A new note on an extension of absolute summability

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Abstract In this paper, we have an absolute matrix generalization theorem by using quasi-β-power increasing sequences. This theorem also includes some new and known results dealing with some basic summability methods.

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1 Introduction

A positive sequence (b_n) is said to be almost increasing if there exists a positive increasing sequence (c_n) and two positive constants A and B such that $Ac_n \leq b_n \leq Bc_n$ (see [1]). A positive sequence $a = (a_n)$ is said to be a quasi-β-power increasing if there exists a constant $K = K(\beta, a) \geq 1$ such that

$$Kn^{\beta}a_n \geq m^{\beta}a_m$$

holds for $n \geq m$. If $Kn^{\beta}a_n \geq m^{\beta}a_m$ stays with $\beta = 0$ then a is called a quasi increasing sequence. It should be noted that every almost increasing sequence is a quasi-β-power increasing sequence for any nonnegative β, but the converse need not be true as can be seen by taking $a_n = n^{-\beta}$.

Let $\sum a_n$ be a given series with partial sums (s_n) and let (p_n) be a sequence of positive real numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty,$$

$(P_{-i} = p_{-i} = 0, \quad i \geq 1)$. The sequence-to-sequence transformation

$$w_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v,$$

defines the sequence (w_n) of the weighted mean or simply the $(\bar{N}, p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [4]). The series $\sum a_n$ is said to be summable $|\bar{N}, p_n; \delta|_k$, $k \geq 1$ and $\delta \geq 0$, if (see [3])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{\delta k+k-1} |w_n - w_{n-1}|^k < \infty.$$

If we take $\delta = 0$, then we have $|\bar{N}, p_n|_k$ summability method (see [2]).

Given a normal matrix $A = (a_{nv})$, i.e., a lower triangular matrix of nonzero diagonal entries, we associate two lower semimatrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ as follows:

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n, v = 0, 1, ...$$

and

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1, 2, ...$$

It may be noted that $\bar{A}$ and $\hat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v = \sum_{v=0}^{n} \bar{a}_{nv} a_v$$

and

$$\bar{\Delta} A_n(s) = \sum_{v=0}^{n} \hat{a}_{nv} a_v.$$

Let $A = (a_{nv})$ be a normal matrix. Then A defines a sequence-to-sequence transformation, mapping of the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \quad n = 0, 1, ...$$

A series $\sum a_n$ is said to be summable $|A, p_n; \delta|_k$, $k \geq 1$, and $\delta \geq 0$ if (see [5])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{\delta k+k-1} \left| \bar{\Delta} A_n(s) \right|^k < \infty,$$

where

$$\bar{\Delta} A_n(s) = A_n(s) - A_{n-1}(s).$$

If we take $\delta = 0$, then $|A, p_n; \delta|_k$ summability method is reduced to $|A, p_n|_k$ summability method. That is, A series $\sum a_n$ is said to be summable $|A, p_n|_k$, $k \geq 1$, if (see [6])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{k-1} \left| \bar{\Delta} A_n(s) \right|^k < \infty.$$

If we take weighted mean matrix,

$$a_{nv} = \begin{cases} \frac{p_v}{P_n}, & 0 \leq v \leq \text{n} \\ 0 & v > n, \end{cases}$$

where (p_n) is a sequence of positive numbers with $P_n = p_0 + p_1 + p_2 + ... + p_n \to \infty$ as $n \to \infty$, then $|A, p_n; \delta|_k$ summability method is reduced to $|\bar{N}, p_n; \delta|_k$ summability method and also if we put $\delta = 0$ and $a_{nv} = \frac{p_v}{P_n}$, then $|A, p_n; \delta|_k$ summability method is reduced to $\left|\bar{N}, p_n\right|_k$ summability method.
Let ω be the class of all matrices $A = (a_{nv})$ satisfying

$$\begin{aligned} &A \quad \text{is a normal matrix,} \\ &a_{n,v} \geq 0, \quad n, v = 0, 1, ..., \\ &\bar{a}_{n0} = 1, \quad n = 0, 1, ..., \\ &a_{n-1,v} \geq a_{nv}, \quad n \geq v + 1. \end{aligned}$$

