Transactions of A. Razmadze
Mathematical Institute
Vol. **1??** (202?), ?–??

CHAIN RULES AND THEIR GENERALIZATIONS FOR NABLA DERIVATIVES ON MEASURE CHAINS

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Abstract. This paper provides some standard versions of the chain rule for nabla derivatives on measure chains, which differ from the chain rules from known calculus. Further, the mean value theorem together with the Fréchet derivatives are used to generalize the chain rules to measure chains in the Banach space framework. As a consequence of these chain rules, the nabla derivative of the inverse function and the substitution rule in a more general setup of measure chains is proved. To illustrate our results, we provide examples on suitable time scales.

1. Introduction

Time scale calculus, a unifying theory introduced by Hilger in 1988, bridges the gap between continuous and discrete analysis. It provides a framework that encompasses differential calculus, difference calculus and q-calculus, allowing mathematicians to study dynamic equations on arbitrary time scales. This theory detailed in [8, 9] is particularly valuable in modeling real-world phenomena that exhibit both continuous and discrete behavior, such as population dynamics, economic systems and control theory. Emerging areas of interest arising from the study of time scales calculus are reported in [35]. The versatility of time scale calculus lies in its ability to extend classical results to more generalized settings, offering new insights and solutions to problems that were previously approached separately in continuous and discrete domains. This unified approach not only simplifies the theoretical underpinnings but also enhances the applicability of mathematical models across various fields. The recent additions of Riemann intergrability and Riemann–Stieltjes integrability presented in [28, 29] bring forth new developments in calculus on time scales. This calculus forms the basis of the celebrated dynamic equations on time scales with several papers marking remarkable discoveries in this field. The general results on the existence of solutions for nonlinear dynamic equations with impulses and nonlocal initial conditions are discussed in [37]. The existence of solutions of first-order initial value problems and continuous dependence of solutions are studied in [11]. Further, the dynamic local and nonlocal initial value problems using the measure of non-compactness is studied in [36]. In [33], the stability analysis of first-order nonlinear impulsive time-varying delay dynamic systems on time scales is done via the Gronwall inequality and the Banach contraction principle. Following this, in [32] the stability results for n^{th}-order linear dynamic equations on time scales are obtained. Furthermore, several studies have examined the stability results for integro–dynamic equations. For instance, [31] explores the Ulam-type stability for nonlinear impulsive Volterra–Fredholm integro-dynamic adjoint equations. In addition, [30] studies the Bielecki–Hyer–Ulam stability of nonlinear impulsive fractional Hammerstein and mixed integro-dynamic systems.

Nabla derivatives introduced in [6] provide a means to generalize the concept of backward differentiation. In [5], the representation of polynomials for nabla dynamic equations on time scales is derived and related to those for delta dynamic equations. The concept of nabla calculus received a thorough treatment in [4] and was well contrasted with the earlier known Hilger delta calculus. Later, the study of calculus of variations with the nabla derivative gained momentum reported in [3, 23]. The dynamic calculus with nabla derivatives forms the fundamental basis for the study of nabla dynamic equations on time scales. The existence and convergence results for the nabla dynamic equation on

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2020 *Mathematics Subject Classification.* 26E70, 34N99.
Key words and phrases. Chain rule; Fréchet derivative; measure chains; nabla derivative; time scales.
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time scales via upper and lower solutions are discussed in [38]. Further, [13] explores the existence and uniqueness of solutions to an initial value problem involving generalized Hausdorff derivatives called structural nabla derivatives. Unlike the delta derivative, which generalizes the forward difference operator, the nabla derivative is concerned with backward differences, making it particularly useful for certain types of dynamic equations and boundary value problems as seen in [12, 20, 21]. The study of stability properties of nabla dynamic equations along with the demonstration of an application to the cobweb model is done in [19]. The oscillatory behavior of nabla dynamic equations is studied in [24]. Some important inequalities useful in the qualitative theory of such equations are established in [16].

The development of nabla calculus allows for the study of dynamic systems where past states influence future behavior, which is essential in applications such as financial modeling, biological systems, and engineering. By extending the principles of discrete backward differentiation to arbitrary time scales, nabla calculus offers a robust framework for analyzing systems that operate in nonuniform time domains. This approach not only enriches the theory of time scale calculus but also provides powerful tools for solving complicated dynamic equations that arise in various scientific and engineering disciplines as found in [17, 18]. Recently, [26] presented the existence and uniqueness results for fuzzy nabla dynamic equations on time scales - a must read for researchers interested in contributing in this area. Quite recently, generalized Hardy-type inequalities via nabla calculus on time scales were explored in [27] opening further scope for study of calculus on time scales via nabla derivatives. Noteworthy here, is that parallel to the theory of dynamic systems on time scales, [18] goes a step further and studies dynamical systems with nabla half derivatives on time scales.

A striking point in the study of time scale calculus outlined in [1] is that the chain rule for derivatives does not carry on to an arbitrary time scale. This fact is demonstrated in [25] and some versions of the chain rules are offered for delta derivative on time scales. Later, in [14], these chain rules are generalized to a class of new functions called 'pair differentiable' functions on time scales. The chain rules are fundamental tools in the study of dynamic equations on time scales. For instance, it is seen in [15] that the chain rule is applied in studying the oscillation properties of first-order nonlinear dynamic equations on time scales. The interest in the study of chain rules on time scales continues to this very day as seen in [22], in which new generalizations of the chain rule involving the Stieltjes derivatives and integrals are presented. While the current work develops chain rules for nabla derivatives on measure chains, it is worth noting that in related contexts such as [7] involving transmission problems with fractional boundary conditions, semigroup theory has been used to establish the existence and uniqueness, an Arendt–Batty-type criteria has ensured strong stability, spectral analysis has ruled out exponential decay, and the Borichev–Tomilov framework has yielded polynomial decay rates.

The chain rules for nabla derivatives on measure chains are physically motivated by the need to model systems where current behavior depends on past states, such as in backward control problems, economic systems with memory, and irreversible thermodynamic processes. On measure chains, the nabla chain rule plays a crucial role in accurately describing pre-jump dynamics and enabling backward integration across nonuniform time domains. This is particularly important in systems with impulses, hysteresis, or non-anticipative behavior, where backward-looking formulations are essential for the stability analysis and variational problems. In contrast, chain rules for delta derivatives are forward-looking and more suited to initial value problems, feed forward control, and real-time modeling scenarios. While both nabla and delta chain rules extend classical calculus to arbitrary time scales, their physical interpretations and consequences differ: delta rules govern forward-evolving processes, whereas nabla rules are essential for understanding systems driven by historical data or backward-propagating dynamics.

