Characterizations for the G-Fractional Integral Operators in Generalized G-Morrey Spaces on $\mathbb{R}_+$

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Abstract. In this paper we study the boundedness of the G-fractional integral operators J_G^α on $\mathbb{R}_+ = [0, \infty)$ in the generalized G-Morrey spaces $\mathcal{M}_{p,\nu,\lambda}(\mathbb{R}_+)$. We give a characterization for the strong and weak type boundedness of J_G^α on the generalized G-Morrey spaces, respectively.

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1 Introduction

As is known, the maximal functions, singular integrals and potentials generated by different differential operators are important in applications and different questions of harmonic analysis and therefore, their study is very topical. Non-accidentally, there exits extensive literature devoted to different properties of the aforementioned object of harmonic analysis. We only cite those works that relate to the question considered in the paper. The reader can find detailed information in the mentioned paper [3].

Based on our investigation, Gegenbauer differential operator G was introduced in [2] $G \equiv G_\lambda = (x^2-1)\frac{d^2}{dx^2} + (2\lambda+1)x\frac{d}{dx},\ x \in [1, \infty),\ \lambda \in (0, \frac{1}{2})$. The shift operator A_{cht}^λ generated by G is given as follows [10]

$$A_{cht}^\lambda f(chx) = \frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda)\Gamma(\frac{1}{2})}\int_0^\pi f(chxcht - shxshtcos\varphi)(sin\varphi)^{2\lambda-1}d\varphi,$$

where $F(x) = \int_0^\infty e^{-t}t^{x-1}dt,\ x > 0$ is a Gamma function.

Let $H_r = (0,r),\ r \in (0,\infty)$. For any $E \subset \mathbb{R}_+ = [0,\infty)$ $\mu E = |E|_\lambda = \int_E sh^{2\lambda} t dt$. The Gegenbauer maximal operator M_G and the Gegenbauer potential I_G^α are introduced in [12] as follows:

$$M_G f(chx) = \sup_{r>0} \frac{1}{|H_r|_\lambda} \int_{H_r} A_{cht}^\lambda |f(chx)| sh^{2\lambda} t dt,$$

$$I_G^\alpha f(chx) = \frac{1}{\Gamma(\frac{\alpha}{2})} \int_0^\infty \left(\int_0^\infty r^{\frac{\alpha}{2}-1} h_r(cht) dr \right) A_{cht}^\lambda f(chx) sh^{2\lambda} t dt,$$

where

$$h_r(cht) = \int_1^\infty e^{-\nu(\nu+2\lambda)r} P_\nu^\lambda(cht)(\nu^2-1)^{\lambda-\frac{1}{2}} d\nu,\ 0 < \alpha < 2\lambda+1,$$

and

$$P_\nu^\lambda(cht) = \frac{\Gamma(\nu+2\lambda) cos\pi\lambda}{\Gamma(\lambda)\Gamma(\lambda+1)} (2cht)^{-\nu-2\lambda} {}_2F_1(\frac{\nu}{2}+\lambda, \frac{\nu}{2}+\lambda+\frac{1}{2}; \nu+\lambda+1; (cht)^{-2})$$

is eigen function of the operator G, and ${}_2F_1(\alpha,\beta;\gamma;x)$ is Gauss hypergeometric function.

We also consider the Gegenbauer fractional maximal operator M_G^α and Gegenbauer fractional integral J_G^α introduced in [16] as follows:

$$M_G^\alpha f(chx) = \sup_{r>0} \frac{1}{|H_r|_\lambda^{1-\frac{\alpha}{\gamma}}} \int_{H_r} A_{cht}^\lambda |f(chx)| sh^{2\lambda} t dt,\ 0 \le \alpha < \gamma \le 2\lambda+1,$$

$$J_G^\alpha f(chx) = \int_0^\infty |H_y|_\lambda^{\frac{\alpha}{\gamma}-1} A_{chy}^\lambda f(chx) sh^{2\lambda} y dy,\ 0 < \alpha < \gamma \le 2\lambda+1.$$

Note that $M_G^0 \equiv M_G$.

The operators M_G^α, I_G^α and J_G^α play important roles in harmonic analysis and applications.

In the present work, we study Spanne-Guliyev type boundedness of the operator J_G^α from $\mathcal{M}_{p,\omega_1,\gamma}(\mathbb{R}_+)$ to $\mathcal{M}_{q,\omega_2,\gamma}(\mathbb{R}_+)$, $1 < p < q < \infty$, and from $\mathcal{M}_{1,\omega_1,\gamma}(\mathbb{R}_+)$ to the weak $W\mathcal{M}_{q,\omega_2,\gamma}(\mathbb{R}_+)$, $1 < q < \infty$. We also study Adams-Guliyev type boundedness of the operator J_G^α from $\mathcal{M}_{p,\omega^{\frac{1}{p}},\gamma}(\mathbb{R}_+)$ to

$\mathcal{M}_{q,\omega^{\frac{1}{q}},\gamma}(\mathbb{R}_+),\ 1 < p < q < \infty$ and from the space $\mathcal{M}_{1,\omega_1,\gamma}(\mathbb{R}_+)$ to the weak space $W\mathcal{M}_{q,\omega^{\frac{1}{q}},\gamma}(\mathbb{R}_+),\ 1 < q < \infty$.
We shall give a characterization for the Spanne-Guliyev and Adams-Guliyev boundedness type of the operator J_G^{α} on the generalized G-Morrey space, including weak versions. These results are analogues of the paper [3]. Here and further ω, ω_1 and ω_2 will be positive measurable functions on $\mathbb{R}_+$.

Further, $A \lesssim B$ will mean that there exits some positive constant C, which may depend on some nonessential parameters such that $A \lesssim CB$. If $A \lesssim B$ and $B \lesssim A$, then we write $A \approx B$ and say that A and B are equivalent.

2 Notations and preleminary results

We denote by $L_{p,\lambda}(\mathbb{R}_+),\ 1 \le p \le \infty$ the space of functions $\mu_\lambda(x) = sh^{2\lambda}x-$ measurable on $\mathbb{R}_+$ with the finite norm

$$\|f\|_{L_{p,\lambda}(\mathbb{R}_+)} = \left(\int_{\mathbb{R}_+} |f(cht)|^p d\mu_\lambda(t)\right)^{\frac{1}{p}},\ 1 \le p < \infty,\ d\mu_\lambda(t) = sh^{2\lambda}tdt,$$

$$\|f\|_{L_{\infty,\lambda}} = \|f\|_{L_\infty} = \underset{t\in\mathbb{R}_+}{esssup}|f(cht)|,\ p = \infty.$$

Let $f \in L_{p,\lambda}(\mathbb{R}_+),\ 1 \le p < \infty$. Then for any $y \in \mathbb{R}_+$ the following inequality (see [10], Lemma 2)

$$\|A^{\lambda}_{chy} f\|_{L_{p,\lambda}(\mathbb{R}_+)} \le \|f\|_{L_{p,\lambda}(\mathbb{R}_+)} \tag{2.1}$$

holds.

We also denote by $WL_{p,\lambda}(\mathbb{R}_+)$ the weak space defined as the set of locally integrable functions on $\mathbb{R}_+$ with the finite norm

$$\|f\|_{WL_{p,\lambda}(\mathbb{R}_+)} = \underset{t>0}{sup}\, t(|\{x \in \mathbb{R}_+ : |f(chx)| > t\}|_\lambda)^{\frac{1}{p}}\,.$$

The main objective of this paper is to obtain the results similar to [3].

In the study of local properties of solutions of partial differential equations, together with weighted Lebesgue spaces, Morrey spaces play an important role. They were introduced by C.Morrey in 1938 [18].

Consider the Riesz potential (see [1])

$$I_\alpha f(x) = \int\limits_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \ 0 < \alpha < n.$$

In [1], Adams studied the boundedness of the Riesz potential on the Morrey space and proved the following theorem.

Theorem A [1]. Let $0 < \alpha < n$ and $0 \le \lambda < n, \ 1 \le p < (n-\lambda)/\alpha$.

(1) If $1 < p < (n-\lambda)/\alpha$, then the condition

$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$$

is necessary and sufficient for the boundedness of I_α from $M_{p,\lambda}(\mathbb{R}^n)$ to $M_{q,\lambda}(\mathbb{R}^n)$.

(2) If $p = 1$, then the condition

$$1 - \frac{1}{q} = \frac{\alpha}{n-\lambda}$$

is necessary and sufficient for the boundedness of I_α from $M_{1,\lambda}(\mathbb{R}^n)$ to $WM_{q,\lambda}(\mathbb{R}^n)$.

In the work [5], there was introduced Gegenbauer-Morrey (G-Morrey) space associated with the Gegenbauer differential operator G as the set of locally integrable functions $f(chx), \ x \in \mathbb{R}_+$, with the finite norm

$$\|f\|_{L_{p,\nu,\lambda}(\mathbb{R}_+)} = \mathop{sup}_{x,r \in \mathbb{R}_+} \left(r^{-\nu} \int_0^r A^\lambda_{cht} |f(chx)|^p d\mu_\lambda(t) \right)^{\frac{1}{p}},$$

$1 \le p < \infty, \ 0 < \lambda < 1/2$ and $0 \le \nu \le 2\lambda + 1$, and also the weak space $WL_{p,\nu.\lambda}(\mathbb{R}_+)$ with the finite norm

$$\|f\|_{WL_{p,\nu,\lambda}(\mathbb{R}_+)} = \mathop{sup}_{r>0} r \sup_{\substack{x \in \mathbb{R}_+ \\ t>0}} \left(t^{-\nu} |\{y \in [0,t) : A^\lambda_{chy} |f(chx)| > r\}|_\lambda \right)^{\frac{1}{p}}.$$

The Hardy-Littlewood-Sobolev theorem for the Gegenbauer potential I_G^α in the G-Morrey space, which is analogous to the Theorem A is proved in [5].