2 The Known Results

Sulaiman [7] proved the following theorem dealing with $|\bar{N}, p_n|_k$ summability factors of infinite series.
Theorem 2.1 If the sequences (X_n) is a quasi-β-power increasing sequence $0 < \beta < 1$, and (λ_n) is a sequence of constants both satisfying conditions

$$\sum_{n=1}^{m} \frac{1}{n} P_n = O(P_m), \tag{1}$$

$$\lambda_n \to 0 \quad as \quad n \to \infty, \tag{2}$$

$$\sum_{n=1}^{\infty} n X_n(\beta) |\Delta|\Delta\lambda_n|| < \infty, \tag{3}$$

and

$$\sum_{n=1}^{m} \frac{1}{n(n^\beta X_n)^{k-1}} |t_n|^k = O(m^\beta X_m), \tag{4}$$

$$\sum_{n=1}^{m} \frac{p_n}{P_n} \frac{1}{(n^\beta X_n)^{k-1}} |t_n|^k = O(m^\beta X_m), \tag{5}$$

then the series $\sum_{n=1}^{\infty} a_n \lambda_n$ is summable $|\bar{N}, p_n|_k$, $k \geq 1$.

3 The Main Results

The aim of this paper is to generalize Sulaiman's theorem concerning $|\bar{N}, p_n|_k$ summability factors of infinite series to $|A, p_n; \delta|_k$ summability method which is more general matrix summability method. So, we have generalized Theorem 2.1 by using normal matrix as follow:

Theorem 3.1 Let $A \in \omega$ satisfying

$$a_{nn} = O\left(\frac{p_n}{P_n}\right) \tag{6}$$

$$\sum_{v=1}^{n-1} \frac{1}{v}|\hat{a}_{n,v}| = O(a_{nn}). \tag{7}$$

Let (X_n) be a quasi-β-power increasing sequence $0 < \beta < 1$ and (λ_n) be a sequence of constants. If the condition (1), (2) and (3) of Theorem 2.1 and

$$\sum_{n=v+1}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k} |\hat{a}_{n,v+1}| = O\left(\left(\frac{P_v}{p_v}\right)^{\delta k}\right) \quad m \to \infty, \tag{8}$$

$$\sum_{n=v+1}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k} |\Delta_v \hat{a}_{nv}| = O\left(\frac{p_v}{P_v}\left(\frac{P_v}{p_v}\right)^{\delta k}\right) \quad m \to \infty, \tag{9}$$

$$\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k} \frac{1}{n(n^\beta X_n)^{k-1}}|t_n|^k = O(m^\beta X_m), \quad m \to \infty, \tag{10}$$

$$\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k} \frac{p_n}{P_n}\frac{1}{(n^\beta X_n)^{k-1}}|t_n|^k = O(m^\beta X_m) \quad m \to \infty, \tag{11}$$

are satisfied, then the series

$$\sum_{n=1}^{\infty} a_n \lambda_n$$

is summable $|A, p_n; \delta|_k$, $k \geq 1$, $\delta \geq 0$.

The following lemma is required to prove our theorem.

Lemma 3.2 [7] Let (X_n) be a quasi-β-power increasing sequence such that $0 < \beta < 1$, $\lambda_n \to 0$ as $n \to \infty$ and $\sum_{n=1}^{\infty} nX_n(\beta)|\Delta|\Delta\lambda_n|| < \infty$ are satisfied. Then

$$n^{\beta+1}X_n|\Delta\lambda_n| = O(1) \quad as \quad n \to \infty,$$

$$\sum_{n=1}^{m} n^\beta X_n|\Delta\lambda_n| < \infty,$$

$$n^\beta X_n|\lambda_n| = O(1) \quad as \quad n \to \infty,$$

where $X_n(\beta) = n^\beta X_n$.

4 Proof of Theorem 3.1

Let (W_n) denotes the A-transform of the series $\sum a_n\lambda_n$. Then, by the definition, we have that

$$\bar{\Delta}W_n = \sum_{v=1}^{n} \hat{a}_{nv}a_v\lambda_v = \sum_{v=1}^{n} va_v v^{-1}\lambda_v.$$

Applying Abel's transformation to this sum, we have that

$$\bar{\Delta} W_n = \sum_{v=1}^{n-1} \Delta_v \left(\hat{a}_{nv}\lambda_v v^{-1}\right) \sum_{r=1}^{v} r a_r + \hat{a}_{nn}\lambda_n n^{-1} \sum_{v=1}^{n} v a_v$$

$$\bar{\Delta} W_n = \sum_{v=1}^{n-1} (v+1) t_v \left(\frac{1}{v(v+1)} \hat{a}_{nv}\lambda_v + \frac{1}{(v+1)} \Delta_v \hat{a}_{nv}\lambda_v + \frac{1}{(v+1)} \hat{a}_{n,v+1}\Delta\lambda_v \right) + \frac{n+1}{n} a_{nn}\lambda_n t_n$$