In this paper, we first develop certain standard versions of the chain rules for nabla derivatives in more general setup of measure chains. It is pointed out that we use a more careful analysis as compared to that in [8, 10] for developing the first version of the chain rule, taking into account that if a function f is continuous in its domain and as s approached t, we may have $f(s) = f(t)$, prompting us to be careful with arbitrary division, paving way for another approach. In addition, the second chain rule in Theorem 4.2 is strikingly different from its delta derivative counterpart. We then generalize the standard version of the chain rule [8, Theorem 1.90] followed by another version instrumental in leading to important consequences of the chain rule - namely the derivative of the inverse and the substitution rule. Motivated by [34], we generalize the chain rule [8, Theorem 1.90] for nabla derivatives on measure chains which involve Fréchet differentiable functions and functions of several arguments necessitating the existence of partial derivatives.

The rest of this paper is organized as follows: In Section 2, we give all necessary preliminaries that are required and used throughout this paper. The Hilger type of differentiability are derived for nabla derivatives in Section 3. In Section 4, we derive some standard versions of the chain rule for nabla derivatives on measure chains, while the fundamental consequences of the chain rule – namely the nabla derivative of the inverse function and the substitution rule are obtained in Section 5. In the subsequent Section 6, the chain rules are generalized to a bigger class of functions on "larger spaces". The paper concludes by discussing the results obtained in Section 7.

2. Preliminaries

In this section, we discuss the basic terminologies and some related examples. We start by defining measure chains. These following definitions and results are taken from [1, 2, 8, 34].

Definition 1. (Measure Chain) Let $\mathbb{T}$ be a given set. A triple $(\mathbb{T}, \le, v)$ is called a (strong) measure chain provided the following axioms hold:

(1) Axiom 1: For all $p, q, r \in \mathbb{T}$, the relation $\le$ satisfies the following:
  (i) Reflexivity: $p \le p$;
  (ii) Transitivity: $p \le q$ and $q \le r \implies p \le r$;
  (iii) Antisymmetry: $p \le q$ and $q \le p \implies p = q$;
  (iv) Total: either $p \le q$ or $q \le p$.
(2) Axiom 2: A nonempty subset of $\mathbb{T}$ bounded below has the greatest lower bound (i.e., the chain $(\mathbb{T}, \le)$ is conditionally complete).
(3) Axiom 3: For all $p, q, r \in \mathbb{T}$, the mapping $v\colon \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ satisfies the following:
  (i) Cocycle property: $v(p, q) + v(q, r) = v(p, r)$;
  (ii) Strong isotony: $p > q \implies v(p, q) > 0$;
  (iii) Continuity: v is continuous.

Remark 1. In a measure chain $(\mathbb{T}, \le, v)$, if $\mathbb{T}$ is a closed subset of $\mathbb{R}$, "$\le$" is the usual order relation on $\mathbb{R}$, and $v(p, q) = p - q$ for all $p, q \in \mathbb{T}$, then $\mathbb{T}$ is called a time scale.

We first introduce the forward and backward jump operators.

Definition 2. Let $(\mathbb{T}, \le, v)$ be a measure chain. For $t \in \mathbb{T}$, the operator $\sigma\colon \mathbb{T} \to \mathbb{T}$ defined by $\sigma(t) = \inf\{s \in \mathbb{T}\colon s > t\}$ is called the forward jump operator.

Definition 3. Let $(\mathbb{T}, \le, v)$ be a measure chain. For $t \in \mathbb{T}$, the operator $\rho\colon \mathbb{T} \to \mathbb{T}$ defined by $\rho(t) = \sup\{s \in \mathbb{T}\colon s < t\}$ is called the backward jump operator.

Definition 4. A point $t \in \mathbb{T}$ is said to be right-scattered, left-scattered, isolated, right-dense, left-dense, dense accordingly if $\sigma(t) > t$, $t > \rho(t)$, $\rho(t) < t < \sigma(t)$, $\sigma(t) = t < \sup \mathbb{T}$, $\inf \mathbb{T} < t = \rho(t)$, and $\sigma(t) = t = \rho(t)$ respectively.

We now introduce the graininess functions on measure chains.

Definition 5. The forward graininess function $\mu\colon \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) = v(\sigma(t), t)$.

Definition 6. The backward graininess function $\nu\colon \mathbb{T} \to [0, \infty)$ is defined by $\nu(t) = v(t, \rho(t))$.

Note 1. Because of the strong isotony property of v, both μ and ν are strictly positive at scattered points.

Before we define the nabla derivative, we need the concept of $\mathbb{T}_\kappa$ operator to ensure uniqueness of the derivative explained in Remark 2.

Definition 7. Let $\mathbb{T}$ be a time scale. If $\mathbb{T}$ has a right-scattered minimum m, then we define a new set $\mathbb{T}_\kappa$ as $\mathbb{T}_\kappa = \mathbb{T} \setminus \{m\}$. Otherwise, $\mathbb{T}_\kappa = \mathbb{T}$. That is,

$$\mathbb{T}_\kappa = \begin{cases} \mathbb{T} \setminus [\inf \mathbb{T}, \sigma(\inf \mathbb{T})) & \text{if } -\infty < \inf \mathbb{T}, \\ \mathbb{T} & \text{if } \inf \mathbb{T} = -\infty. \end{cases}$$

Definition 8. Let $(\mathbb{T}, \leq, v)$ be a measure chain and let X be the Banach space with norm $\|\cdot\|$. A mapping $\phi\colon \mathbb{T} \to X$ is nabla differentiable at $t \in \mathbb{T}_\kappa$ provided there exists $\phi^\nabla(t) \in X$ such that for each $\varepsilon > 0$ there exists a neighborhood N of t such that

$$\|\phi(s) - \phi(\rho(t)) - \phi^\nabla(t)v(s, \rho(t))\| \leq \varepsilon |v(s, \rho(t))| \quad \text{for each } s \in N.$$

In this case, $\phi^\nabla(t)$ is said to be the nabla derivative of ϕ at t.

Remark 2. Suppose that $\inf \mathbb{T} > -\infty$ and $\phi^\nabla(t)$ is defined at a point $t \in \mathbb{T} \setminus \mathbb{T}_\kappa$. Then the unique point $t \in \mathbb{T} \setminus \mathbb{T}_\kappa$ is $\inf \mathbb{T}$. Hence for any $\varepsilon > 0$ there exists a neighborhood $N = \{t\}$ of t such that $\phi(\rho(t)) = \phi(s) = \phi(\rho(\inf \mathbb{T})) = \phi(\inf \mathbb{T})$, $s \in N$. Then, for any $a \in X$ and for each $s \in N$, we have

$$\|\phi(s) - \phi(\rho(t)) - av(s, \rho(t))\| = \|\phi(\inf \mathbb{T}) - \phi(\inf \mathbb{T}) - av(\inf \mathbb{T}, \inf \mathbb{T})\| \leq \varepsilon |v(s, \rho(t))|.$$

Thus each $a \in X$ is the nabla derivative of ϕ if $t \notin \mathbb{T}_\kappa$.