Theorem B [5, Theorem 2.1]. Let $0 < \alpha < 2\lambda + 1, \ 0 < \nu < 2\lambda + 1 - \alpha$ and $1 \le p < (2\lambda + 1 - \nu)/\alpha$.

(i) If $1 < p < (2\lambda + 1 - \nu)/\alpha$, then the condition

$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{2\lambda + 1 - \nu}$$

is necessary and sufficient for the boundedness of I_G^α from $L_{p,\nu,\lambda}(\mathbb{R}_+)$ to $L_{q,\nu,\lambda}(\mathbb{R}_+)$.

(ii) If $p = 1 < (2\lambda + 1 - \nu)/\alpha$, then the condition

$$1 - \frac{1}{q} = \frac{\alpha}{2\lambda + 1 - \nu}$$

is necessary and sufficient for the boundedness of I_G^α from $L_{1,\nu,\lambda}(\mathbb{R}_+)$ to $L_{q,\nu,\lambda}(\mathbb{R}_+)$.

If we take $\nu = 0$ in the Theorem B, then we obtain the following result.

Theorem C [8, Theorem 3]. Suppose that $0 < \lambda < 1/2$, $0 < \alpha < 2\lambda + 1$, and that $1 \le p < (2\lambda + 1)/\alpha$.

(a) If $1 < p < (2\lambda + 1)/\alpha$, then the condition

$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{2\lambda + 1}$$

is necessary and sufficient for the boundedness of I_G^α from $L_{p,\lambda}(\mathbb{R}_+)$ to $L_{q,\lambda}(\mathbb{R}_+)$.

(b) If $p = 1$, then the condition

$$1 - \frac{1}{q} = \frac{\alpha}{2\lambda + 1}$$

is necessary and sufficient for the boundedness of I_G^α from $L_{1,\lambda}(\mathbb{R}_+)$ to $WL_{q,\lambda}(\mathbb{R}_+)$.

We denote $H_r = (0, r) \subset \mathbb{R}_+$. Further, we need the following relation (see [8])

$$|H_r|_\lambda = \int_{H_r} sh^{2\lambda} t dt \approx \left(sh\frac{r}{2}\right)^\gamma, \tag{2.2}$$

where $0 < \lambda < 1/2$ and

$$\gamma = \gamma_\lambda(r) = \begin{cases} 2\lambda + 1, & \text{if } \ 0 < r < 2, \\ 4\lambda, & \text{if } \ 2 \le r < \infty. \end{cases}$$

According to (2.2), in [16] the following concept was introduced.

Definition 2.1 Let $1 \le p < \infty,\ 0 < \lambda < 1/2$ and $0 \le \nu \le \gamma$. We denote by $L_{p,\nu,\lambda}(\mathbb{R}_+)$ G-Morrey space associated with the Gegenbauer differential operator G as the set of locally integrable functions $f(chx),\ x \in \mathbb{R}_+$ with the finite norm

$$\begin{aligned}
\|f\|_{L_{p,\nu,\lambda}(\mathbb{R}_+)} &= \sup_{\substack{x\in\mathbb{R}_+\\ r>0}} \left(|H_r|_\lambda^{-\frac{\nu}{\gamma}} \int_0^r A_{chy}^\lambda \, |f(chx)|^p \, d\mu_\lambda(y) \right)^{\frac{1}{p}} \\
&= \sup_{\substack{x\in\mathbb{R}_+\\ r\in(0,2)}} \left(|H_r|_\lambda^{-\frac{\nu}{2\lambda+1}} \int_0^r A_{chy}^\lambda |f(chx)|^p d\mu_\lambda(y) \right)^{\frac{1}{p}} \\
&+ \sup_{\substack{x\in\mathbb{R}_+\\ r\in[2,\infty)}} \left(|H_r|_\lambda^{-\frac{\nu}{4\lambda}} \int_0^r A_{chy}^\lambda |f(chx)|^p d\mu_\lambda(y) \right)^{\frac{1}{p}}.
\end{aligned}$$

Thus, $L_{p,\nu,\lambda}(\mathbb{R}_+) = \{f \in L_{p,\lambda}^{loc}(\mathbb{R}_+) : \|f\|_{L_{p,\nu,\lambda}(\mathbb{R}_+)} < \infty\}$.

In [17] it has been proven that if $L_{p,0,\lambda}(\mathbb{R}_+) = L_{p,\lambda}(\mathbb{R}_+),\ L_{p,\gamma,\lambda}(\mathbb{R}_+) = L_\infty(\mathbb{R}_+)$ for $\nu < 0$ or $\nu > \gamma$ then $L_{p,\nu,\lambda}(\mathbb{R}_+) = \Theta(\mathbb{R}_+)$, where Θ is the set of functions equivalent to zero on $\mathbb{R}_+$.

Definition 2.2. Let $1 \le p < \infty$ and $0 \le \nu \le \gamma$. We denote by $WL_{p,\nu,\lambda}(\mathbb{R}_+)$ the weak space $L_{p,\nu,\lambda}(\mathbb{R}_+)$ defined as the set of locally integrable functions $f(chx),\ x \in \mathbb{R}_+$ with the finite norm

$$\begin{aligned}
\|f\|_{WL_{p,\nu,\lambda}(\mathbb{R}_+)} &= \sup_{r>0} r \sup_{t>0,\ x\in\mathbb{R}_+} \left(\left(sh\frac{t}{2} \right)^{-\nu} \left| \{y \in [0,t) : A_{chy}^\lambda \, |f(chx)| > r\} \right|_\lambda \right)^{\frac{1}{p}} \\
&= \sup_{r>0} r \sup_{t>0,\ x\in\mathbb{R}_+} \left(\left(sh\frac{t}{2} \right)^{-\nu} \int\limits_{\left\{y\in[0,t): A_{chy}^\lambda |f(chx)|>r\right\}} sh^{2\lambda} y dy \right)^{\frac{1}{p}}.
\end{aligned}$$

Remark 2.3. In [12, Corollary 3.1] it has been proven that

$$|J_G^\alpha f(chx)| \lesssim \int_0^\infty \frac{|A_{chy}^\lambda f(chx)|}{(shy)^{2\lambda+1-\alpha}} d\mu_\lambda(y), \qquad 0 < \alpha < 2\lambda + 1.$$

According to (2.2), we can write

$$|J_G^\alpha f(chx)| \lesssim \int_0^\infty \frac{|A_{chy}^\lambda f(chx)|}{(shy)^{2\lambda+1-\alpha}} d\mu_\lambda(y) \lesssim \int_0^2 \frac{|A_{chy}^\lambda f(chx)|}{\left(sh\frac{y}{2}\right)^{2\lambda+1-\alpha}} d\mu_\lambda(y)$$

$$+\int_2^\infty \frac{|A_{chy}^\lambda f(chx)|}{\left(sh\frac{y}{2}\right)^{4\lambda-\alpha}} d\mu_\lambda(y) \lesssim \int_0^\infty \frac{A_{chy}^\lambda |f(chx)|}{|H_y|_\lambda^{1-\frac{\alpha}{\gamma}}} d\mu_\lambda(y) = J_G^\alpha\left(|f|\right)(chx). \quad (2.3)$$

From the inequality (see [16, Proposition 2.4]) $t \leq sht \leq e^A t$, where $t \in [0, A]$ and $A > 0$, it follows that Definition 1.1 and Definition 1.2 are equivalent to G-Morrey space introduced in [5].

This approach seemed natural to us, since it is an absolutely continuous measure of the intervals (0,2) and [2,∞), respectively. Therefore, the relation (2.2) was used in the formulation of the theorem as well as its proof.

Theorem 2.4 [16]. Let $0 < \alpha < \gamma_\lambda(r)$, $0 < \nu < \gamma_\lambda(r) - \alpha$ and $1 \leq p < (\gamma_\lambda(r) - \nu)/\alpha$.

(i) If $1 < p < (\gamma_\lambda(r) - \nu)/\alpha$, then the condition

$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{\gamma_\lambda(r) - \nu}$$

is necessary and sufficient for the boundedness of J_G^α from $L_{p,\nu,\lambda}(\mathbb{R}_+)$ to $L_{q,\nu,\lambda}(\mathbb{R}_+)$.

(ii) If $p = 1 < (\gamma_\lambda(r) - \nu)/\alpha$, then the condition

$$1 - \frac{1}{q} = \frac{\alpha}{\gamma_\lambda(r) - \nu}$$

is necessary and sufficient for the boundedness of J_G^α from $L_{1,\nu,\lambda}(\mathbb{R}_+)$ to $WL_{q,\nu,\lambda}(\mathbb{R}_+)$.