$$\bar{\Delta} W_n = \sum_{v=1}^{n-1} \hat{a}_{nv}\lambda_v v^{-1} t_v + \sum_{v=1}^{n-1} \Delta_v \hat{a}_{nv}\lambda_v t_v + \sum_{v=1}^{n-1} \hat{a}_{n,v+1} t_v \Delta\lambda_v + a_{nn}\lambda_n t_n \frac{n+1}{n}$$

by the formula for the difference of products of sequences (see [4]) we have

$$\bar{\Delta} W_n = W_{n,1} + W_{n,2} + W_{n,3} + W_{n,4}.$$

To complete the proof of Theorem 3.1, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k + k - 1} \mid W_{n,r} \mid^k < \infty, \quad \text{for} \quad r = 1, 2, 3, 4.$$

Firstly, in view of the hypothesis of Theorem 3.1 and Lemma 3.2, we have

$$\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k + k - 1} \mid W_{n,1} \mid^k = \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k + k - 1} \left| \sum_{v=1}^{n-1} \hat{a}_{n,v}\lambda_v t_v v^{-1} \right|^k$$

$$\leq \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k + k - 1} \left\{ \sum_{v=1}^{n-1} |\hat{a}_{n,v}| |\lambda_v|^k |t_v|^k \frac{1}{v} \right\} \times \left\{ \sum_{v=1}^{n-1} \frac{1}{v} |\hat{a}_{n,v}| \right\}^{k-1}$$

$$= O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k + k - 1} a_{nn}^{k-1} \sum_{v=1}^{n-1} |\hat{a}_{n,v}| \lambda_v|^k |t_v|^k \frac{1}{v}$$

$$= O(1) \sum_{v=1}^{m} \frac{1}{v} \frac{|\lambda_v| (v^\beta X_v |\lambda_v|)^{k-1} |t_v|^k}{(v^\beta X_v)^{k-1}} \sum_{n=v+1}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k} |\hat{a}_{n,v}|$$

$$= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^{\delta k} \frac{1}{v} \frac{|\lambda_v| |t_v|^k}{(v^\beta X_v)^{k-1}}$$

$$= O(1) \sum_{v=1}^{m-1} \left(\sum_{r=1}^{v} \left(\frac{P_r}{p_r}\right)^{\delta k} \frac{1}{r} \frac{|t_r|^k}{(r^\beta X_r)^{k-1}} \right) \Delta|\lambda_v| + O(1) \left(\sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^{\delta k} \frac{1}{v} \frac{|t_v|^k}{(v^\beta X_v)^{k-1}} \right) |\lambda_m|$$

$$= O(1) \sum_{v=1}^{m-1} v^\beta X_v |\Delta\lambda_v| + O(1) m^\beta X_m |\lambda_m| = O(1) \quad \text{as} \quad m \to \infty,$$

in view of the hypothesis of Theorem 3.1 and Lemma 3.2, we have

$$\begin{aligned}
&\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \mid W_{n,2}\mid^k = \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left|\sum_{v=1}^{n-1}\frac{v+1}{v}\Delta_v\hat{a}_{nv}\lambda_v t_v\right|^k \\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\sum_{v=1}^{n-1}|\Delta_v\hat{a}_{nv}||\lambda_v|^k|t_v|^k \times \left\{\sum_{v=1}^{n-1}|\Delta_v\hat{a}_{nv}|\right\}^{k-1} \\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k} a_{nn}^{k-1}\sum_{v=1}^{n-1}|\Delta_v\hat{a}_{nv}||\lambda_v|^k|t_v|^k \\
&= O(1)\sum_{v=1}^{m}|\lambda_v|^k|t_v|^k \times \sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}|\Delta_v\hat{a}_{nv}| \\
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}\frac{p_v}{P_v}|\lambda_v||t_v|^k\frac{(v^\beta X_v|\lambda_v|)^{k-1}}{(v^\beta X_v)^{k-1}} \\
&= O(1)\sum_{v=1}^{m-1}\left(\sum_{r=1}^{v}\left(\frac{P_r}{p_r}\right)^{\delta k}\frac{p_r}{P_r}\frac{|t_r|^k}{(r^\beta X_r)^{k-1}}\right)\Delta|\lambda_v| + O(1)\left(\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k}\frac{p_n}{P_n}\frac{|t_n|^k}{(n^\beta X_n)^{k-1}}\right)|\lambda_m| \\
&= O(1)\sum_{v=1}^{m-1}v^\beta X_v|\Delta\lambda_v| + O(1)m^\beta X_m|\lambda_m| \\
&= O(1) \quad \text{as} \quad m\to\infty,
\end{aligned}$$