Note 2. Throughout this paper, we shall denote $\mathbb{T}$ to be a measure chain $(\mathbb{T}, \leq, v)$ unless explicitly stated that $\mathbb{T}$ is a time scale. Further, X will denote the Banach space with norm $\|\cdot\|$.

Example. Consider the time scale $\mathbb{T} = \mathbb{N}_0^{1/2} = \{\sqrt{n}\colon n \in \mathbb{N}\} \cup \{0\}$. Let $t \in \mathbb{T}$. Then $t = \sqrt{n}$ for some $n \in \mathbb{N}$. Here $\rho(t) = \sup\{\sqrt{k} \in \mathbb{N}_0^{1/2}\colon k < n\} = \sqrt{n-1} = \sqrt{t^2 - 1}$. This leads us to calculating the backward graininess as $\nu(t) = t - \rho(t) = t - \sqrt{t^2 - 1}$.

Definition 9. A function $\phi\colon \mathbb{T} \to X$ is said to be ld-continuous provided it is continuous at left-dense points in $\mathbb{T}$ and its right-sided limits exist at right-dense points in $\mathbb{T}$.

Definition 10. A function $\Phi\colon \mathbb{T} \to X$ is said to be an antiderivative of $\phi\colon \mathbb{T} \to X$ provided $\Phi^\nabla(t) = \phi(t)$ for all $t \in \mathbb{T}_\kappa$.

Definition 11. Let Φ be an antiderivative of an ld-continuous function ϕ. We define the indefinite nabla integral of ϕ as $\int \phi(t)\nabla t = \Phi(t) + C$, where C is an arbitrary constant. Further, we define the Cauchy nabla integral of ϕ by $\int_p^q \phi(t)\nabla t = \Phi(q) - \Phi(p)$ for all $p, q \in \mathbb{T}$.

Example. On the time scale $\mathbb{T} = \mathbb{Z}$, we find that $\left(\frac{\alpha^{t+1}}{\alpha-1}\right)^\nabla = \frac{\alpha^{t+1} - \alpha^t}{\alpha - 1} = \alpha^t$. Thus we get $\int \alpha^t \nabla t = \frac{\alpha^{t+1}}{\alpha-1} + C$, where C is an arbitrary constant.

Let $\mathscr{L}(X, Y)$ denote the set of all bounded, linear and continuous operators from X to Y. The concept of Fréchet differentiability is required for generalizing the chain rules. We present it below.

Remark 3. Let X and Y be normed spaces. By $o(X, Y)$ we mean the set of maps $r : X \to Y$ for which there is some map $\alpha : X \to Y$ such that

(i) $r(x) = \alpha(x)\|x\|$ for each $x \in X$;
(ii) $\alpha(0) = 0$;
(iii) α is continuous at 0.

The elements of $o(X, Y)$ will be called remainders.

Definition 12. Let ϕ be a function defined on an open subset U of a Banach space X into the Banach space Y. We say that ϕ is Fréchet differentiable at $x_0 \in U$ provided there is $L \in \mathscr{L}(X, Y)$ and $r \in o(X, Y)$ such that

$$\phi(x) = \phi(x_0) + L(x - x_0) + r(x - x_0), \quad x \in U.$$

The operator L will be called the Fréchet derivative of ϕ at x_0. In this case, write $\mathscr{F}\phi(x_0) = L$.

Remark 4. The Fréchet derivative is unique whenever it exists. Further, a linear combination, composition and product of Fréchet differentiable functions is Fréchet differentiable.

Remark 5. A function $\phi\colon U \to Y$ is Fréchet differentiable at x_0 if and only if there is $\Phi\colon U \to \mathscr{L}(X, Y)$ continuous at x_0 and for which

$$\phi(x) - \phi(x_0) = \Phi(x)(x - x_0), \quad x \in U.$$

Definition 13. Let ϕ be a function defined on an open subset U of a Banach space X into the Banach space Y. We say that ϕ is Gâteaux differentiable at $x_0 \in U$ provided there is $T \in \mathscr{L}(X, Y)$ such that

$$\lim_{t \to 0} \frac{\phi(x_0 + tv) - \phi(x_0)}{t} = Tv$$

for every $v \in X$. The operator T is called the Gâteaux derivative of ϕ at x_0. In this case, write $\phi'(x_0) = T$.

Remark 6. Every Fréchet differentiable function is Gâteaux differentiable, but the converse is not true.

3. Auxiliary Results

In this section, we discuss some auxiliary results that are required for computing the nabla derivative on measure chains.

Theorem 3.1. *If $\phi\colon \mathbb{T} \to X$ is nabla differentiable at $t \in \mathbb{T}_\kappa$, then it is continuous at t.*

Proof. Assume that ϕ is nabla differentiable at $t \in \mathbb{T}_\kappa$ and let $0 < \varepsilon < 1$. Define $\varepsilon^* = \frac{\varepsilon}{1 + \|\phi^\nabla(t)\| + 2|\nu(t)|}$. Then $0 < \varepsilon^* < 1$. By the definition of nabla differentiability, there exists a neighborhood N (without loss of generality, assume $\operatorname{diam}(N) \le \varepsilon^*$) of t such that

$$\|\phi(s) - \phi(\rho(t)) - \phi^\nabla(t) v(s, \rho(t))\| \le \varepsilon^* |v(s, \rho(t))| \text{ for each } s \in N. \tag{3.1}$$

Then using the cocycle property and (3.1), we see that

$$\begin{aligned}
\|\phi(t) - \phi(s)\| &= \|\{\phi(t) - \phi(\rho(t)) - \nu(t)\phi^\nabla(t)\} \\
&\quad - \{\phi(s) - \phi(\rho(t)) - v(s, \rho(t))\phi^\nabla(t)\} + v(t, s)\phi^\nabla(t)\| \\
&\le \varepsilon^* |v(s, \rho(t))| + \varepsilon^* \nu(t) + |v(t, s)| \|\phi^\nabla(t)\| \\
&\le \varepsilon^* |v(s, t) + v(t, \rho(t))| + \varepsilon^* \nu(t) + |v(t, s)| \|\phi^\nabla(t)\| \\
&\le \varepsilon^* [\nu(t) + v(s, t) + \nu(t) + \|\phi^\nabla(t)\|] \\
&< \varepsilon^* [1 + \|\phi^\nabla(t)\| + 2\nu(t)] \\
&= \varepsilon.
\end{aligned}$$

As $\varepsilon > 0$ is arbitrary this implies, $\lim_{s \to t} \phi(s) = \phi(t)$. Thus, ϕ is continuous at t. □

From ordinary calculus we know that if a function is continuous, then it may not be differentiable. But the following theorem is a sort of converse of Theorem 3.1 which holds for the nabla derivatives on measure chains.