Note that from (2.3) under the conditions of Theorem 2.4 the boundedness of the Gegenbauer potential I_G^α follows from the boundedness G-fractional Gegenbauer integral J_G^α.

3 Generalized G-Morrey spaces

In this section we give the following generalization of the G-Morrey spaces.

Definition 3.1 Let $1 \leq p < \infty$ and $\omega(x,r)$ be a positive measurable function on $\mathbb{R}_+ \times (0,\infty)$. We defined generalized G-Morrey space $\mathcal{M}_{p,\omega,\gamma}(\mathbb{R}_+)$ associated with the Gegenbauer differential operator G for all functions $f \in L^{loc}_{p,\lambda}(\mathbb{R}_+)$ by the finite norm

$$\begin{aligned}\|f\|_{\mathcal{M}_{p,\omega,\gamma}(\mathbb{R}_+)} &= \sup_{\substack{x\in\mathbb{R}_+\\ r>0}} \omega(x,r)^{-1}\left(sh\frac{r}{2}\right)^{-\frac{\gamma}{p}}\|f\|_{L_{p,\lambda}(H_r)}\\ &= \sup_{\substack{x\in\mathbb{R}_+\\ r\in(0.2)}} \omega(x,r)^{-1}\left(sh\frac{r}{2}\right)^{-\frac{2\lambda+1}{p}}\|f\|_{L_{p,\lambda}(H_r)}\\ &+ \sup_{\substack{x\in\mathbb{R}_+\\ r\in[2,\infty)}} \omega(x,r)^{-1}\left(sh\frac{r}{2}\right)^{-\frac{4\lambda}{p}}\|f\|_{L_{p,\lambda}(H_r)}.\end{aligned}$$

We also defined the weak generalized G-Morrey space $W\mathcal{M}_{p,\omega,\gamma}(\mathbb{R}_+)$ of locally integrable functions $f(chx)$, $x \in \mathbb{R}_+$ with the finite norm

$$\begin{aligned}\|f\|_{W\mathcal{M}_{p,\omega,\gamma}(\mathbb{R}_+)} &= \sup_{\substack{x\in\mathbb{R}_+\\ r>0}} \omega(x,r)^{-1}\left(sh\frac{r}{2}\right)^{-\frac{\gamma}{p}}\|f\|_{WL_{p,\lambda}(H_r)}\\ &= \sup_{\substack{x\in\mathbb{R}_+\\ r\in(0.2)}} \omega(x,r)^{-1}\left(sh\frac{r}{2}\right)^{-\frac{2\lambda+1}{p}}\|f\|_{WL_{p,\lambda}(H_r)}\\ &+ \sup_{\substack{x\in\mathbb{R}_+\\ r\in[2,\infty)}} \omega(x,r)^{-1}\left(sh\frac{r}{2}\right)^{-\frac{4\lambda}{p}}\|f\|_{WL_{p,\lambda}(H_r)}.\end{aligned}$$

By analogy with [19] in the work [13] was introduced the following notation.

Definition 3.2. Let $0 \leq p < \infty$ and $\varphi(r) : \mathbb{R}_+ \to \mathbb{R}_+$ be a Lebesgue measurable function. The generalized Gegenbauer-Morrey (G-Morrey) space $M_{p,\varphi,\lambda}(\mathbb{R}_+)$ associated with the Gegenbauer differential operator G are the set of locally integrable functions $f(chx)$, $x \in \mathbb{R}_+$ with the finite

$$\|f\|_{M_{p,\varphi,\lambda}(\mathbb{R}_+)} \equiv \|f\|_{M_{p,\varphi,\lambda}} := \sup_{\substack{x\in\mathbb{R}_+\\ r>0}} \left(\frac{1}{\varphi(r)}\int_{H_r} A^{\lambda}_{cht}|f(chx)|^p d\mu_\lambda(t)\right)^{\frac{1}{p}},$$

and the weak G-Morrey space $WM_{p,\varphi,\lambda}(\mathbb{R}_+)$ are the set of locally integrable functions $f(chx),\ x \in \mathbb{R}_+$ with the finite norm

$$\|f\|_{WM_{p,\varphi,\lambda}(\mathbb{R}_+)} \equiv \|f\|_{WM_{p,\varphi,\lambda}}$$
$$= \sup_{r>0} r \sup_{\substack{x\in\mathbb{R}_+ \\ t>0}} \left(\frac{1}{\varphi(t)} \left| \left\{ y \in H_t : A^{\lambda}_{cht} |f(chx) > r \right\} \right| \right)^{\frac{1}{p}}$$
$$= \sup_{r>0} r \sup_{\substack{x\in\mathbb{R}_+ \\ r>0}} \left(\frac{1}{\varphi(t)} \int_{\{y\in H_t : A^{\lambda}_{cht}|f(chx)>r\}} d\mu_{\lambda}(y) \right)^{\frac{1}{p}}.$$

Let $0 < \delta \leq 1$. Assume that for any $r > 0$, $\varphi(r)$ satisfies the conditions.

$$r \leq t \leq 2r \Rightarrow \varphi(t) \approx \varphi(r), \tag{a}$$

$$\int_r^{\infty} \frac{\varphi(t)}{t^{\nu\delta+1}} dt \lesssim \begin{cases} r^{-(2\lambda+1)\delta}\varphi(r),\ \nu = 2\lambda+1,\ 0 < r < 2, \\ r^{-4\lambda\delta}\varphi(r),\ \nu = 4\lambda,\ 2 \leq r < \infty. \end{cases} \tag{b}$$

Theorem D [13]. Let $0 < \lambda < 1/2,\ 0 < \alpha < 2\lambda+1,\ 1 \leq p < \alpha/(2\lambda+1)$ and $1/p - 1/q = \alpha/(2\lambda+1)$. Assume that φ satisfies the conditions (a) and (b). Then

(i) If $p > 1$ and $f \in M_{p,\varphi,\lambda}(\mathbb{R}_+)$, then

$$\|J^{\alpha}_G f\|_{M_{q,\varphi^{p/q},\lambda}} \lesssim \|f\|_{M_{p,\varphi,\lambda}},$$

(ii) If $p = 1$ and $f \in M_{1,\varphi,\lambda}(\mathbb{R}_+)$, then

$$\|J^{\alpha}_G f\|_{WM_{q,\varphi,\lambda}(\mathbb{R}_+)} \lesssim \|f\|_{M_{1,\varphi,\lambda}(\mathbb{R}_+)}.$$

Lemma 3.3 [15]. Let $\omega(x, r)$ be a positive measurable function on $(\mathbb{R}_+)\times(0,\infty)$.

(i) If

$$\sup_{t<r<\infty} \left(sh\frac{r}{2} \right)^{-\frac{\gamma}{p}} \omega(x,r)^{-1} = \infty \tag{3.1}$$

is true for some $t > 0$ and any $x \in \mathbb{R}_+$, then $\mathcal{M}_{p,\omega,\gamma}(\mathbb{R}_+) = \Theta(\mathbb{R}_+)$.

(ii) If

$$\sup_{0<r<t} \omega(x,r)^{-1} = \infty \tag{3.2}$$

is true for some $t>0$ and any $x \in \mathbb{R}_+$, then $\mathcal{M}_{p,\omega,\gamma}(\mathbb{R}_+) = \Theta(\mathbb{R}_+)$.

Remark 3.4. We denote by Ω_p^γ the sets of all positive measurable functions ω on $\mathbb{R}_+ \times (0,\infty)$ such that for all $t \in (0,\infty)$
$\sup_{x\in\mathbb{R}_+} \left\| \frac{(sh\frac{r}{2})^{-\frac{\gamma}{p}}}{\omega(x,r)} \right\|_{L_\infty(t,\infty)} < \infty$ and $\sup_{x\in\mathbb{R}_+} \|\omega(x,r)^{-1}\|_{L_\infty(0,t)} < \infty$, respectively.
In the following as we keep in mind Lemma 3.3, we also assume $\omega \in \Omega_p^\gamma$.

A function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be almost increasing (resp. almost decreasing) if $\omega(r) \lesssim \omega(s)$ for $r \le s$. Let $1 \le p < \infty$. Denote by Φ_p^γ the set of all almost decreasing functions $\omega : \mathbb{R}_+ \to \mathbb{R}_+$, such that $t \to (sh\frac{t}{2})^{\frac{\gamma}{p}}\omega(t)$ is almost increasing.

Lemma 3.5 [15, Lemma 2.4]. Let $\omega \in \Phi_p^\gamma$, $1 \le p < \infty$, $H_0 = (0,r_0)$ and χ_{H_0} be the characteristic function of the interval H_0. Then $\chi_{H_0} \in \mathcal{M}_{p,\omega,\gamma(\mathbb{R}_+)}$.

