Transactions of A. Razmadze
Mathematical Institute
Vol. **1??** (202?), ?–??

TRIPLE WEAK SOLUTION TO THE $p(z)$-LAPLACIAN-LIKE PROBLEM WITH NON-FLOWING BOUNDARY CONDITIONS

YOUSSEF FADIL[1*], MOHAMED EL OUAARABI[2] AND MOHAMED OKESSOU[1]

Abstract. This paper considers a class of $p(z)$-Laplacian-like problems with indefinite weight and no flux boundary. We use variational techniques and Bonanno-Marano's critical point theorem to prove that there are at least three weak solutions to the problem in generalized Sobolev spaces.

1. Introduction

Let $\mathscr{G}$ be a smooth bounded domain in $\mathbb{R}^N (N \geq 2)$, where there is a Lipschitz boundary denoted by $\partial\mathscr{G}$ and $p, q, s \in C_+(\overline{\mathscr{G}})$ satisfy some supplementary conditions mentioned later, λ is a real parameter. The main objective of this manuscript is to prove that there are three possible solutions to a problem related to capillary phenomena

$$\begin{cases} -\mathscr{M}\Big(\Theta(\omega)\Big)\Big[\operatorname{div}\Big(|\nabla\omega|^{p(z)-2}\nabla\omega + \dfrac{|\nabla\omega|^{2p(z)-2}\nabla\omega}{\sqrt{1+|\nabla\omega|^{2p(z)}}}\Big) - |\omega|^{p(z)-2}\omega\Big] = \lambda h(z)|\omega|^{q(z)-2}\omega & \text{in } \mathscr{G}, \\ \omega = \text{constant} & \text{on } \partial\mathscr{G} \\ \displaystyle\int_{\partial\mathscr{G}} \Big(|\nabla\omega|^{p(z)-2} + \dfrac{|\nabla\omega|^{2p(z)-2}}{\sqrt{1+|\nabla\omega|^{2p(z)}}}\Big)\dfrac{\partial\omega}{\partial\nu} d\Gamma = 0 & \text{on } \partial\mathscr{G}. \end{cases} \tag{1.1}$$

Where

$$\Theta(\omega) := \int_{\mathscr{G}} \frac{1}{p(z)}\Big(|\nabla\omega|^{p(z)} + \sqrt{1+|\nabla\omega|^{2p(z)}} + |\omega|^{p(z)}\Big)dz,$$

h is in a generalized Lebesgue space $L^{s(z)}(\mathscr{G})$ and $\mathscr{M} : \mathbb{R}^+ \to \mathbb{R}^+$ are functions that satisfy some conditions which will be stated later.

A relatively new and rapidly growing area of mathematics is differential equations with variable exponents. They are an extension of classical differential equations where the exponent of the unknown function is a variable rather than a constant; for example, we refer to [12–14,18]. This seemingly small change can lead to a variety of new and challenging problems with applications in many different scientific and engineering fields, such as elastic mechanics (Zhikov [9]), thermoheological and electrorheological fluids (Ružička [8]), (Rajagopal-Ružička [7]), image restoration (Chen-Levine-Rao [3]).

Capillarity is the phenomenon whereby the surface of a liquid is raised or lowered where it touches a solid. For example, the surface of water in a clean drinking glass will be slightly higher at the edges where it touches the glass than in the centre. Recently, studying capillarity has become increasing popular. This growing interest is not only due to natural attractions. In addition to phenomena such as droplets, bubbles, and waves, they are also used in industrial, biomedical, and pharmaceutical applications and in microfluidic systems.We refer to: Zhou-Ge [10] and Ge [4], for more information about these operators.

2020 *Mathematics Subject Classification.* 35J50, 35G30, 35J35, 46E35.
Key words and phrases. Variable exponent Sobolev spaces, $p(z)$-Laplacian like problem, no-flux boundary condition, critical point theorem.
*Corresponding author.