Theorem 3.2. *If ϕ is continuous at $t \in \mathbb{T}_\kappa$ and t is left-scattered, then ϕ is nabla differentiable at t with $\phi^\nabla(t) = \frac{\phi(t) - \phi(\rho(t))}{\nu(t)}$.*

Proof. Under the given assumptions, and by continuity, we have

$$\lim_{s \to t} \frac{\phi(s) - \phi(\rho(t))}{v(s, \rho(t))} = \frac{\phi(t) - \phi(\rho(t))}{v(t, \rho(t))} = \frac{\phi(t) - \phi(\rho(t))}{\nu(t)}.$$

Hence, for given $\varepsilon > 0$ there exists a neighborhood V of t such that

$$\left\| \frac{\phi(s) - \phi(\rho(t))}{v(s, \rho(t))} - \frac{\phi(t) - \phi(\rho(t))}{\nu(t)} \right\| \le \varepsilon \text{ for each } s \in V.$$

It follows that,

$$\left\| [\phi(s) - \phi(\rho(t))] - \left(\frac{\phi(t) - \phi(\rho(t))}{\nu(t)} \right) v(s, \rho(t)) \right\| \le \varepsilon |v(s, \rho(t))| \text{ for all } s \in V.$$

Hence $\phi^\nabla(t) = \frac{\phi(t) - \phi(\rho(t))}{\nu(t)}$. □

Example. Let $\mathbb{T} = \mathbb{Z}$ and let $\phi(t) = t^2$, $t \in \mathbb{T}$. Since each point $t \in \mathbb{T}$ is left-scattered, by Theorem 3.2, we have

$$\phi^\nabla(t) = \frac{\phi(t) - \phi(\rho(t))}{\nu(t)} = \frac{t^2 - (\sqrt{t^2-1})^2}{t - \sqrt{t^2-1}} = \frac{(t + \sqrt{t^2-1})(t - \sqrt{t^2-1})}{t - \sqrt{t^2-1}} = t + \sqrt{t^2-1}.$$

The next theorem enables us to have an expression for the nabla derivative at left-dense points.

Theorem 3.3. *Suppose $t \in \mathbb{T}_\kappa$ is left-dense. Then a function ϕ is nabla differentiable at t if and only if $\lim_{s\to t} \dfrac{\phi(s) - \phi(t)}{v(s,t)}$ exists as a finite number. Moreover, $\phi^\nabla(t) = \lim_{s\to t} \frac{\phi(s)-\phi(t)}{v(s,t)}$.*

Proof. Assume that ϕ is nabla differentiable at t and $t \in \mathbb{T}_\kappa$ is left-dense (i.e., $\rho(t) = t$). Let $\varepsilon > 0$ be given. Since ϕ is nabla differentiable at t, by definition, there exists a neighborhood N of t such that

$$\|\phi(s) - \phi(t) - \phi^\nabla(t) v(s,t)\| \le \varepsilon |v(s,t)| \quad \text{for each } s \in N.$$

Then, it follows that $\left\| \frac{\phi(s)-\phi(t)}{v(s,t)} - \phi^\nabla(t) \right\| \le \varepsilon$ for $s \in N$ with $s \neq t$. Thus, $\phi^\nabla(t) = \lim_{s\to t} \dfrac{\phi(s)-\phi(t)}{v(s,t)}$. Conversely, suppose that $t \in \mathbb{T}_\kappa$ is left-dense and $\lim_{s\to t} \frac{\phi(s)-\phi(t)}{v(s,t)} = L$ exists. Then, given $\varepsilon > 0$ there exists a neighborhood N of t such that $\left\| \frac{\phi(s)-\phi(t)}{v(s,t)} - L \right\| \le \varepsilon$. Since t is left-dense, it follows that

$$\|\phi(s) - \phi(\rho(t)) - Lv(s,\rho(t))\| \le \varepsilon |v(s,\rho(t))| \quad \text{for each } s \in N,$$

which implies that ϕ is nabla differentiable and $\phi^\nabla(t) = \lim_{s\to t} \frac{\phi(s)-\phi(t)}{v(s,t)}$. □

Remark 7. From Theorem 3.2 and Theorem 3.3, it follows that for a nabla differentiable function ϕ, we have $\phi(\rho(t)) = \phi(t) - \nu(t)\phi^\nabla(t)$, $t \in \mathbb{T}_\kappa$.

4. Standard Versions of the Chain Rule for Nabla Derivatives on Measure Chains

We begin this section by demonstrating why the chain rule from ordinary calculus does not carry over to nabla derivatives. Precisely, if $\phi, \psi \colon \mathbb{T} \to \mathbb{T}$ are functions such that ψ is nabla differentiable at t and ϕ is nabla differentiable at $\psi(t)$, then it may still happen that $(\phi \circ \psi)^\nabla(t) \neq \phi^\nabla(\psi(t))\psi^\nabla(t)$. To see this, let $\mathbb{T} = \mathbb{Z}$ and define $\phi, \psi \colon \mathbb{Z} \to \mathbb{Z}$ by $\phi(t) = t^2 = \psi(t)$ for all $t \in \mathbb{Z}$. Then $(\phi \circ \psi)(t) = t^4$. Since every $t \in \mathbb{Z}$ is isolated, we have

$$(\phi \circ \psi)^\nabla(t) = \frac{(\phi\circ\psi)(t) - (\phi\circ\psi)(\rho(t))}{\nu(t)} = \frac{(\phi\circ\psi)(t) - (\phi\circ\psi)(t-1)}{t - \rho(t)} = 4t^3 - 6t^2 + 4t - 1.$$

Further, $\phi^\nabla(t) = 2t - 1$ and therefore $\phi^\nabla(\psi(t)) = 2t^2 - 1$. Also, $\psi^\nabla(t) = 2t - 1$. Thus, we obtain

$$\phi^\nabla(\psi(t))\psi^\nabla(t) = 4t^3 - 2t^2 - 2t + 1 \neq 4t^3 - 6t^2 + 4t - 1 = (\phi \circ \psi)^\nabla(t),$$

unless when $t = \frac{1}{2}$ or $t = 1$. Since $t = \frac{1}{2} \notin \mathbb{Z}$, we find that the known chain rule does not carry over to nabla derivatives on $\mathbb{Z} \setminus \{1\}$, in this demonstration. This brings us to having a chain rule for nabla derivatives on measure chains. We present the "first version" of such a rule.