Moreover,

$$\frac{1}{\omega(r_0)} \le \|\chi_{H_0}\|_{W\mathcal{M}_{p,\omega,\gamma}(\mathbb{R}_+)} \le \|\chi_{H_0}\|_{\mathcal{M}_{p,\omega,\gamma}(\mathbb{R}_+)} \lesssim \frac{1}{\omega(r_0)}.$$

4 G-fractional integral operator in the spaces $\mathcal{M}_{p,\omega,\lambda}(\mathbb{R}_+)$

4.1 Spanne-Guliyev type result

Denote $E_\gamma = \begin{cases} (0,2) & \text{if} \quad \gamma = 2\lambda+1 \\ [2,\infty) & \text{if} \quad \gamma = 4\lambda \end{cases}$ and let

$$\left(sh\frac{t}{2}\right)^{\frac{\gamma}{q}} \int_t^\infty \left(sh\frac{s}{2}\right)^{-\frac{\gamma}{q}-1} \|f\|_{L_{p,\lambda}(H_s)} \left(ch\frac{s}{2}\right) ds, \; t \in E_\gamma$$
$$= \begin{cases} \left(sh\frac{t}{2}\right)^{\frac{2\lambda+1}{q}} \int_t^\infty (sh\frac{s}{2})^{-\frac{2\lambda+1}{q}-1} \|f\|_{L_{p,\lambda}(H_s)} (ch\frac{s}{2}) ds, \; t \in (0,2), \\ (sh\frac{t}{2})^{\frac{4\lambda}{q}} \int_t^\infty (sh\frac{s}{2})^{-\frac{4\lambda}{q}-1} \|f\|_{L_{p,\lambda}(H_s)} (ch\frac{s}{2}) ds, \qquad t \in [2,\infty). \end{cases}$$

The following result is an analogue of the Theorem 4.2 from [3].

Theorem 4.1 Let $1 \le p < \infty,\ 0 < \alpha < \gamma_\lambda(r),\ f \in L^{loc}_{p,\lambda}(\mathbb{R}_+)$ and

$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{\gamma_\lambda(r)}. \tag{4.1}$$

Then for $p > 1$

$$\|J^\alpha_G f\|_{L_{q,\lambda}(H_t)} \lesssim \left(sh\frac{t}{2}\right)^{\frac{\gamma}{q}} \int_t^\infty \left(sh\frac{s}{2}\right)^{-\frac{\gamma}{q}-1} \|f\|_{L_{p,\lambda}(H_s)} \left(ch\frac{s}{2}\right) ds,\ t \in E_\gamma \tag{4.2}$$

and for $p = 1$

$$\|J^\alpha_G f\|_{WL_{q,\lambda}(H_t)} \lesssim \left(sh\frac{t}{2}\right)^{\frac{\gamma}{q}} \int_t^\infty \left(sh\frac{s}{2}\right)^{-\frac{\gamma}{q}-1} \|f\|_{L_{1,\lambda}(H_s)} \left(ch\frac{s}{2}\right) ds,\ t \in E_\gamma \tag{4.3}$$

Proof. For a given interval $H_t = (0, t) \subset \mathbb{R}_+$, we split the function f as $f = f_1 + f_2$, where $f_1 = f\chi_{H_t},\ f_2 = f\chi_{(H_t)^c} = f\chi_{(t,\infty)}$. Then

$$J^\alpha_G f(chx) = J^\alpha_G f_1(chx) + J^\alpha_G f_2(chx).$$

Let $1 < p < \infty,\ 0 < \alpha < \gamma/p$ and (4.1) hold. Since $f_1 \in L_{p,\lambda}(\mathbb{R}_+)$, by the boundedness of the operator J^α_G from $L_{p,\lambda}(\mathbb{R}_+)$ to $L_{q,\lambda}(\mathbb{R}_+)$ (see Theorem 2.4 by $\nu = 0$) it follows that

$$\|J^\alpha_G f_1\|_{L_{q,\lambda}(\mathbb{R}_+)} \lesssim \|f_1\|_{L_{p,\lambda}(\mathbb{R}_+)} = \|f\|_{L_{p.\lambda}(H_t)}$$

$$\lesssim \left(sh\frac{t}{2}\right)^{\frac{\gamma}{q}} \int_t^\infty \left(sh\frac{s}{2}\right)^{-\frac{\gamma}{q}-1} \|f\|_{L_{p.\lambda}(H_s)} \left(ch\frac{s}{2}\right) ds,\ t \in E_\gamma. \tag{4.4}$$

Estimate the integral $J^\alpha_G f_2$.

$$\begin{aligned} J^\alpha_G f_2(chx) &= \int_t^\infty A^\lambda_{chy}|f(chx)| \left(sh\frac{y}{2}\right)^{\alpha-\gamma} d\mu_\lambda(y) \\ &\lesssim \int_t^\infty A^\lambda_{chy}|f(chx)| \left(\int_y^\infty \left(sh\frac{s}{2}\right)^{\alpha-\gamma-1} \left(ch\frac{s}{2}\right) ds\right) d\mu_\lambda(y) \\ &= \int_t^\infty \left(\int_{t<y<s} A^\lambda_{chy}|f(chx)| d\mu_\lambda(y)\right) \left(sh\frac{s}{2}\right)^{\alpha-\gamma-1} \left(ch\frac{s}{2}\right) ds \\ &\lesssim \int_t^\infty \left(\int_{H_s} A^\lambda_{chy}|f(chx)| d\mu_\lambda(y)\right) \left(sh\frac{s}{2}\right)^{\alpha-\gamma-1} \left(ch\frac{s}{2}\right) ds. \end{aligned}$$

By using Hölder's inequality, (2.1) and (2.2), we have

$$\begin{aligned}\int_{H_s} A^\lambda_{chy}|f(chx)|d\mu_\lambda(y) &\lesssim \left(\int_{H_s} A^\lambda_{chy}|f(chx)|^p d\mu_\lambda(y)\right)^{\frac{1}{p}} \left(\int_{H_s} d\mu_{\lambda(y)}\right)^{\frac{1}{p'}} \\ &\lesssim \left(sh\frac{s}{2}\right)^{\frac{\gamma}{p'}} \|A^\lambda_{chy} f\|_{L_{p,\lambda}(H_s)} \\ &\lesssim \left(sh\frac{s}{2}\right)^{\gamma(1-\frac{1}{p})} \|f\|_{L_{p,\lambda}(H_s)},\end{aligned}$$

where $p + p' = pp'$.

Then, we obtain

$$\begin{aligned}J^\alpha_G f_2(chx) &\lesssim \int_t^\infty \left(sh\frac{s}{2}\right)^{\alpha-\frac{\gamma}{p}-1} \|f\|_{L_{p,\lambda}(H_s)} \left(ch\frac{s}{2}\right) ds \\ &= \int_t^\infty \left(sh\frac{s}{2}\right)^{-\frac{\gamma}{q}-1} \|f\|_{L_{p,\lambda}(H_s)} \left(ch\frac{s}{2}\right) ds.\end{aligned} \tag{4.5}$$

From (4.5) and (2.2), we get

$$\begin{aligned}&\|J^\alpha_G f_2\|_{L_{q,\lambda}(H_t)} \lesssim \|\chi_{H_t}\|_{L_{q,\lambda}(\mathbb{R}_+)} \int_t^\infty \left(sh\frac{s}{2}\right)^{-\frac{\gamma}{q}-1} \|f\|_{L_{p,\lambda}(H_s)} \left(ch\frac{s}{2}\right) ds \\ &= \|\chi_{H_t}\|_{L_{q,\lambda}(H_t)} \int_t^\infty \left(sh\frac{s}{2}\right)^{-\frac{\gamma}{q}-1} \|f\|_{L_{p,\lambda}(H_s)} \left(ch\frac{s}{2}\right) ds \\ &= |H_t|^{\frac{1}{q}}_\lambda \int_t^\infty \left(sh\frac{s}{2}\right)^{-\frac{\gamma}{q}-1} \|f\|_{L_{p,\lambda}(H_s)} \left(sh\frac{s}{2}\right) ds \\ &\approx (sh\frac{t}{2})^{\frac{\gamma}{q}} \int_t^\infty \left(sh\frac{s}{2}\right)^{-\frac{\gamma}{q}-1} \|f\|_{L_{p,\lambda}(H_s)} \left(sh\frac{s}{2}\right) ds,\ t \in E_\gamma.\end{aligned} \tag{4.6}$$

By combining (4.4) and (4.6), we obtain (4.2).

Finally, in the case of $p = 1$, by the boundedness of the operator J^α_G from $L_{1,\lambda}(\mathbb{R}_+)$ to $WL_{q,\lambda}(\mathbb{R}_+)$, (see Theorem 2.4 by $\nu = 0$) it follows that

$$\begin{aligned}&\|J^\alpha_G f_1\|_{WL_{q,\lambda}(H_t)} \lesssim \|f_1\|_{L_{1,\lambda}(H_t)} = \|f\|_{L_{1,\lambda}(H_t)} \\ &\lesssim \left(sh\frac{t}{2}\right)^{\frac{\gamma}{q}} \int_t^\infty \left(sh\frac{s}{2}\right)^{-\frac{\gamma}{q}-1} \|f\|_{L_{1,\lambda}(H_s)} \left(sh\frac{s}{2}\right) ds,\ t \in E_\gamma.\end{aligned} \tag{4.7}$$

Note that the inequality (4.6) is also true in the case of $p = 1$. Then by (4.6) and (4.7), we get the inequality (4.3). □

We will use the following statement on the boundedness of the weighted Hardy operator

$$H_w g(t) = \int_t^\infty g(s)w(s)ds, \quad 0 < t < \infty.$$

The following theorem is true:

Theorem 4.2 [3]. Let v_1, v_2 and w be weights on $\mathbb{R}_+$ and $v_1(t)$ be bounded outside a neighborhood of the origin. The inequality

$$\underset{t>0}{esssup}\, v_2(t) H_w g(t) \leq C \underset{t>0}{esssup}\, v_1(t) g(t) \tag{4.8}$$

holds for some $C > 0$ for all nonnegative and nondecreasing g on $\mathbb{R}_+$ if and only if

$$B := \underset{t>0}{esssup} \int_t^\infty \frac{w(s)ds}{\underset{s<r<\infty}{esssup}\, v_1(r)} < \infty.$$

Moreover, the value C=B is the best constant for (4.8).

