_{V_{k_j}\cap\Omega_{k+\nu}}(x) = T_{\mathcal{K}}^1 g + T_{\mathcal{K}}^2 g,$$

we have

$$\Omega_k\setminus\Omega_{k+\nu} = \bigcup_{i=0}^{\nu-1}\Omega_{k+i}\setminus\Omega_{k+i+1}. \tag{3.9}$$

It is easy to see that if $x \in V_{k_j}$, then

$$\alpha_{j,k} \leq \frac{1}{\mu(V_{k_j})} \int_{V_{k_j}} g d\mu \leq Mg(x) \tag{3.10}$$

and from (3.9), (3.10), we get for $x \in G$

$$T^1_{\mathcal{K}} g(x) = \sum_{j,k} \alpha_{j,k} \chi_{V_{k_j} \setminus \Omega_{k+\nu}}(x) \leq Mg(x) \sum_k \chi_{\Omega_k \setminus \Omega_{k+\nu}}(x)$$

$$\leq Mg(x) \sum_{i=0}^{\nu-1} \sum_{k:\,(k,j)\in\mathcal{K}} \chi_{\Omega_{k+i} \setminus \Omega_{k+i+1}} \leq \nu Mg(x).$$

Consequently, we have

$$\|T^1_{\mathcal{K}} g\|_X \leq \nu \|Mg\|_X \leq \nu \|M\|_X \|g\|_X. \tag{3.11}$$

We have

$$T^2_{\mathcal{K}} g(x) = \sum_{i=0}^{\nu-1} \sum_{(k,j)\in\mathcal{K}_i} \alpha_{j,k} \chi_{V_{k_j} \cap \Omega_{k+\nu}} = \sum_{i=0}^{\nu-1} T^2_{\mathcal{K}_i} g(x). \tag{3.12}$$

For $x \in G$, we have

$$T^2_{\mathcal{K}_i} g(x) = \sum_{(k,j)\in\mathcal{K}_i} \alpha_{j,k} \chi_{V_{k_j} \cap \Omega_{k+\nu}} = \sum_{(k,j)\in\mathcal{K}_i} \alpha_{j,k} \sum_{l=\nu}^{\infty} \chi_{V_{k_j} \cap (\Omega_{k+l} \setminus \Omega_{k+l+1})}(x)$$

$$= \sum_{l=\nu}^{\infty} \sum_{(k,j)\in\mathcal{K}_i} \alpha_{j,k} \chi_{V_{k_j} \cap (\Omega_{k+l} \setminus \Omega_{k+l+1})}(x). \tag{3.13}$$

Applying (3.4) and the fact that $\{V_{k_j} : (k,j) \in \mathcal{K}_i\}$ is sparse collection, we obtain for all $l \geq \nu$,

$$\left\| \sum_{(j,k)\in\mathcal{K}_i} \alpha_{j,k} \chi_{V_{k_j} \cap (\Omega_{k+l} \setminus \Omega_{k+l+1})} \right\|_X$$

$$\leq C_{[\infty]} \left(\max_{(k,j)\in\mathcal{K}_i} \frac{\mu(V_{k_j} \cap (\Omega_{k+l} \setminus \Omega_{k+l+1})}{\mu(V_{k_j})} \right)^{\gamma} \|T_{\mathcal{K}_i} g\|_X \leq C_{[\infty]} \frac{\widetilde{C}^{\gamma}}{\widetilde{A}^{(l-1)\gamma}} \|T_{\mathcal{K}_i} g\|_X. \tag{3.14}$$

Combining above inequalities with (3.13), (3.14) and (3.7) we obtain

$$\|T^2_{\mathcal{K}_i} g\|_X \leq \frac{1}{2\nu} \|T_{\mathcal{K}_i} g\|_X \leq \frac{1}{2\nu} \|T_{\mathcal{K}} g\|_X \quad (i = 0, 1, ..., \nu - 1). \tag{3.15}$$

Combining (3.15), (3.11) and (3.12) we obtain

$$\|T_{\mathcal{K}} g\|_X \leq \nu \|\mathcal{M}\|_X \|g\|_X + \frac{1}{2} \|T_{\mathcal{K}} g\|_X.$$

Since $\mathcal{K}$ is finite, we obtain, $\|T_{\mathcal{K}} g\|_X < \infty$. Hence,

$$\|T_{\mathcal{K}} g\|_X \leq 2\nu \|\mathcal{M}\|_X \|f\|_X$$

and this completes the proof of the implication $(ii) \Rightarrow (i)$. $\square$

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Faculty of Exact and Natural Sciences, Javakhishvili Tbilisi State University, 13, University St., Tbilisi, 0143, Georgia
Email address: tengiz.kopaliani@tsu.ge

Faculty of Exact and Natural Sciences, Javakhishvili Tbilisi State University, 13, University St., Tbilisi, 0143, Georgia
Email address: aniozbetelashvili29@gmail.com