Theorem 4.1. *(The Chain Rule I) Assume that $\psi \colon \mathbb{T} \to \mathbb{R}$ is nabla differentiable on $\mathbb{T}_\kappa$ and $\phi \colon \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Then there exists c in the real interval $[\rho(t), t]$ such that*

$$(\phi \circ \psi)^\nabla(t) = \phi'(\psi(c))\psi^\nabla(t).$$

Proof. Consider the function $\eta : \mathbb{R} \to \mathbb{R}$ defined by

$$\eta(y) = \begin{cases} \dfrac{\phi(y) - \phi(\psi(\rho(t)))}{y - \psi(\rho(t))} & \text{if} \quad y \neq \psi(\rho(t)) \\ \phi'(\psi(\tilde{c})) & \text{if} \quad y = \psi(\rho(t)). \end{cases} \tag{4.1}$$

By the virtue of the mean value theorem and the continuity of ψ, η is well-defined as the existence of $\tilde{c}$ is guaranteed for each fixed t. Further, η is continuous at each such t by the condition on ϕ. Now, we show that

$$\frac{\phi(\psi(s)) - \phi(\psi(\rho(t)))}{s - \rho(t)} = \eta(\psi(s)). \frac{\psi(s) - \psi(\rho(t))}{s - \rho(t)}. \tag{4.2}$$

If $\psi(s) \neq \psi(\rho(t))$, then (4.2) becomes

$$\frac{\phi(\psi(s)) - \phi(\psi(\rho(t)))}{s - \rho(t)} = \frac{\phi(\psi(s)) - \phi(\psi(\rho(t)))}{\psi(s) - \psi(\rho(t))} . \frac{\psi(s) - \psi(\rho(t))}{s - \rho(t)}.$$

If $\psi(s) = \psi(\rho(t))$, then using (4.1) in (4.2), we get

$$\frac{\phi(\psi(s)) - \phi(\psi(\rho(t)))}{s - \rho(t)} = \phi'(\psi(c)). \frac{\psi(s) - \psi(\rho(t))}{s - \rho(t)},$$

which on applying $\psi(s) = \psi(\rho(t))$, simply returns $0 = 0$, validating the truth of (4.2). Since η and ψ are continuous at each t, it follows that $\eta \circ \psi$ is continuous at t and hence $\lim_{s\to t} \eta(\psi(s)) = \eta(\psi(t))$. This, by the definition of η in (4.1) together with the mean value theorem in ordinary calculus, gives the existence of $c \in [\rho(t), t]$ such that $\eta(\psi(t)) = \phi'(\psi(c))$. Thus

$$\begin{aligned}(\phi \circ \psi)^\nabla(t) &= \lim_{s\to t} \frac{\phi(\psi(s)) - \phi(\psi(\rho(t)))}{s - \rho(t)} \\ &= \lim_{s\to t} \left[\eta(\psi(s)). \frac{\psi(s) - \psi(\rho(t))}{s - \rho(t)}\right] \\ &= \lim_{s\to t} \eta(\psi(s)). \lim_{s\to t} \frac{\psi(s) - \psi(\rho(t))}{s - \rho(t)} \\ &= \phi'(\psi(c))\psi^\nabla(t).\end{aligned}$$

□

The following example illustrates this result.

Example. Let $\mathbb{T} = \mathbb{Z}$. Take $\phi(t) = t^2 = \psi(t)$, $t \in \mathbb{T}$. We shall find a constant c as in Theorem 4.1 such that $(\phi \circ \psi)^\nabla(3) = \phi'(\psi(c))\psi^\nabla(3)$. We have already seen at the start of this section that $(\phi\circ\psi)^\nabla(t) = 4t^3 - 6t^2 + 4t - 1$ and $\psi^\nabla(t) = 2t - 1$. Also, $\phi'(t) = 2t$. Then $(\phi\circ\psi)^\nabla(3) = \phi'(\psi(c))\psi^\nabla(3)$ implies that $c^2 = \frac{65}{10}$ which yields $c = \sqrt{\frac{13}{2}} \in [\rho(3), 3] = [2, 3]$, as guaranteed in Theorem 4.1.

In the following chain rule, we relax the condition on ψ being continuous on the whole of $\mathbb{R}$. We consider $\phi\colon \mathbb{R} \to \mathbb{R}$ and $\psi\colon \mathbb{T} \to \mathbb{R}$. The chain rule presented in Theorem 4.2 is not only an extension of the chain rule introduced by Pötzsche in [25], to measure chains, but also corresponds to the nabla derivative version.

Theorem 4.2. *(The Chain Rule II) Let $\mathbb{T}$ be a measure chain. Suppose that $\phi\colon \mathbb{R} \to \mathbb{R}$ is continuously differentiable and $\psi\colon \mathbb{T} \to \mathbb{R}$ is nabla differentiable. Then $\phi \circ \psi\colon \mathbb{T} \to \mathbb{R}$ is nabla differentiable and*

$$(\phi \circ \psi)^\nabla(t) = \left\{ \int_0^1 \phi'(\psi(\rho(t)) + \theta\nu(t)\psi^\nabla(t))d\theta \right\} \psi^\nabla(t).$$

Proof. Applying the substitution rule from ordinary calculus, we obtain

$$\begin{aligned}(\phi \circ \psi)(s) - (\phi \circ \psi)(\rho(t)) &= \phi(\psi(s)) - \phi(\psi(\rho(t))) \\ &= \int_{\psi(\rho(t))}^{\psi(s)} \phi'(\tau)d\tau \\ &= [\psi(s) - \psi(\rho(t))] \int_0^1 \phi'(\theta\psi(s) + (1-\theta)\psi(\rho(t)))d\theta. \qquad (4.3)\end{aligned}$$

Let $t \in \mathbb{T}_\kappa$ and $\varepsilon > 0$ be given. Since ψ is nabla differentiable at t, there exists a neighborhood N_1 of t such that for each $s \in N_1$

$$|\psi(s) - \psi(\rho(t)) - \psi^\nabla(t)v(s, \rho(t))| \leq \varepsilon^*|v(s, \rho(t))|, \qquad (4.4)$$

where

$$\varepsilon^* = \frac{\varepsilon}{1 + 2\int_0^1 \phi'(\theta\psi(t) + (1-\theta)\psi(\rho(t)))d\theta} = \frac{\varepsilon}{1 + 2\left[\phi(\psi(t)) - \phi(\psi(\rho(t)))\right]}. \tag{4.5}$$