Theorem 4.3. Let $1 \leq p < \infty$, $0 < \alpha < \gamma/p$, and the condition (4.1) hold. Also, let the pair (ω_1, ω_2), where $\omega_1 \in \Omega_p^\gamma$, $\omega_2 \in \Omega_q^\gamma$ satisfy the condition

$$\int_t^\infty \frac{\underset{r<s<\infty}{essinf} \omega_1(x,s)(sh\frac{s}{2})^{\frac{\gamma}{p}}}{(sh\frac{r}{2})^{\frac{\gamma}{q}+1}} \left(ch\frac{r}{2}\right) dr \lesssim \omega_2(x,t), \ t \in E_\gamma. \tag{4.9}$$

Then for $p > 1$ the operator J_G^α is bounded from $\mathcal{M}_{p,\omega_1,\gamma}(\mathbb{R}_+)$ to $\mathcal{M}_{q,\omega_2,\gamma}(\mathbb{R}_+)$ and for $p = 1$ the operator J_G^α is bounded from $\mathcal{M}_{1,\omega_1,\gamma}(\mathbb{R}_+)$ to $W\mathcal{M}_{q,\omega_2,\gamma}(\mathbb{R}_+)$.

Proof. By Theorems 4.1 and 4.2 with $v_2(r) = \omega_2(x,r)^{-1}$, $v_1(r) = \left(sh\frac{r}{2}\right)^{-\frac{\gamma}{p}} \omega_1(x,r)^{-1}$, $w(r) = \left(sh\frac{r}{2}\right)^{-\frac{\gamma}{q}-1} ch\frac{r}{2}$, $g(r) = \|f\|_{L_{p,\lambda}(H_r)}$, we have

$$\begin{aligned}
\|J_G^\alpha f\|_{\mathcal{M}_{q,\omega_2,\gamma}} &= \sup_{\substack{x\in\mathbb{R}_+\\ t\in E_\gamma}} \omega_2(x,t)^{-1} \left(sh\frac{t}{2}\right)^{-\frac{\gamma}{q}} \|J_G^\alpha f\|_{L_{q,\lambda}(H_t)} \\
&\lesssim \sup_{\substack{x\in\mathbb{R}_+\\ t\in E_\gamma}} \omega_2(x,t)^{-1} \int_t^\infty \left(sh\frac{r}{2}\right)^{-\frac{\gamma}{q}-1} \left(ch\frac{r}{2}\right) \|f\|_{L_{p,\lambda}(H_r)} dr \\
&\lesssim \sup_{\substack{x\in\mathbb{R}_+\\ t\in E_\gamma}} \omega_1(x,t)^{-1} \left(sh\frac{t}{2}\right)^{-\frac{\gamma}{p}} \|f\|_{L_{p,\lambda}(H_t)} = \|f\|_{\mathcal{M}_{p,\omega_1,\gamma}},
\end{aligned}$$

and for $p>1$, we have

$$\begin{aligned}\|J_G^{\alpha}\|_{W\mathcal{M}_{q,\omega_2,\gamma}} &\lesssim \sup_{\substack{x\in\mathbb{R}_+\\ t\in E_\gamma}} \omega_2(x,t)^{-1}\int_t^{\infty}\left(sh\frac{r}{2}\right)^{-\frac{\gamma}{q}-1}\left(ch\frac{r}{2}\right)\|f\|_{L_{1,\lambda}(H_r)}dr\\ &\lesssim \sup_{\substack{x\in\mathbb{R}_+\\ t\in E_\gamma}} \omega_1(x,t)^{-1}\left(sh\frac{t}{2}\right)^{-\gamma}\|f\|_{L_{1,\lambda}(H_t)}=\|f\|_{\mathcal{M}_{1,\omega_1,\gamma}}\end{aligned}$$

for $p=1$. □

Remark 4.4. Note that the analogue of Theorems 4.1 and 4.3 is proved by another method for the Gegenbauer potential I_G^{α} in [14].

In order to prove our main result, we need the following estimate.

Lemma 4.5. Let $H_0=(0,r_0)$, then

$$\left(sh\frac{r}{2}\right)^{\alpha}\lesssim J_G^{\alpha}\chi_{H_0}(chx)$$

for every $x\in H_0$.

Proof. By (2.2), we have

$$\begin{aligned}J_G^{\alpha}\chi_{H_0}(chx) &= \int_0^{\infty}\frac{A_{cht}^{\lambda}\chi_{H_0}(chx)}{(sh\frac{t}{2})^{\gamma-\alpha}}d\mu_\lambda(t)=\int_0^{r_0}\left(sh\frac{t}{2}\right)^{\alpha-\gamma}sh^{2\lambda}tdt\\ &\gtrsim (sh\frac{r_0}{2})^{\alpha-\gamma}\left(sh\frac{r_0}{2}\right)^{\gamma}\gtrsim\left(sh\frac{r_0}{2}\right)^{\alpha}.\end{aligned}$$

□

The following theorem is one of our main results.

Theorem 4.6. Let $0<\alpha<\gamma,\ p,q\in[1,\infty),\ \omega_1\in\Omega_p^{\gamma},\ \omega_2\in\Omega_q^{\gamma},\ 1\le p<\frac{\gamma}{\alpha}$ and $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{\gamma}$.

If $\omega_1\in\Phi_p^{\gamma}$ satisfies the condition

$$\int_t^{\infty}\left(sh\frac{r}{2}\right)^{\alpha-1}\left(ch\frac{r}{2}\right)\omega_1(r)dr\lesssim\left(sh\frac{t}{2}\right)^{\alpha}\omega_1(t),\ t\in E_\gamma \tag{4.10}$$

then the condition

$$\left(sh\frac{t}{2}\right)^{\alpha}\omega_1(t)\lesssim\omega_2(t),\ t\in E_\gamma \tag{4.11}$$

is necessary and sufficient for the boundedness of J_G^α from $\mathcal{M}_{1,\omega_1,\gamma}(\mathbb{R}_+)$ to $W\mathcal{M}_{q,\omega_2,\gamma}(\mathbb{R}_+)$ by $p=1$.

Moreover, if $1<p<\gamma/\alpha$, then the condition (4.11) is necessary and sufficient for the boundedness J_G^α from $\mathcal{M}_{p,\omega_1,\gamma}(\mathbb{R}_+)$ to $\mathcal{M}_{q,\omega_2,\gamma}(\mathbb{R}_+)$.

Proof. **Sufficiency.** Let (4.11) hold, then (4.10) implies (4.9) and by Theorem 4.3 J_G^α is bounded from $\mathcal{M}_{p,\omega_1,\gamma}(\mathbb{R}_+)$ to $\mathcal{M}_{q,\omega_2,\gamma}(\mathbb{R}_+)$

Necessity. Now let J_G^α be bounded from $\mathcal{M}_{p,\omega_1,\gamma}(\mathbb{R}_+)$ to $\mathcal{M}_{q,\omega_2,\gamma}(\mathbb{R}_+)$. Let $H_0=(0,r_0)$ and $x\in H_0$, then by Lemmas 3.4 and 4.5, we have

$$\begin{aligned}\left(sh\frac{t_0}{2}\right)^\alpha &\lesssim |H_0|_\lambda^{-\frac{1}{q}}\|J_G^\alpha\chi_{H_0}\|_{L_{q,\lambda(H_0)}}\lesssim\omega_2(t_0)\|J_G^\alpha\chi_{H_0}\|_{\mathcal{M}_{q,\omega_2,\gamma}}\\ &\lesssim \omega_2(t_0)\|\chi_{H_0}\|_{\mathcal{M}_{p,\omega_1,\gamma}}\lesssim\frac{\omega_2(t_0)}{\omega_1(t_0)},\end{aligned}$$

or

$$\left(sh\frac{t_0}{2}\right)^\alpha\lesssim\frac{\omega_2(t_0)}{\omega_1(t_0)}\Leftrightarrow\left(sh\frac{t_0}{2}\right)^\alpha\omega_1(t_0)\lesssim\omega_2(t_0)$$

for all $t_0>0$.

Since this is true for every $t_0>0$, we are done . □

Remark 4.7. If we take $\omega_1(t)=(sh\frac{t}{2})^{\frac{\nu-\gamma}{p}}$ and $\omega_2(t)=(sh\frac{t}{2})^{\frac{\mu-\gamma}{q}}$ in Theorem 4.6, then the conditions (4.11) and (4.10) are equivalent to $0<\nu<\gamma-\alpha p$ and $\frac{\nu}{p}=\frac{\mu}{q}$ respectively.

Corollary 4.8 [17]. Let $0<\alpha<\gamma,\ 1\le p<\gamma/\alpha,\ 0<\nu<\gamma-\alpha p$, and (4.1) hold. Then the operator J_G^α is bounded from $L_{1,\nu,\lambda}(\mathbb{R}_+)$ to $WL_{q,\mu,\lambda}(\mathbb{R}_+)$ if and only if $\nu=\mu/q$.