Similar problems have been studied by several authors including in [2] Afrouzi-Shokooh-Chung, studied the existence and multiplicity of weak solutions for the following problem

$$-\operatorname{div}\Big(|\nabla w|^{p(z)-2}\nabla w+\frac{|\nabla w|^{2p(z)-2}\nabla w}{\sqrt{1+|\nabla w|^{2p(z)}}}\Big)=\lambda f(z,w) \text{ in } \mathscr{G},$$
$$w=0 \text{ on } \partial\mathscr{G},$$

Applying variational techniques.

The paper is split into three sections. In Section 2, we're going to go over some basic preliminaries concerning the functional framework in which we're going to deal with our problem. In section 3, we prepare the proof of the main theorem through the presentation of several conditions, as well as the proof of the existence of results for the problem (1.1).

2. Preliminary results and definitions

We need some basic properties of generalized Lebesgue-Sobolev spaces to discuss the problem (1.1). $L^{q(z)}(\mathscr{G})$ and $W^{1,q(z)}(\mathscr{G})$, see [16] for more details. Let $\mathscr{G}$ be a smooth bounded domain in $\mathbb{R}^N(N\geq 2)$, with a Lipschitz boundary denoted by $\partial\mathscr{G}$. Set

$$C_+(\overline{\mathscr{G}})=\Big\{q : q\in C(\overline{\mathscr{G}}) \text{ with } q(z)>1 \text{ for any } z\in\overline{\mathscr{G}}\Big\}.$$

For each $\eta>0,\ q\in C_+(\overline{\mathscr{G}})$, we define

$$q^+ := \max\Big\{q(z),\ z\in\overline{\mathscr{G}}\Big\} \text{ and } q^- := \min\Big\{q(z),\ z\in\overline{\mathscr{G}}\Big\}$$

and

$$[\eta]^q := \max\{\eta^{q^-},\eta^{q^+}\},\qquad [\eta]_q := \min\{\eta^{q^-},\eta^{q^+}\}.$$

So

$$[\eta]^{\frac{1}{q}} := \max\{\eta^{\frac{1}{q^-}},\eta^{\frac{1}{q^+}}\},\qquad [\eta]_{\frac{1}{q}} := \min\{\eta^{\frac{1}{q^-}},\eta^{\frac{1}{q^+}}\}.$$

Let us define by

$$\delta(z) := Sup\{\delta>0|B(z,\delta)\subseteq\mathscr{G}\}\ for\ all\ z\in\mathscr{G}$$

where B is a ball of radius δ centered at x . It's easy to see that there exists $x_0\in\mathscr{G}$ such that $B(z_0,D)\subseteq\mathscr{G}$, where $D=sup_{z\in\mathscr{G}}\delta(z)$.

For every $q\in C_+(\overline{\mathscr{G}})$, we define

$$L^{q(z)}(\mathscr{G})=\Big\{\omega:\mathscr{G}\to\mathbb{R} \text{ is measurable: } \int_{\mathscr{G}}|\omega(z)|^{q(z)}dz<+\infty\Big\},$$

equipped with the Luxemburg norm

$$|\omega|_{q(z)}=\inf\Big\{C>0:\int_{\mathscr{G}}\Big|\frac{\omega(z)}{C}\Big|^{q(z)}dz\leq 1\Big\}.$$

Next, we define $\rho_{q(z)}(\omega)=\int_{\mathscr{G}}|\omega(z)|^{q(z)}dz,\ \forall\ \omega\in L^{q(z)}(\mathscr{G})$. This gives us the following.

Proposition 1. *[15]. For all $(\omega_n),\omega\in L^{q(z)}(\mathscr{G})$, the following assertions hold:*

$$|\omega|_{q(z)}<1(resp.=1;>1) \text{ equivalent } \rho_{q(z)}(\omega)<1(resp.=1;>1), \tag{2.1}$$

$$|\omega|_{q(z)}>1 \text{ implies } |\omega|^{q^-}_{q(z)}\leq\rho_{q(z)}(\omega)\leq|\omega|^{q^+}_{q(z)}, \tag{2.2}$$

$$|\omega|_{q(z)}<1 \text{ implies } |\omega|^{q^+}_{q(z)}\leq\rho_{q(z)}(\omega)\leq|\omega|^{q^-}_{q(z)}, \tag{2.3}$$

$$\lim_{n\to\infty}|\omega_n-\omega|_{q(z)}=0 \text{ equivalent } \lim_{n\to\infty}\rho_{q(z)}(\omega_n-\omega)=0. \tag{2.4}$$