Moreover, as ϕ' is continuous on $\mathbb{R}$, it is uniformly continuous on closed subsets of $\mathbb{R}$, and since ψ is continuous (being nabla differentiable), there exists a neighborhood N_2 of t such that for each $s \in N_2$

$$|\phi'(\theta\psi(s) + (1-\theta)\psi(\rho(t))) - \phi'(\theta\psi(t) + (1-\theta)\psi(\rho(t)))| \leq \frac{\varepsilon}{2(\varepsilon^* + |\psi^\nabla(t)|)}. \tag{4.6}$$

To see this, note that

$$\begin{aligned} |\theta\psi(s) + (1-\theta)\psi(\rho(t)) - \theta\psi(t) + (1-\theta)\psi(\rho(t))| &= \theta|\psi(s) - \psi(t)| \\ &\leq |\psi(s) - \psi(t)| \end{aligned}$$

holds for all $0 \leq \theta \leq 1$. Let $s \in N = N_1 \cap N_2$. For convenience, we put

$$\alpha = \theta\psi(s) + (1-\theta)\psi(\rho(t)) \quad \text{and} \quad \beta = \theta\psi(t) + (1-\theta)\psi(\rho(t)).$$

It follows from the definition of β that

$$\int_0^1 \phi'(\beta)d\theta = \int_0^1 \phi'(\psi(\rho(t)) + \theta\nu(t)\psi^\nabla(t))d\theta. \tag{4.7}$$

Further, using the definition of β in (4.5), we get

$$\int_0^1 \phi'(\beta)d\theta \leq \frac{\varepsilon}{2\varepsilon^*}. \tag{4.8}$$

Also, from (4.6), we see that

$$\int_0^1 \phi'(\alpha)d\theta \leq \int_0^1 \phi'(\beta)d\theta + \frac{\varepsilon}{2(\varepsilon^* + |\psi^\nabla(t)|)}. \tag{4.9}$$

Therefore, using (4.3)-(4.4), (4.7)-(4.9), for $s \in N$, we get

$$\begin{aligned} &\left|(\phi\circ\psi)(s) - (\phi\circ\psi)(\rho(t)) - v(s,\rho(t))\psi^\nabla(t)\int_0^1 \phi'(\beta)d\theta\right| \\ &= \left|\phi(\psi(s)) - \phi(\psi(\rho(t))) - v(s,\rho(t))\psi^\nabla(t)\int_0^1 \phi'(\beta)d\theta\right| \\ &= \left|[\psi(s) - \psi(\rho(t))]\int_0^1 \phi'(\alpha)d\theta - v(s,\rho(t))\psi^\nabla(t)\int_0^1 \phi'(\beta)d\theta\right| \\ &= \left|[\psi(s) - \psi(\rho(t)) - \psi^\nabla(t)v(s,\rho(t))]\int_0^1 \phi'(\alpha)d\theta + v(s,\rho(t))\psi^\nabla(t)\int_0^1 (\phi'(\alpha) - \phi'(\beta))d\theta\right| \\ &\leq |\psi(s) - \psi(\rho(t)) - \psi^\nabla(t)v(s,\rho(t))|\int_0^1 |\phi'(\alpha)|d\theta + |v(s,\rho(t))||\psi^\nabla(t)|\int_0^1 |\phi'(\alpha) - \phi'(\beta)|d\theta \\ &\leq \varepsilon^*|v(s,\rho(t))|\left[\int_0^1 \phi'(\beta)d\theta + \frac{\varepsilon}{2(\varepsilon^* + |\psi^\nabla(t)|)}\right] + |v(s,\rho(t))||\psi^\nabla(t)|\frac{\varepsilon}{2(\varepsilon^* + |\psi^\nabla(t)|)} \\ &\leq \varepsilon^*|v(s,\rho(t))|\left[\frac{\varepsilon}{2\varepsilon^*} + \frac{\varepsilon}{2(\varepsilon^* + |\psi^\nabla(t)|)}\right] + |v(s,\rho(t))||\psi^\nabla(t)|\frac{\varepsilon}{2(\varepsilon^* + |\psi^\nabla(t)|)} \\ &= \varepsilon|v(s,\rho(t))|. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$(\phi \circ \psi)^{\nabla}(t) = \psi^{\nabla}(t) \int_0^1 \phi'(\beta) d\theta = \left\{ \int_0^1 \phi'(\psi(\rho(t)) + \theta\nu(t)\psi^{\nabla}(t)) d\theta \right\} \psi^{\nabla}(t).$$

□

Let us now illustrate this version of the chain rule. Similar to [8, Example 1.91] we have the following.

Example. Let $\phi\colon \mathbb{R} \to \mathbb{R}$ and $\psi\colon \mathbb{Z} \to \mathbb{R}$ be defined by $\phi(t) = e^t$ and $\psi(t) = t^2$. Then $\phi'(t) = e^t$ and $\psi^{\nabla}(t) = \frac{\psi(t)-\psi(\rho(t))}{t-\rho(t)} = \psi(t) - \psi(t-1) = 2t - 1$. On one hand

$$\begin{aligned}\left\{ \int_0^1 \phi'(\psi(\rho(t)) + \theta\nu(t)\psi^{\nabla}(t)) d\theta \right\} \psi^{\nabla}(t) &= (2t-1) \int_0^1 \exp[(t-1)^2 + \theta(2t-1)] d\theta \\ &= (2t-1)\exp[(t-1)^2] \int_0^1 \exp[\theta(2t-1)] d\theta \\ &= \exp[(t-1)^2][\exp(2t-1) - 1].\end{aligned}$$

On the other hand

$$\begin{aligned}(\phi \circ \psi)^{\nabla}(t) = \nabla\phi(\psi(t)) &= \phi(\psi(t)) - \phi(\psi(t-1)) \\ &= \phi(t^2) - \phi((t-1)^2) \\ &= \exp(t^2) - \exp((t-1)^2) \\ &= \exp[(t-1)^2][\exp(2t-1) - 1].\end{aligned}$$

Thus $(\phi \circ \psi)^{\nabla}(t) = \left\{ \int_0^1 \phi'(\psi(\rho(t)) + \theta\nu(t)\psi^{\nabla}(t)) d\theta \right\} \psi^{\nabla}(t)$.

The following version has major consequences which are discussed in the next section. In this version of the chain rule, we work with two measure chains with nabla derivatives on each of these measure chains.