Moreover, if $1<p<\gamma/\alpha$, then the operator J_G^α is bounded from $L_{p,\nu,\lambda}(\mathbb{R}_+)$ to $L_{q,\mu,\lambda}(\mathbb{R}_+)$ if and only if $\nu/p=\mu/q$.

4.2 Adams-Guliyev type results

The following pointwise estimate plays a key role in proving our main results.

Theorem 4.9. Let $1\le p<\infty,\ \ 0<\alpha<\gamma$ and $f\in L_{p,\lambda}^{loc}(\mathbb{R}_+)$. Then

$$J_G^\alpha f(chx)\lesssim\left(sh\frac{t}{2}\right)^\alpha M_Gf(chx)+\int_t^\infty\left(sh\frac{r}{2}\right)^{\alpha-\frac{\gamma}{p}-1}\left(ch\frac{r}{2}\right)\|f\|_{L_{p,\lambda}(H_r)}dr,\ t\in E_\gamma \quad (4.12)$$

Proof. Write $f = f_1 + f_2$, where $f_1 = f\chi_{(H_t)}$ and $f_2 = f\chi_{(H_t)^c}$. Then

$$J_G^\alpha f(chx) = J_G^\alpha f_1(chx) + J_G^\alpha f_2(chx).$$

First, we show that

$$|J_G^\alpha f_1(chx)| \lesssim \left(sh\frac{t}{2}\right)^\alpha M_G f(chx), \tag{4.13}$$

for all $x \in \mathbb{R}_+$

In fact,

$$\begin{aligned} J_G^\alpha f_1(chx) &= \int_0^t \frac{A_{cht}^\lambda |f(chx)|}{(sh\frac{y}{2})^{\gamma-\alpha}} d\mu_\lambda(y) \\ &\lesssim \sum_{k=0}^\infty \int_{2^{-k-1}t}^{2^{-k}t} \frac{A_{cht}^\lambda |f(chx)|}{(sh\frac{y}{2})^{\gamma-\alpha}} d\mu_\lambda(y) \\ &\le \sum_{k=0}^\infty \frac{sh^\alpha\left(\frac{t}{2^{k+1}}\right)}{sh^\gamma\left(\frac{t}{2^{k+1}}\right)} \int_0^{\frac{t}{2^k}} A_{chy}^\lambda |f(chx)| d\mu_\lambda(y) \\ &\lesssim \left(sh\frac{t}{2}\right)^\alpha M_G f(chx) \sum_{k=0}^\infty 2^{-k\alpha} \lesssim \left(sh\frac{t}{2}\right)^\alpha M_G f(chx). \end{aligned}$$

For $J_G^\alpha\ f_2(chx)$ with $x \in \mathbb{R}_+$, from (4.5), we have

$$\begin{aligned} J_G^\alpha\ f_2(chx) &\lesssim \int_t^\infty \left(sh\frac{r}{2}\right)^{-\frac{\gamma}{q}-1} \left(ch\frac{r}{2}\right) \|f\|_{L_{p,\lambda}(H_r)} dr \\ &= \int_t^\infty \left(sh\frac{r}{2}\right)^{\alpha-\frac{\gamma}{p}-1} \left(ch\frac{r}{2}\right) \|f\|_{L_{p,\lambda}(H_r)} dr,\ t \in E_\gamma. \end{aligned}$$

From this and (4.13), we obtain (4.12). □

Theorem E [15]. Suppose $1 \le p < \infty$, $\omega_1 \in \Omega_p^\gamma$, $\omega_2 \in \Omega_q^\gamma$. Moreover, let the condition (4.1) holds, also the pair (ω_1, ω_2) satisfy the condition

$$\sup_{s>r} \left(sh\frac{s}{2}\right)^\alpha \omega_1(x,s) \lesssim \omega_2(x,r).$$

Then M_G^α is bounded from $\mathcal{M}_{p,\omega_1,\gamma}(\mathbb{R}_+)$ to $\mathcal{M}_{q,\omega_2,\gamma}(\mathbb{R}_+)$ for $p > 1$ and M_G^α is bounded from $\mathcal{M}_{1,\omega,\gamma}(\mathbb{R}_+)$ to $W\mathcal{M}_{q,\omega_2,\gamma}(\mathbb{R}_+)$.

By using Theorem E when $\alpha = 0$ (then $q = p$) and $\omega_1(x,r) = \omega_2(x,r) = \omega(x,r)$, we get the following corollary.

Corollary 4.10. Let $1 \le p < \infty$ and ω satisfy the condition

$$\sup_{s>r>0} \omega(x,s) \lesssim \omega(x,r). \tag{4.14}$$

Then M_G is bounded from $\mathcal{M}_{p,\omega,\gamma}(\mathbb{R}_+)$ to $\mathcal{M}_{p,\omega,\gamma}(\mathbb{R}_+)$ for $p > 1$ and M_G is bounded from $\mathcal{M}_{1,\omega,\gamma}(\mathbb{R}_+)$ to $W\mathcal{M}_{1,\omega,\gamma}(\mathbb{R}_+)$ for $p = 1$.

Theorem 4.11.(Adams-Guliyev type results). Let $1 \le p < q < \infty$, $0 < \alpha < \gamma/p$ and let $\omega \in \Omega_p^\gamma$ satisfy condition (4.14) and

$$\int_r^\infty \left(sh\frac{t}{2}\right)^{\alpha-1} \left(ch\frac{t}{2}\right) \omega(x,t)^{\frac{1}{p}} dt \lesssim \left(sh\frac{r}{2}\right)^{-\frac{\alpha p}{q-p}}, \ r > 0. \tag{4.15}$$

Then for $p > 1$, the operator J_G^α is bounded from $\mathcal{M}_{p,\omega^{\frac{1}{p}},\gamma}(\mathbb{R}_+)$ to $\mathcal{M}_{q,\omega^{\frac{1}{q}},\gamma}(\mathbb{R}_+)$ and for $p = 1$ the operator J_G^α is bounded from $\mathcal{M}_{1,\omega,\gamma}(\mathbb{R}_+)$ to $W\mathcal{M}_{q,\omega^{\frac{1}{q}},\gamma}(\mathbb{R}_+)$

Proof. Let $1 \le p < \infty$ and $f \in \mathcal{M}_{p,\omega,\gamma}(\mathbb{R}_+)$. By Theorem 4.9 the inequality (4.12) is valid. Then from condition (4.15) and inequality (4.12), we get

$$\begin{aligned}
J_G^\alpha f(chx) &\lesssim \left(sh\frac{t}{2}\right)^\alpha M_G f(chx) + \int_t^\infty \left(sh\frac{r}{2}\right)^{\alpha-\frac{\gamma}{p}-1} \left(ch\frac{r}{2}\right) \|f\|_{L_{p,\lambda}(H_r)} dr \\
&\lesssim \left(sh\frac{t}{2}\right)^\alpha M_G f(chx) + \int_t^\infty \frac{\left(sh\frac{r}{2}\right)^{\alpha-1} \omega(x,r)^{\frac{1}{p}} \|f\|_{L_{p,\lambda}(H_r)}}{\left(sh\frac{r}{2}\right)^{\frac{\gamma}{p}} \omega(x,r)^{\frac{1}{p}}} \left(ch\frac{r}{2}\right) dr \\
&\lesssim \left(sh\frac{t}{2}\right)^\alpha M_G f(chx) + \|f\|_{M_{p,\omega^{\frac{1}{p}},\gamma}(\mathbb{R}_+)} \int_t^\infty \frac{\omega(x,r)^{\frac{1}{p}} \left(ch\frac{r}{2}\right)}{\left(sh\frac{r}{2}\right)^{1-\alpha}} dr
\end{aligned}$$

$$\lesssim \left(sh\frac{t}{2}\right)^\alpha M_G f(chx) + \left(sh\frac{t}{2}\right)^{-\frac{\alpha p}{q-p}} \|f\|_{\mathcal{M}_{p,\omega^{\frac{1}{p}},\gamma}}(\mathbb{R}_+), \ t \in E_\gamma. \tag{4.16}$$

The right-hand side attains its minimum at

$$sh\frac{t}{2} = \left(\frac{p}{q-p} \frac{\|f\|_{\mathcal{M}_{p,\omega^{\frac{1}{p}},\gamma}(\mathbb{R}_+)}}{M_G f(chx)}\right)^{\frac{q-p}{\alpha q}}$$

for all $x \in \mathbb{R}_+$, we have

$$J_G^\alpha f(chx) \lesssim (M_G f(chx))^{\frac{p}{q}} \, \|f\|^{1-\frac{p}{q}}_{\mathcal{M}_{p,\omega^{\frac{1}{p}},\gamma}(\mathbb{R}_+)}.$$