in view of the hypothesis of Theorem 3.1 and Lemma 3.2, we obtain that

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\mid W_{n,3}\mid^k=\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left|\sum_{v=1}^{n-1}\hat{a}_{n,v+1}\Delta\lambda_v t_v\right|^k$$

$$\leq\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\Delta\lambda_v||t_v|\right)^k$$

$$=O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|\frac{|\Delta\lambda_v|}{(v^\beta X_v)^{k-1}}|t_v|^k\times\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|v^\beta X_v|\Delta\lambda_v|\right)^{k-1}$$

$$=O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}a_{nn}^{k-1}\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|\frac{|\Delta\lambda_v|}{(v^\beta X_v)^{k-1}}|t_v|^k\times\left(\sum_{v=1}^{n-1}v^\beta X_v|\Delta\lambda_v|\right)^{k-1}$$

$$=O(1)\sum_{v=1}^{m}\frac{|\Delta\lambda_v|}{(v^\beta X_v)^{k-1}}|t_v|^k\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}|\hat{a}_{n,v+1}|$$

$$=O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}\frac{v|\Delta\lambda_v|}{(v^\beta X_v)^{k-1}}\frac{|t_v|^k}{v}$$

$$=O(1)\sum_{v=1}^{m-1}\left(\sum_{r=1}^{v}\left(\frac{P_r}{p_r}\right)^{\delta k}\frac{1}{r}\frac{|t_r|^k}{(r^\beta X_r)^{k-1}}\right)\Delta(v|\Delta\lambda_v|)+O(1)\left(\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k}\frac{1}{n}\frac{|t_n|^k}{(n^\beta X_n)^{k-1}}\right)m|\Delta\lambda_m|$$

$$=O(1)\sum_{v=1}^{m-1}v^\beta X_v\Delta(v|\Delta\lambda_v|)+O(1)m^{\beta+1}X_m|\Delta\lambda_m|$$

$$=O(1)\sum_{v=1}^{m-1}v^\beta X_v|(v+1)\Delta|\Delta\lambda_v|-|\Delta\lambda_v||+O(1)m^{\beta+1}X_m|\Delta\lambda_m|$$

$$=O(1)\sum_{v=1}^{m-1}v^\beta X_v|\Delta\lambda_v|+O(1)\sum_{v=1}^{m-1}v^{\beta+1}X_v|\Delta|\Delta\lambda_v|+O(1)m^{\beta+1}X_m|\Delta\lambda_m|=O(1)\quad\text{as}\quad m\to\infty.$$

Finally,in view of the hypothesis of Theorem 3.1 and Lemma 3.2, we get

$$\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\mid W_{n,4}\mid^k\leq\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}a_{nn}^{k-1}a_{nn}|\lambda_n|^k|t_n|^k$$

$$=O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k+k+1}\left(\frac{p_n}{P_n}\right)^{k-1}\left(\frac{p_n}{P_n}\right)\frac{|t_n|^k}{(n^\beta X_n)^{k-1}}(n^\beta X_n|\lambda_n|)^{k-1}|\lambda_n|$$

$$=O(1)\sum_{n=1}^{m-1}\left(\sum_{v=1}^{n}\left(\frac{P_v}{p_v}\right)^{\delta k}\frac{p_v}{P_v}\frac{|t_v|^k}{(v^\beta X_v)^{k-1}}\right)\Delta|\lambda_n|+O(1)\left(\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k}\frac{p_n}{P_n}\frac{|t_n|^k}{(n^\beta X_n)^{k-1}}\right)|\lambda_m|$$

$$=O(1)\sum_{n=1}^{m-1}n^\beta X_n|\Delta\lambda_n|+O(1)m^\beta X_m|\lambda_m|=O(1)\quad as\quad m\to\infty.$$

This completes the proof of Theorem 3.1.

5 Conclusions

1. If we take $\delta=0$ and $a_{nv}=\frac{p_v}{P_n}$ in Theorem 3.1, then we have a result for $\left|\bar{N},p_n\right|_k$ summability method.
2. If we take $a_{nv}=\frac{p_v}{P_n}$ in Theorem 3.1, then we have a result for $\left|\bar{N},p_n;\delta\right|_k$ summability method.
3. If we take $\delta=0$ in Theorem 3.1, then we have a result for $|A,p_n|_k$ summability method (see [8]).

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