Theorem 4.3. *(The Chain Rule III) Let $(\mathbb{T}, \leq, v)$ and $(\widetilde{\mathbb{T}}, \widetilde{\leq}, \widetilde{v})$ be two measure chains related by a function $g\colon \mathbb{T} \to \widetilde{\mathbb{T}} \subset \mathbb{R}$. Assume that $f\colon \widetilde{\mathbb{T}} \to \mathbb{R}$ and that $\rho, \widetilde{\rho}, \nabla, \widetilde{\nabla}$ are backward jump operators and nabla derivatives on $\mathbb{T}$ and $\widetilde{\mathbb{T}}$ respectively. Suppose that g has the property that $g(\rho(t)) = \widetilde{\rho}(g(t))$. If $g^{\nabla}(t)$ and $f^{\widetilde{\nabla}}(g(t))$ exist, then $(f \circ g)^{\nabla} = (f^{\widetilde{\nabla}} \circ g) g^{\nabla}$.*

Proof. Let $\varepsilon > 0$ be given. Define $\varepsilon^* = \frac{\varepsilon}{1+|g^{\nabla}(t)|+|f^{\widetilde{\nabla}}(g(t))|}$. Then $\varepsilon^* \in (0, 1)$. Since $g^{\nabla}(t)$ and $f^{\widetilde{\nabla}}(t)$ exist there exists neighborhoods N of t and V of $g(t)$ on which

$$|g(s) - g(\rho(t)) - g^{\nabla}(t)v(s, \rho(t))| \leq \varepsilon^* |v(s, \rho(t))| \quad \text{for each} \quad s \in N, \tag{4.10}$$

and

$$|f(r) - f(\widetilde{\rho}(g(t))) - f^{\widetilde{\nabla}}(g(t))\widetilde{v}(r, \widetilde{\rho}(g(t)))| \leq \varepsilon^* |\widetilde{v}(r, \widetilde{\rho}(g(t)))| \quad \text{for each} \quad r \in V. \tag{4.11}$$

Since g is nabla differentiable and $t \in \mathbb{T}_{\kappa}$, it is continuous at t and as such there exists a neighborhood U of t such that $s \in U$ implies $g(s) \in V$. Now, put $W = N \cap U$ and let $s \in W$. Then $s \in N$ and $g(s) \in V$. Therefore, using (4.10)-(4.11), we get

$$\begin{aligned}&|(f \circ g)(s) - (f \circ g)(\rho(t)) - f^{\widetilde{\nabla}}(g(t))g^{\nabla}(t)v(s, \rho(t))| \\ &= |f(g(s)) - f(g(\rho(t))) - f^{\widetilde{\nabla}}(g(t))\widetilde{v}(g(s), \widetilde{\rho}(g(t))) + f^{\widetilde{\nabla}}(g(t))[\widetilde{v}(g(s), g(\rho(t))) - g^{\nabla}(t)v(s, \rho(t))]| \\ &\leq \varepsilon^* |\widetilde{v}(g(s), g(\rho(t)))| + |f^{\widetilde{\nabla}}(g(t))[g(s) - g(\rho(t)) - g^{\nabla}(t)v(s, \rho(t))]| \\ &\leq \varepsilon^* |g(s) - g(\rho(t)) - g^{\nabla}(t)v(s, \rho(t)) + g^{\nabla}(t)v(s, \rho(t))| + \varepsilon^* |f^{\widetilde{\nabla}}(g(t))||v(s, \rho(t))| \\ &\leq \varepsilon^* \{\varepsilon^* |v(s, \rho(t))| + |g^{\nabla}(t)||v(s, \rho(t))| + |f^{\widetilde{\nabla}}(g(t))||v(s, \rho(t))|\}\end{aligned}$$

$$\leq \varepsilon^*\{1 + |g^\nabla(t)| + |f^{\widetilde{\nabla}}(g(t))|\}|v(s,\rho(t))|$$
$$= \varepsilon|v(s,\rho(t))|.$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$(f \circ g)^\nabla = (f^{\widetilde{\nabla}} \circ g)g^\nabla.$$

□

We give an illustration of Theorem 4.3 in the following example.

Example. Let $\mathbb{T} = \mathbb{N}_0$. With the notations of Theorem 4.3, let $g(t) = t^2$, $\widetilde{\mathbb{T}} = g(\mathbb{T})$, and $f(t) = 2t^2+3$. Then, $\widetilde{\mathbb{T}} = \mathbb{N}_0^2$ and $(f \circ g)(t) = 2t^4 + 3$. Since each $t \in \mathbb{T}$ is left-scattered, we have

$$(f \circ g)^\nabla(t) = \frac{(f \circ g)(t) - (f \circ g)(\rho(t))}{\nu(t)} = f(g(t)) - f(g(t-1)) = 8t^3 - 12t^2 + 8t - 2.$$

To find the nabla derivative of $f \circ g$, we first calculate

$$g^\nabla(t) = \frac{g(t) - g(\rho(t))}{t - \rho(t)} = 2t - 1.$$

For $\mathbb{N}_0^2$, $\widetilde{\rho}(t) = (\sqrt{t} - 1)^2$. Then

$$f^{\widetilde{\nabla}}(t) = \frac{f(t) - f(\widetilde{\rho}(t))}{t - \widetilde{\rho}(t)} = \frac{8t^{3/2} - 12t + 8t^{1/2} - 2}{2\sqrt{t} - 1}.$$

Thus $f^{\widetilde{\nabla}}(g(t)) = \frac{8t^3-12t^2+8t-2}{2t-1}$. Hence

$$(f^{\widetilde{\nabla}} \circ g)(t)g^\nabla(t) = 8t^3 - 12t^2 + 8t - 2 = (f \circ g)^\nabla(t).$$

5. Consequences of the Chain Rule

In this section, we present two major consequences of Theorem 4.3, namely "the derivative of the inverse" and "the substitution rule".

Theorem 5.1. *(Derivative of the Inverse) Suppose that $g\colon \mathbb{T} \to X$ is strictly monotone and $\widetilde{\mathbb{T}} = g(\mathbb{T})$ is the induced measure chain with the jump operator $\widetilde{\rho}(x) = g(\rho(t))$ for $x = g(t)$. Then $g \circ \rho = \widetilde{\rho} \circ g$ on $\mathbb{T}$. Assume that $t \in \mathbb{T}_\kappa$ such that $g^\nabla(t)$ exists and $(g^{-1})^{\widetilde{\nabla}}$ exists at $x = g(t)$. Then $g^\nabla(t) \neq 0$ and $\frac{1}{g^\nabla(t)} = (g^{-1})^{\widetilde{\nabla}}(x)$.*

Proof. By Theorem 4.3, $(f \circ g)^\nabla = (f^{\widetilde{\nabla}} \circ g)g^\nabla$. Let $f = g^{-1}\colon \widetilde{T} \to \mathbb{T}$. Then

$$\begin{aligned}(g^{-1} \circ g)^\nabla &= ((g^{-1})^{\widetilde{\nabla}} \circ g)g^\nabla \\ t^\nabla &= ((g^{-1})^{\widetilde{\nabla}} \circ g)g^\nabla \\ 1 &= ((g^{-1})^{\widetilde{\nabla}} \circ g)g^\nabla.\end{aligned}$$