Hence, the statement of the theorem follows from the boundedness of the G-maximal operator M_G in $\mathcal{M}_{p,\omega,\gamma}(\mathbb{R}_+)$ provided by Corollary 4.10 by virtue of condition (4.14)

$$\begin{aligned}
\|J_G^\alpha f\|_{\mathcal{M}_{q,\omega^{\frac{1}{q}},\gamma}} &\lesssim \|f\|_{\mathcal{M}_{p,\omega^{\frac{1}{p}},\gamma}} \sup_{\substack{x\in\mathbb{R}_+\\ r>0}} \omega(x,r)^{-\frac{1}{q}} \left(sh\frac{r}{2}\right)^{-\frac{\gamma}{q}} \|M_G f\|^{\frac{p}{q}}_{L_{p,\lambda}(H_r)} \\
&= \|f\|_{\mathcal{M}_{p,\omega^{\frac{1}{p}},\gamma}} \left(\sup_{\substack{x\in\mathbb{R}_+\\ r>0}} \omega(x,r)^{-\frac{1}{p}} \left(sh\frac{r}{2}\right)^{-\frac{\gamma}{p}} \|M_G f\|_{L_{p,\lambda}(H_r)}\right)^{\frac{p}{q}} \\
&= \|f\|_{\mathcal{M}_{p,\omega^{\frac{1}{p}},\gamma}} \|M_G f\|^{\frac{p}{q}}_{\mathcal{M}_{p,\omega^{\frac{1}{p}},\gamma}} \lesssim \|f\|_{\mathcal{M}_{p,\omega^{\frac{1}{p}},\gamma}}.
\end{aligned}$$

for $1 < p < q < \infty$ and

$$\begin{aligned}
\|J_G^\alpha f\|_{W\mathcal{M}_{q,\omega^{\frac{1}{q}}}} &\lesssim \|f\|^{1-\frac{1}{q}}_{\mathcal{M}_{1,\omega,\gamma}} \sup_{\substack{x\in\mathbb{R}_+\\ r>0}} \omega(x,r)^{-\frac{1}{q}} \left(sh\frac{r}{2}\right)^{-\frac{\gamma}{q}} \|M_G f\|^{\frac{1}{q}}_{WL_{1,\lambda}(H_r)} \\
&\lesssim \|f\|^{1-\frac{1}{q}}_{\mathcal{M}_{1,\omega,\gamma}} \|f\|^{\frac{1}{q}}_{\mathcal{M}_{1,\omega,\gamma}} = \|f\|_{\mathcal{M}_{1,\omega,\gamma}},
\end{aligned}$$

for $p = 1 < q < \infty$. □

The following theorem is one of our main results.

Theorem 4.12. Let $0 < \alpha < \gamma$, $1 \le p < q < \infty$ and (4.14) hold.

If $\omega \in \Phi_p^\gamma$ satisfies condition

$$\int_r^\infty \left(sh\frac{t}{2}\right)^{\alpha-1} \left(ch\frac{t}{2}\right) \omega(t)^{\frac{1}{p}} dt \lesssim \omega(r)^{\frac{1}{p}} \left(sh\frac{r}{2}\right)^{\alpha}, \tag{4.17}$$

then the condition

$$\omega(r)^{\frac{1}{p}} \left(sh\frac{r}{2}\right)^{\alpha} \lesssim \left(sh\frac{r}{2}\right)^{-\frac{\alpha p}{q-p}} \tag{4.18}$$

is necessary and sufficient for the boundedness J_G^α from $\mathcal{M}_{1,\omega,\gamma}(\mathbb{R}_+)$ to $W\mathcal{M}_{q,\omega^{\frac{1}{q}},\gamma}(\mathbb{R}_+)$ if $p = 1$ and from $\mathcal{M}_{p,\omega^{\frac{1}{p}},\gamma}(\mathbb{R}_+)$ to $\mathcal{M}_{q,\omega^{\frac{1}{q}},\gamma}(\mathbb{R}_+)$ if $1 < p < \infty$.

Proof. **Sufficiency.** Let (4.18) hold, then from (4.17) implies (4.15) and by Theorem 4.11 J_G^α is bounded from $\mathcal{M}_{p,\omega^{\frac{1}{p}},\gamma}(\mathbb{R}_+)$ to $\mathcal{M}_{q,\omega^{\frac{1}{q}},\gamma}(\mathbb{R}_+)$ if $1<p<\infty$ and $\mathcal{M}_{1,\omega,\gamma}(\mathbb{R}_+)$ to $W\mathcal{M}_{q,\omega^{\frac{1}{q}},\gamma}(\mathbb{R}_+)$ if $p=1$.

Necessity. Let J_G^α be bounded from $\mathcal{M}_{p,\omega^{\frac{1}{p}},\gamma}(\mathbb{R}_+)$ to $\mathcal{M}_{q,\omega^{\frac{1}{q}},\gamma}(\mathbb{R}_+)$ if $1<p<q<\infty$ and from $\mathcal{M}_{1,\omega,\gamma}(\mathbb{R}_+)$ to $\mathcal{M}_{q,\omega^{\frac{1}{q}},\gamma}(\mathbb{R}_+)$ if $p=1$. Let $H_0=(0,t_0)$ and $x\in H_0$. By Lemma 4.5, we have $\left(sh\frac{t_0}{2}\right)^\alpha \lesssim J_G^\alpha \chi_{H_0}(chx)$. Therefore, by Lemma 3.5 and Lemma 4.5, we get

$$\begin{aligned}\left(sh\frac{t_0}{2}\right)^\alpha &\lesssim |H_0|_\lambda^{-\frac{1}{q}}\|J_G^\alpha\chi_{H_0}\|_{L_{q,\lambda}(H_0)} \lesssim \omega(t_0)^{\frac{1}{q}}\|J_G^\alpha\chi_{H_0}\|_{\mathcal{M}_{q,\omega^{\frac{1}{q}},\gamma}} \\ &\lesssim \omega(t_0)^{\frac{1}{q}}\|\chi_{H_0}\|_{\mathcal{M}_{p,\omega^{\frac{1}{p}},\gamma}} \lesssim \omega(t_0)^{\frac{1}{q}-\frac{1}{p}},\end{aligned}$$

or

$$\left(sh\frac{t_0}{2}\right)^\alpha \omega(t_0)^{\frac{1}{p}-\frac{1}{q}} \lesssim 1 \Longleftrightarrow \left(sh\frac{t_0}{2}\right)^\alpha \omega(t_0)^{\frac{1}{p}} \lesssim \left(sh\frac{t_0}{2}\right)^{-\frac{\alpha p}{q-p}},$$

for all $t_0>0$.
Since this is true for every $x\in\mathbb{R}_+$ and $t_0>0$, we are done. □

Theorem 4.13 (Adams-Gunawan type result). Let $0<\alpha<\gamma,\ 1\le p<q<\infty$ and $\omega\in\Omega_p^\gamma$ satisfy the condition (4.14) and

$$\left(sh\frac{t}{2}\right)^\alpha \omega(x,t)+\int_t^\infty \left(sh\frac{r}{2}\right)^{\alpha-1}\left(ch\frac{r}{2}\right)\omega(x,r)dr \lesssim \omega(x,t)^{\frac{p}{q}},\ t\in E_\gamma. \tag{4.19}$$

Then for $p>1$, the operator J_G^α is bounded from $\mathcal{M}_{p,\omega^{\frac{1}{p}},\gamma}(\mathbb{R}_+)$ to $\mathcal{M}_{q,\omega^{\frac{1}{q}},\gamma}(\mathbb{R}_+)$ and for $p=1$, the operator J_G^α is bounded from $\mathcal{M}_{1,\omega,\gamma}(\mathbb{R}_+)$ to $W\mathcal{M}_{q,\omega^{\frac{1}{q}},\gamma}(\mathbb{R}_+)$.

Proof. Let $1\le p<\infty$ and $f\in\mathcal{M}_{p,\omega,\gamma}(\mathbb{R}_+)$. By Theorem 4.9, the inequality (4.12) is valid. Then from condition (4.14) and inequality (4.12), we get

$$\begin{aligned}|J_G^\alpha f(chx)| &\lesssim \left(sh\frac{t}{2}\right)^\alpha M_G f(chx)+\int_t^\infty \left(sh\frac{r}{2}\right)^{\alpha-\frac{\gamma}{p}-1}\left(ch\frac{r}{2}\right)\|f\|_{L_{p,\lambda}(H_r)}dr \\ &\lesssim \left(sh\frac{t}{2}\right)^\alpha M_G f(chx)+\|f\|_{\mathcal{M}_{p,\omega,\gamma}}\int_t^\infty \left(sh\frac{r}{2}\right)^{\alpha-1}\left(ch\frac{r}{2}\right)\omega(x,r)dr,\ t\in E_\gamma.\end{aligned} \tag{4.20}$$

From (4.19), we have

$$\omega(x,t)\left(sh\frac{t}{2}\right)^{\alpha} \lesssim \omega(x,t)^{p/q} \Rightarrow \left(sh\frac{t}{2}\right)^{\alpha} \lesssim \omega(x,t)^{p/q-1}, \tag{4.21}$$

and

$$\int_t^{\infty}\left(sh\frac{r}{2}\right)^{\alpha-1}\left(ch\frac{r}{2}\right)\omega(x,r) \lesssim \omega(x,t)^{p/q}$$
$$\Rightarrow \|f\|_{\mathcal{M}_{p,\omega,\gamma}}\int_t^{\infty}\left(sh\frac{r}{2}\right)^{\alpha-1}\left(ch\frac{r}{2}\right)\omega(x,r)dr \lesssim \|f\|_{\mathcal{M}_{p,\omega,\gamma}}\omega(x,t)^{p/q} \tag{4.22}$$