Proposition 2. *[4]. Let p and q be measurable functions such that $p \in L^{\infty}(\mathscr{G})$, and $1 \le p(z)q(z) \le \infty$, for a.e. $z \in \mathscr{G}$. Let $w \in L^{q(z)}(\mathscr{G}), w \ne 0$. Then*

$$\left[|w|_{p(z)q(z)}\right]_p \le \left||w|^{p(z)}\right|_{q(z)} \le \left[|w|_{p(z)q(z)}\right]^p.$$

Recall that the critical Sobolev exponent is defined as :

$$p^*(z) := \frac{Np(z)}{N-p(z)}, \quad \text{if } p(z) < N \text{ or } p^*(z) := +\infty, \quad \text{if } p(z) \ge N.$$

Remark 1. [6] . We shall denote the conjugate exponent of the function $s(z)$ by $s'(z)$ and set $\beta(z) := \frac{s(z)q(z)}{s(z)-q(z)}$. Then there exist continuous and compact embeddings $X \hookrightarrow L^{s'(z)q(z)}(\mathscr{G})$ and $X \hookrightarrow L^{\beta(z)}(\mathscr{G})$ and the constant $k > 0$ such that

$$|w|_{s'(z)q(z)} \le k\|w\|. \tag{2.5}$$

In the sequel, we recall the definition of a weak solution for problem (1.1) in order to formulate its variational approach.

Proposition 3. *[17]. The spaces $L^{q(z)}(\mathscr{G})$ is a separable and reflexive Banach spaces.*

Proposition 4. *[17]. The conjugate space of $L^{q(z)}(\mathscr{G})$ is $L^{q'(z)}(\mathscr{G})$ where $\frac{1}{q(z)} + \frac{1}{q'(z)} = 1$ for all $z \in \mathscr{G}$. For any $\omega \in L^{q(z)}(\mathscr{G})$ and $\phi \in L^{q'(z)}(\mathscr{G})$, we have the inequality of the Hölder type*

$$\Big|\int_{\mathscr{G}} \omega\phi \, dz\Big| \le \left(\frac{1}{q^-} + \frac{1}{q'^-}\right) |\omega|_{q(z)}|\phi|_{q'(z)} \le 2|\omega|_{q(z)}|\phi|_{q'(z)}. \tag{2.6}$$

Remark 2. If $T_1, T_2 \in C_+(\overline{\mathscr{G}})$ with $T_1(z) \le T_2(z)$ for any $z \in \overline{\mathscr{G}}$, then there exists the continuous embedding $L^{T_2(z)}(\mathscr{G}) \hookrightarrow L^{T_1(z)}(\mathscr{G})$.

Let now $q \in C_+(\overline{\mathscr{G}})$ and we define $W^{1,q(z)}(\mathscr{G})$ as

$$W^{1,q(z)}(\mathscr{G}) = \Big\{\omega \in L^{q(z)}(\mathscr{G}) \text{ such that } |\nabla\omega| \in L^{q(z)}(\mathscr{G})\Big\},$$

fitted with the norm

$$\|\omega\| = |\omega|_{q(z)} + |\nabla\omega|_{q(z)}.$$

Proposition 5. *[15, 17]. The space $\Big(W^{1,q(z)}(\mathscr{G}), \|\cdot\|\Big)$ is a reflexive and separable Banach space.*

We will try to find a weak solution in this paper in order to solve (1.1) in the following space

$$\mathscr{W} := \Big\{\omega \in W^{1,q(z)}(\mathscr{G}) : \quad \omega|_{\partial\mathscr{G}} = constant\Big\}. \tag{2.7}$$

The space $\mathscr{W}$ is a closed subspace of the separable and reflexive Banach space $W^{1,q(z)}(\mathscr{G})$, so $\mathscr{W}$ is also separable and reflexive Banach space (See [12]) with the norm $\|\cdot\|$.

Proposition 6. *[13]. If $\mathscr{X}$ is a reflexive Banach space, $\mathscr{Y}$ is a Banach space, $J : \mathscr{Z} \longrightarrow \mathscr{Y}$ is completely continuous, and $\mathscr{Z} \subset \mathscr{X}$ is nonempty, closed and convexe then J is compact.*

Proposition 7. *(Bonanno-Marano, [[1], Theorem 3.6]) .*
Let $\mathscr{W}$ be a reflexive real Banach space and $\Phi : \mathscr{W} \to \mathbb{R}$ a coercive, continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on $\mathscr{W}$. Let $\Psi : \mathscr{W} \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that

$$\textbf{(1):} \inf_{z\in\mathscr{W}} \Phi(z) = \Phi(0) = \Psi(0) = 0.$$

Assume that there exist $r > 0$ and $\bar{x} \in \mathscr{W}$, with $r < \Phi(\bar{x})$, such that:

$$\textbf{(2):} \quad \frac{\sup_{\Phi(z)\le r} \Psi(z)}{r} < \frac{\Psi(\bar{x})}{\Phi(\bar{x})};$$

(3): *for each* $\lambda \in \Lambda_r := \left(\frac{\Phi(\bar{x})}{\Psi(\bar{x})}, \frac{r}{\sup_{\Phi(z) \le r} \Psi(z)} \right)$, *the functional* $\Phi - \lambda\Psi$ *is coercive.*

Then for each $\lambda \in \Lambda_r$, *the functional* $\Phi - \lambda\Psi$ *has at least three distinct critical points in* $\mathscr{W}$.

3. Assumptions and main result

Firstly, we introduce the definition of a weak solution for (1.1).

Definition 1. We call that $\omega \in \mathscr{W}$ is a weak solution of problem (1.1) if

$$\mathscr{M}\Big(\Theta(\omega)\Big) \Big[\int\limits_{\mathscr{G}} \Big(|\nabla\omega|^{p(z)-2} + \frac{|\nabla\omega|^{2p(z)-2}}{\sqrt{1+|\nabla\omega|^{2p(z)}}} \Big) \nabla\omega\nabla v dz + \int\limits_{\mathscr{G}} |\omega|^{p(z)-2}\omega v dz \Big] = \lambda \int\limits_{\mathscr{G}} h(z)|\omega|^{q(z)-2}\omega v \, dz$$

for all $v \in \mathscr{W}$.

We will use the following assumptions :

(A_1): $\mathscr{M} : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a continuous and non-decreasing function satisfying there exists $r_2 \ge r_1 > 0$ and $\beta \ge \alpha > 1$ such that

$$r_1 t^{\alpha-1} \le M(t) \le r_2 t^{\beta-1}, \forall t \in \mathbb{R}^+$$

(A_2): Suppose that $h \in L^{s(z)}(\mathscr{G})$ and satisfy

$$h(z) := \begin{cases} \le 0, & \text{for } z \in \mathscr{G} \backslash B(z_0, D), \\ \ge h_0, & \text{for } z \in B\left(z_0, \frac{D}{2}\right), \\ > 0, & \text{for } z \in B(z_0, D) \backslash B\left(z_0, \frac{D}{2}\right), \end{cases}$$

where $B(z_0, D)$ is the ball centered at x_0 and of radius D, and h_0 is a positive constant.

$$(A_3) \;\; \textbf{:} \; 1 < \inf_{z \in \overline{\mathscr{G}}} q(z) \le \inf_{z \in \overline{\mathscr{G}}} p(z) < \alpha \inf_{z \in \overline{\mathscr{G}}} p(z), \text{ and } \sup_{z \in \overline{\mathscr{G}}} p(z) < N < s(z), \quad \forall z \in \mathscr{G}.$$

In the sequel, let

$$L := m\left(D^N - \left(\frac{D}{2}\right)^N\right), m := \frac{\pi^{\frac{N}{2}}}{\frac{N}{2}\Gamma\left(\frac{N}{2}\right)}.$$

Here Γ is the Euler function. Let $k > 0$ be the best constant for which the inequality (2.2) below holds.Our main result can be written as below.