Summing (4.21) and (4.22), we get

$$|J_G^{\alpha}f(chx)| \lesssim \omega(x,t)^{\frac{p}{q}-1}M_Gf(chx) + \omega(x,t)^{\frac{p}{q}}\|f\|_{\mathcal{M}_{p,\omega,\gamma}}$$

If $\omega(x,t)^{\frac{p}{q}-1}M_Gf(chx) \leq \omega(x,t)^{\frac{p}{q}}\|f\|_{\mathcal{M}_{p,\omega,\gamma}}$, then $\omega(x,t) \geq M_Gf(chx)\|f\|^{-1}_{\mathcal{M}_{p,\omega,\gamma}}$ and we have

$$\begin{aligned}|J_G^{\alpha}f(chx)| &\lesssim \omega(x,t)^{\frac{p}{q}-1}M_Gf(chx) \lesssim M_Gf(chx)\left(\frac{\|f\|_{\mathcal{M}_{p,\omega,\gamma}}}{M_Gf(chx)}\right)^{1-\frac{p}{q}}\\ &= (M_Gf(chx))^{\frac{p}{q}}\|f\|^{1-\frac{p}{q}}_{\mathcal{M}_{p,\omega,\gamma}}.\end{aligned}$$

And if $\omega(x,t)^{\frac{p}{q}-1}M_Gf(chx) \geq \omega(x,t)^{\frac{p}{q}}\|f\|_{\mathcal{M}_{p,\omega,\gamma}}$, then $\omega(x,t) \leq M_Gf(chx)\|f\|^{-1}_{\mathcal{M}_{p,\omega,\gamma}}$ and we get

$$\begin{aligned}|J_G^{\alpha}f(chx)| &\lesssim \omega(x,t)^{\frac{p}{q}}\|f\|_{\mathcal{M}_{p,\omega,\gamma}} \leq \left(\frac{M_Gf(chx)}{\|f\|_{\mathcal{M}_{p,\omega,\gamma}}}\right)^{\frac{p}{q}}\|f\|_{\mathcal{M}_{p,\omega,\gamma}}\\ &= (M_Gf(chx))^{\frac{p}{q}}\|f\|^{1-\frac{p}{q}}_{\mathcal{M}_{p,\omega,\gamma}}.\end{aligned}$$

Thus,

$$|J_G^{\alpha}f(chx)| \lesssim (M_Gf(chx))^{\frac{p}{q}}\|f\|^{1-\frac{p}{q}}_{\mathcal{M}_{p,\omega,\gamma}}. \tag{4.23}$$

From Corollary 4.10 and (4.23), we obtain (see proof Theorem 4.11)

$$\|J_G^{\alpha}f\|_{\mathcal{M}_{q,\omega^{\frac{1}{q}},\gamma}} \lesssim \|f\|^{1-\frac{p}{q}}_{\mathcal{M}_{p,\omega^{\frac{1}{p}},\gamma}}\|M_Gf\|^{\frac{p}{q}}_{\mathcal{M}_{p,\omega^{\frac{1}{p}},\gamma}} \lesssim \|f\|_{\mathcal{M}_{p,\omega^{\frac{1}{p}},\gamma}},$$

for $1 < p < q < \infty$ and

$$\|J_G^\alpha f\|_{W\mathcal{M}_{q,\omega^{\frac{1}{q}},\gamma}} \lesssim \|f\|_{\mathcal{M}_{1,\omega,\gamma}}^{1-\frac{1}{q}} \|M_G f\|_{\mathcal{M}_{1,\omega,\gamma}}^{\frac{1}{q}} \lesssim \|f\|_{\mathcal{M}_{1,\omega,\gamma}},$$

for $p = 1 < q < \infty$. □

The following theorem is one of our main result.

Theorem 4.14. Let $0 < \alpha < \gamma,\ 1 \le p < q < \infty$ and $\omega \in \Omega_p^\gamma$.

If $\omega \in \Phi_p^\gamma$ satisfies the condition (4.19), then the condition

$$\left(sh\frac{r}{2}\right)^\alpha \omega(r)^{\frac{1}{p}} \lesssim \omega(r)^{\frac{1}{q}} \tag{4.24}$$

is necessary and sufficient for the boundedness of J_G^α from $\mathcal{M}_{1,\omega,\gamma}(\mathbb{R}_+)$ to $W\mathcal{M}_{q,\omega^{\frac{1}{q}},\gamma}(\mathbb{R}_+)$ if $p = 1$ and from $\mathcal{M}_{p,\omega^{\frac{1}{p}},\gamma}(\mathbb{R}_+)$ to $\mathcal{M}_{q,\omega^{\frac{1}{q}},\gamma}(\mathbb{R}_+)$ if $1 < p < q < \infty$.

Proof. Let (4.24) hold. Note that if $\omega \in \Phi_p^\gamma$, then

$$\begin{aligned}
\int_t^\infty \left(sh\frac{r}{2}\right)^{\alpha-1}\left(ch\frac{r}{2}\right)\omega(r)dr &= \int_t^\infty \left(sh\frac{r}{2}\right)^{\frac{\gamma}{p}-\frac{\gamma}{q}-1}\left(ch\frac{r}{2}\right)\omega(r)dr \\
&\ge \left(sh\frac{t}{2}\right)^{\frac{\gamma}{p}}\omega(t)\int_t^\infty \left(sh\frac{r}{2}\right)^{-\frac{\gamma}{q}-1}\left(ch\frac{r}{2}\right)dr \\
&= q\left(sh\frac{t}{2}\right)^\alpha \omega(t), \quad i.e.
\end{aligned}$$

$$\left(sh\frac{t}{2}\right)^\alpha \omega(t) \le \int_t^\infty \left(sh\frac{r}{2}\right)^{\alpha-1}\left(ch\frac{r}{2}\right)\omega(r)dr \lesssim \omega(t)^{p/q},\ t \in E_\gamma.$$

Suppose that $\omega(r) = \varphi(r)^{\frac{1}{p}}$. Then the condition (4.19) is equivalent to

$$\int_t^\infty \left(sh\frac{r}{2}\right)^\alpha\left(ch\frac{r}{2}\right)\omega(r)^{\frac{1}{p}}dr \lesssim \omega(t)^{\frac{1}{q}}. \tag{4.25}$$

Then by Theorem 4.13 the statement of Theorem 4.14 is proved.

Necessity. Now let J_G^{α} be bounded from $\mathcal{M}_{p,\omega^{\frac{1}{p}},\gamma}(\mathbb{R}_+)$ to $\mathcal{M}_{q,\omega^{\frac{1}{q}},\gamma}(\mathbb{R}_+)$. Let $H_0=(0,t_0)$ and $x\in H_0$. By Lemma 4.5, we have $\left(sh\frac{t_0}{2}\right)^{\alpha}\lesssim J_G^{\alpha}\chi_{H_0}(chx)$. Therefore, by Lemma 3.5 and Lemma 4.5, we have

$$\begin{aligned}\left(sh\frac{t_0}{2}\right)^{\alpha} &\lesssim |H_0||_{\lambda}^{-\frac{1}{q}}\|J_G^{\alpha}\chi_{H_0}\|_{L_{q,\lambda}(H_0)}\\ &\lesssim \omega(t_0)^{\frac{1}{q}}\|J_G^{\alpha}\chi_{H_0}\|_{\mathcal{M}_{q,\omega^{\frac{1}{q}},\gamma}}\lesssim \omega(t_0)^{\frac{1}{q}}\|\chi_{H_0}\|_{\mathcal{M}_{p,\omega^{\frac{1}{p}},\gamma}}\lesssim \omega(t_0)^{\frac{1}{q}-\frac{1}{p}},\end{aligned}$$

or

$$\left(sh\frac{t_0}{2}\right)^{\alpha}\omega(t_0)^{\frac{1}{p}-\frac{1}{q}}\lesssim 1\Longleftrightarrow\left(sh\frac{t_0}{2}\right)^{\alpha}\omega(t_0)^{\frac{1}{p}}\lesssim\omega(t_0)^{\frac{1}{q}},$$

for all $t_0>0$.
Since this is true for every $x\in\mathbb{R}_+$ and $t_0>0$, we are done. □

Remark 4.15. If we take $\omega(t)=(sh\frac{t}{2})^{\nu-\gamma}$ at Theorem 4.12, then the condition (4.17) is equivalent to $0<\nu<\gamma-\alpha p$ and condition (4.18) is equivalent to $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{\gamma-\nu}$. Therefore, we get the following corallary.

Corollary 4.16 (Theorem 2.4). Let $0<\alpha<\gamma$, $1\le p<q<\infty$ and $0<\nu<\gamma-\alpha p$. Then the operator J_G^{α} is bounded from $L_{1,\nu,\lambda}(\mathbb{R}_+)$ to $WL_{q,\nu,\lambda}(\mathbb{R}_+)$ if and only if $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{\gamma-\nu}$.

Moreover, if $1<p<q<\infty$, then the operator J_G^{α} is bounded from $L_{p,\nu,\lambda}(\mathbb{R}_+)$ to $L_{q,\nu,\lambda}(\mathbb{R}_+)$ if and only if $1-\frac{1}{q}=\frac{\alpha}{\gamma-\nu}$.

Corollary 4.17 . If we take $\gamma=2\lambda+1$ in Corallary 4.16, we obtain Theorem B.

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