Theorem 3.1. *Assume that assumptions* (A_3) *and* (A_2) *are fulfilled,* $d > 0$ *with*

$$r < \frac{r_1}{\alpha\,(p^+)^\alpha} \left(\left[\frac{2d}{D}\right]_p m \left(D^N - \left(\frac{D}{2}\right)^N\right)\right)^\alpha$$

and

$$\overline{w}_r := \frac{1}{r} \left\{ \frac{p^{+\frac{q^+}{p^-}}}{q^-} [k]^q |f|_{s(z)} \left[[r]^{\frac{1}{p}} \right]^p \right\}$$

$$< \quad \gamma_d := \frac{\frac{1}{q^+} f_0 [d]_q m \left(\frac{D}{2}\right)^N}{\frac{r_2}{\beta(p^-)^\beta} \left(2\left[\frac{2d}{D}\right]^p m \left(D^N - \left(\frac{D}{2}\right)^N\right) + \frac{1}{p^-}|\mathscr{G}| + [d]^p m D^N\right)^\beta}.$$

Then for every $\lambda \in \bar{\Lambda}_r := \left(\frac{1}{\gamma_d}, \frac{1}{\overline{w}_r}\right)$, *problem* (1.1) *has at least three weak solutions.*

Remark 3. For $r = 1$ conditions of Theorem 3.2 become as follows: There exists $d > 0$ such that

$$\frac{\alpha\,(p^+)^\alpha}{r_1} < \left(\left[\frac{2d}{D}\right]_p m\left(D^N - \left(\frac{D}{2}\right)^N\right)\right)^\alpha$$

and

$$\overline{w}_1 = \left\{\frac{p^{+\frac{q^+}{p^-}}}{q^-}[k]^q |f|_{s(z)}\right\} < \gamma_d.$$

Now, we present the proof of Theorem (3.1) by using Proposition (7). First of all, let us define

$$\Psi(w) := \int\limits_{\mathscr{G}} \frac{1}{q(z)} h(z)|w|^{q(z)} dx.$$

The corresponding energy functional of problem (1.1) is defined by $I_\lambda : X \to \mathbb{R}$,

$$I_\lambda(w) = \widehat{\mathscr{M}}(\Theta(w)) - \lambda\Psi(w), \forall w \in X,$$

where $\widehat{\mathscr{M}}(t) = \int_0^t \mathscr{M}(\xi)d\xi$. Put $\Phi(w) = \widehat{\mathscr{M}}(\Theta(w))$, we note that $\Phi \in C^1(z, \mathbb{R})$, moreover

$$\Phi'(w)(v) = \widehat{\mathscr{M}}(\Theta(w))\left[\int\limits_{\mathscr{G}} \left(|\nabla w|^{p(z)-2}\nabla w + \frac{|\nabla w|^{2p(z)-2}\nabla w}{\sqrt{1+|\nabla w|^{2p(z)}}}\right)\nabla v dx + \int\limits_{\mathscr{G}} |w|^{p(z)} wv dx\right]$$

for all $w, v \in \mathscr{W}$. Furthermore, Φ is coercive, since

$$\frac{\sqrt{1+|\nabla w|^{2p(z)}}}{p(z)} \geq \frac{1}{p(z)}|\nabla w|^{p(z)},$$

one has

$$\begin{aligned}\widehat{\mathscr{M}}(\Theta(w)) &\geq r_0 L(w)\\ &\geq \frac{r_1}{\alpha p^+}\int\limits_{\mathscr{G}} \left(|\nabla w|^{p(z)} + |w|^{p(z)}\right) dx.\end{aligned}$$

The above inequality and the statement (2.2) in Proposition 2.1 ensure that for $w \in X$ with $\|w\| > 1$ we have

$$\Phi(w) \geq \frac{r_1}{\alpha p^+}\|w\|^{\alpha p^-},$$

It follows that Φ is coercive. On the other hand, $L \in C^1(\mathscr{W}, \mathbb{R})$. By applying the same argument as in Lapa-Rivera-Broncano [5], it follows that Θ' is strictly monotone. Note that Φ is convex, sequentially weakly lower semi-continuous, and $\Phi' : \mathscr{W} \to \mathscr{W}^*$ is a homeomorphism (see [5]). Using Proposition (2), we conclude that Ψ is well-defined; in fact, for all $w \in \mathscr{W}$,

$$|\Psi(w)| \leq \frac{1}{q^-}\int\limits_{\mathscr{G}} |h(z)||w|^{q(z)}dx \leq \frac{1}{q^-}|h(z)|_{s(z)}||w||^{q(z)}_{s'(z)} \leq \frac{1}{q^-}|h(z)|_{s(z)}[|w|_{g'(z)q(z)}]^q.$$

In addition, because of the inequality (2.5), we have

$$|\Psi(w)| \leq \frac{1}{q^-}|h(z)|_{s(z)}[k||w||]^q,$$

this implies that Ψ is well-defined. Moreover, Ψ' is compact. Indeed, let $w_n \rightharpoonup w$ in $\mathscr{W}$. So w_n converge uniformly to w on $\mathscr{G}$,(see [19]). In the other part

$$\begin{aligned}|< \Psi'(w_n) - \Psi'(w), v >| &\leq \int\limits_{\mathscr{Q}} |f(z)|||w_n|^{q(z)-1} - |w|^{q(z)-1}||v \mid dz\\ &\leq |f(z)|_{s_1(z)}||w_n|^{q(z)-1} - |w|^{q(x)-1}|_{\frac{q(z)}{q(z)-1}}|v|_{\beta(z)}.\end{aligned}$$

By Remark 1 assures that

$$|< \Psi'(w_n) - \Psi'(w), v >| \to 0 \quad \text{as} \quad w \to \infty$$

Consequently, Ψ' is completely continuous. Finally, by proposition 6, Ψ' is compact.

Proof of Theorem 3.1.

As we saw earlier, the functionals Φ and Ψ meet the conditions of Proposition 7. Let $\omega_d \in \mathscr{W}$ such that

$$\omega_d := \begin{cases} 0, & \text{if } z \in \mathscr{G} \backslash B(z_0, D), \\ \frac{2d}{D}(D - |x - x_0|), & \text{if } z \in B(z_0, D) \backslash B\left(z_0, \frac{D}{2}\right), \\ d, & \text{if } z \in B\left(z_0, \frac{D}{2}\right), \end{cases}$$

where $|.|$ denotes the Euclidean norm in $\mathbb{R}^N$. By Proposition 2.1 and the conditions **(2)**, we have

$$\frac{r_1}{\alpha(p^+)^\alpha}\left(\int\limits_{B(z_0,D)\backslash B\left(z_0,\frac{D}{2}\right)} |\nabla\omega_d|\, dx\right)^\alpha \leq \Phi(\omega_d)$$

$$\leq \frac{r_2}{\beta(p^-)^\beta}\left(2\int\limits_{B(z_0,D)\backslash B\left(z_0,\frac{D}{2}\right)} |\nabla\omega_d|^{p(z)}\, dx + |\mathscr{G}| + \int\limits_{B(z_0,D)} |\omega_d|^{p(z)}\, dx\right)^\beta,$$

then

$$\frac{r_1}{\alpha(p^+)^\alpha}\left(\left[\frac{2d}{D}\right]_p m\left(D^N - \left(\frac{D}{2}\right)^N\right)\right)^\alpha \leq \Phi(\omega_d)$$

$$\leq \frac{r_2}{\beta(p^-)^\beta}\left(2\left[\frac{2d}{D}\right]^p m\left(D^N - \left(\frac{D}{2}\right)^N\right) + \frac{1}{p^-}|\mathscr{G}| + [d]^p m D^N\right)^\beta.$$

So

$$\frac{r_1}{\alpha(p^+)^\alpha}\left(\left[\frac{2d}{D}\right]_p m\left(D^N - \left(\frac{D}{2}\right)^N\right)\right)^\alpha \leq \Phi(\omega_d) \leq \frac{r_2}{\beta(p^-)^\beta}\left(2\left[\frac{2d}{D}\right]^p m\left(D^N - \left(\frac{D}{2}\right)^N\right) + \frac{1}{p^-}|\mathscr{G}| + [d]^p m D^N\right)^\beta.$$

Moreover, using (A_2), one has

$$\Psi(\omega_d) \geq \int\limits_{B\left(z_0,\frac{D}{2}\right)} \frac{h(z)}{q(z)}|\omega_d|^{q(z)}\, dx \geq \frac{1}{q^+}h_0[d]_q m\left(\frac{D}{2}\right)^N,$$

and hence

$$\frac{\Psi(\omega_d)}{\Phi(\omega_d)} \geq \frac{\frac{1}{q^-}h_0[d]_q m\left(\frac{D}{2}\right)^N}{\frac{r_2}{\beta(p^-)^\beta}\left(2\left[\frac{2d}{D}\right]^p m\left(D^N - \left(\frac{D}{2}\right)^N\right) + \frac{1}{p^-}|\mathscr{G}| + [d]^p m D^N\right)^\beta} = \gamma_d.$$

Next, from $r < \frac{r_1}{\alpha(p^+)^\alpha}\left(\left[\frac{2d}{D}\right]_p m\left(D^N - \left(\frac{D}{2}\right)^N\right)\right)^\alpha$, we get $r < \Phi(\omega_d)$. Now, for each $w \in \Phi^{-1}((-\infty, r])$, due to Proposition (1), one has

$$\frac{1}{p^+}[\|w\|]_p \leq r \tag{3.1}$$

Proposition 2, inequalities (3.1), and (2.5) now yield

$$\begin{aligned}\Psi(w) \leq \frac{1}{q^-}|h|_{s(z)}||w|^{q(z)}|_{s'(z)} &\leq \frac{1}{q^-}|h|_{s(z)}[k\|w\|]^q \\ &\leq \frac{1}{q^-}|h|_{g(z)}[k]^q\left[(p^+)^{\frac{1}{p^-}}[r]^{\frac{1}{p}}\right]^q \\ &\leq \frac{(p^+)^{\frac{q^+}{p^-}}}{q^-}[k]^q|h|_{s(z)}\left[[r]^{\frac{1}{p}}\right]^q.\end{aligned}$$

As a result

$$\frac{1}{r}\sup_{\Phi(w)\leq r}\Psi(w) \leq \overline{w}_r.$$

Next, we will show that for every positive number λ, the energy function $\Phi - \lambda\Psi$ is coercive. Based on Remark (1), one has

$$\Psi(w) \leq \frac{1}{q^-} \int_{\mathscr{G}} h(z)|w|^{q(z)}dx \leq \frac{1}{q^-}|h|_{s(z)}[k\|w\|]^q. \tag{3.2}$$

For $\|w\| > 1$, Proposition (1) and relation (3.2) give the following

$$\Phi(w) - \lambda\Psi(w) \geq \frac{1}{p^+}\|w\|^{\alpha p^-} - \lambda\frac{1}{q^-}|h|_{s(z)}[k\|w\|]^q.$$

Using the fact that $1 \leq q^- \leq q^+ < \alpha p^-$, we conclude that $\Phi(w) - \lambda\Psi(w)$ is coercive. Since

$$\bar{\Lambda} := \left(\frac{1}{\gamma_d}, \frac{1}{\overline{w}_r}\right) \subseteq \left(\frac{\Phi(\omega_d)}{\Psi(\omega_d)}, \frac{r}{\sup_{\Phi(w)\leq r}\Psi(w)}\right),$$

Proposition (7) establishes that for every $\lambda \in \bar{\Lambda}_r$, the functional $\Phi - \lambda\Psi$ admits at least three critical points in $\mathscr{W}$. These critical points correspond to weak solutions of problem (1.1). This completes the proof of Theorem (3.1).

Acknowledgement

The authors would like to thank the referees and the editor for their careful reading and valuable comments and suggestions on the first draft of the manuscript which improved the presentation and readability.

References

1. G. Bonanno and S. A. Marano, On the structure of the critical set of nondifferentiable functions with a weak compactness condition, Appl. Anal., **89** (2010), 1-10.
2. S.N. Antontsev and S.I. Shmarev, A model porous medium equation with variable exponent of nonlinearity: existence, uniqueness and localization properties of solutions, Nonlinear Anal., **60** (2005), 515-545.
3. Y. Chen, S. Levine, and M. Rao, Variable exponent, linear growth functionals in image processing, SIAM J. Appl. Math., **66** (2006), 1383-1406.
4. D. Edmunds and J. Rakosnik, Sobolev embeddings with variable exponent, Studia Math., 143 (2000), 267- 293.
5. E. C. Lapa, V. P. Rivera, J. Q. Broncano, No-flux boundary problems involving p(z)-Laplacian-like operators, Electron. J. Differential Equations, **219** (2015), 1-10.
6. K. Kefi, p(z)-Laplacian with indefinite weight, Proc. Amer. Math. Soc., **139** (2011), 4351-4360.
7. N.S. Papageorgiou, V.D. Rădulescu, D.D. Repovš, Nonlinear Analysis - Theory and Methods, Springer Monographs in Mathematics, Springer, Cham, 2019.
8. V.D. Rădulescu and D.D. Repovš, Partial Differential Equations with Variable Exponents: Variational Methods and Qualitative Analysis, Chapman and Hall/CRC, Taylor and Francis Group, Boca Raton, FL, 2015.
9. M.M. Rodrigues, Multiplicity of solutions on a nonlinear eigenvalue problem for p(z)-Laplacian-like operators, Mediterr. J. Math., **9** (2012), 211-223.
10. M. Ružička, Electrorheological Fluids: Modeling and Mathematical Theory, Lect. Notes Math. 1748, Springer, Berlin, 2000.
11. Q. M. Zhou and B. Ge, Three Solutions for Inequalities Dirichlet Problem Driven by p(z)-Laplacian-Like, Abstract and Applied Analysis, Hindawi Publishing Corporation, Cairo (2013).
12. M.M. Boureanu, D. Udrea. Existence and multiplicity results for elliptic problems with $p(z)$-growth conditions, Nonlinear Anal. Real World Appl., **14** (2013), 1829-1844.
13. E. Cabanillas, A. G. Aliaga, W. Barahona and G. Rodriguez, Existence of solutions for a class of $p(z)$-Kirchhoff type equation viatopological methods, J. Adv. Appl. Math. and Mech., **2** (2015): 64-72.
14. M. Mihăilescu, . On a class of nonlinear problems involving a $p(z)$-Laplace type operator. Czechoslovak Mathematical Journal ,**58** (2008) , 155-172.
15. X.L. Fan ,D. Zhao On the Spaces $L^{p(z)}(\mathscr{G})$ and $W^{m,p(z)}(\mathscr{G})$. J Math Anal Appl, **263** (2001), 424-446.
16. N.C. Eddine, P.D. Nguyen and M.A. Ragusa. Existence and multiplicity of solutions for a class of critical anisotropic elliptic equations of Schrodinger-Kirchhoff-type, Math. Methods Appl. Sci, **46**(16) (2023), 16782-16801.
17. O. Kováčik , J.Rákosník . On spaces $L^{p(z)}$ and $W^{1,p(z)}$, Czechoslovak Math. J., **41**(4) (1991), 592-618.
18. Y. Fadil, C. Allalou, M. Oukessou. Existence and uniqueness result for a Navier problem involving Leray-Lions type operators in weighted Sobolev spaces, Filomat, **38**(6) (2024), 2143-2155. art.n.8829268, (2023).
19. E. Zeidler, Nonlinar Functional Analysis and Its Applications, **2**, Springer, Berlin, Germany, 1985.

(Received ??.??.20??)

[1]Applied Mathematics and Scientific Computing Laboratory, Faculty of Sciences and Techniques, Sultan Moulay Slimane University, Beni Mellal, Morocco.

[2]Department of Mathematics and Informatics, Faculty of Sciences Ain Chock, Hassan II University, BP 5366, 20100, Casablanca, Morocco.

Email address: mohamedelouaarabi93@gmail.com

Email address: yfadil447@gmail.com