This yields

$$\frac{1}{g^\nabla(t)} = (g^{-1})^{\widetilde{\nabla}}(g(t)) = (g^{-1})^{\widetilde{\nabla}}(x).$$

□

Theorem 5.2. *(Substitution Rule) Assume that $g\colon \mathbb{T} \to X$ is strictly monotone and $\widetilde{\mathbb{T}} = g(\mathbb{T})$ is a measure chain. If $\Phi\colon \mathbb{T} \to X$ is ld-continuous and g is nabla differentiable with ld-continuous nabla derivative, then for $a, b \in \mathbb{T}$, $\int_a^b \Phi(t)g^\nabla(t)\nabla t = \int_{g(a)}^{g(b)} (\Phi \circ g^{-1})(s)\widetilde{\nabla}s$.*

Proof. Since Φ and g^{∇} are ld-continuous, Φg^{∇} is also an ld-continuous. So, it possess an antiderivative Ψ. Then

$$\begin{aligned}\int_a^b \Phi(t)g^{\nabla}(t)\nabla t &= \int_a^b \Psi^{\nabla}(t)\nabla t \\ &= \int_{g(a)}^{g(b)} (\Psi\circ g^{-1})^{\widetilde{\nabla}}(s)\widetilde{\nabla}s \\ &= \int_{g(a)}^{g(b)} (\Psi^{\nabla}\circ g^{-1})(s)(g^{-1})^{\widetilde{\nabla}}(s)\widetilde{\nabla}s \quad \text{(by Theorem 4.3)} \\ &= \int_{g(a)}^{g(b)} ((\Phi g^{\nabla})\circ g^{-1})(s)(g^{-1})^{\widetilde{\nabla}}(s)\widetilde{\nabla}s \\ &= \int_{g(a)}^{g(b)} (\Phi\circ g^{-1})(s)\widetilde{\nabla}s.\end{aligned}$$

□

We shall now illustrate the usefulness of Theorem 5.2.

Example. Let $\mathbb{T} = \mathbb{N}_0^{1/2}$. We shall evaluate $\int_0^t(\sqrt{\tau^2-1}+\tau)3^{\tau^2}\nabla\tau$, $t\in\mathbb{T}$. We have seen earlier that if $\Phi(t) = t^2$, then $\Phi^{\nabla}(t) = \sqrt{t^2-1}+t$. So we take $g(t) = t^2$ and $\Phi(t) = 3^{t^2}$. The substitution rule now allows the following computation

$$\begin{aligned}\int_0^t (\sqrt{\tau^2-1}+\tau)3^{\tau^2}\nabla\tau &= \int_0^t \Phi(\tau)g^{\nabla}(\tau)\nabla\tau \\ &= \int_0^{t^2} (\Phi\circ g^{-1})(\tau)\widetilde{\nabla}\tau \\ &= \int_0^{t^2} 3^{\tau}\widetilde{\nabla}\tau \\ &= \frac{3^{\tau+1}}{2}\Bigg|_0^{t^2} \\ &= \frac{3^{t^2+1}-3}{2}.\end{aligned}$$

6. Generalizations of the Chain Rule for Nabla Derivatives on Measure Chains

In this section, we suppose that $(\mathbb{T},\leq,\nu)$ is a measure chain with backward jump operator ρ and backward graininess function ν^*. Assume that X and Y are Banach Spaces and we denote the norm on these spaces by $\|\cdot\|$. Let $\phi:\mathbb{T}\times X\to Y$. For a fixed $x_0\in X$, we denote the nabla derivative of $\phi(t,x_0)$ by $\nabla\phi(\cdot,x_0)$ and for a fixed $t_0\in\mathbb{T}$, we denote the Fréchet derivative of $\phi(t_0,x)$ by $\mathscr{F}\phi(t_0,\cdot)$.

Theorem 6.1. *For fixed $t_0\in\mathbb{T}_k$, let $\psi\colon\mathbb{T}\to X$ and $\phi\colon\mathbb{T}\times X\to Y$ be functions such that ψ and $\phi(\cdot,\psi(t_0))$ are differentiable at t_0. Suppose the following conditions hold:*

(1) Let $U\subseteq\mathbb{T}$ be a neighborhood of t_0 such that $\phi(t,\cdot)$ is differentiable for $t\in U\cup\{\rho(t_0)\}$;

(2) $\mathscr{F}\phi(\rho(t_0),\cdot)$ *is continuous on the line segment* $\{\psi(t_0)-h\nu^*(t_0)\psi^\nabla(t_0)\in X\colon h\in[0,1]\}$*;*
(3) $\mathscr{F}\phi$ *is continuous at* $(t_0,\psi(t_0))$*.*

Then $F\colon\mathbb{T}\to Y$ *defined by* $F(t)=\phi(t,\psi(t))$ *is nabla differentiable at* t_0 *and*

$$F^\nabla(t_0)=\nabla\phi(t_0,\psi(t_0))+\left(\int_0^1\mathscr{F}\phi\left(\rho(t_0),\psi(t_0)-h\nu^*(t_0)\psi^\nabla(t_0)\right)dh\right)\psi^\nabla(t_0).$$

Proof. We can choose a neighborhood $U_0\subseteq U$ of t_0 such that for $t\in U_0$

$$\nu^*(t_0)\le|\nu(\rho(t_0),t)|. \tag{6.1}$$

This is trivial if t_0 is left-dense (by choosing $U=U_0$) and we can choose $U_0=\{t\in U\colon\rho(t_0)\le t\}$ in case t_0 is left-scattered. Define

$$\Omega(t,h)=\mathscr{F}\phi\left(t,\psi(t_0)+h(\psi(t)-\psi(t_0))\right)\quad\text{for } t\in U_0 \text{ and } h\in[0,1]. \tag{6.2}$$

Note that there exists a constant $C>0$ such that

$$\|\Omega(t_0,h)-\Omega(\rho(t_0),h)\|\le C\nu^*(t_0)\le C|\nu(\rho(t_0),t)|. \tag{6.3}$$

This trivially holds when t_0 is left-dense. In case t_0 is left-scattered, we see that $\Omega(t_0,\cdot)$ has constant value $\mathscr{F}\phi(t_0,\psi(t_0))\in\mathscr{L}(X,Y)$ and since the mapping $\Omega(\rho(t_0),\cdot)$ is continuous on the compact interval [0,1], it is bounded. From this and (6.3), we get

$$C=\frac{1}{\nu^*(t_0)}\left(\sup_{h\in[0,1]}\|\Omega(\rho(t_0),h)\|+\|\mathscr{F}\phi(t_0,\psi(t_0))\|\right).$$

Let $\varepsilon>0$ be given. Choose $\varepsilon_1>0$ and $\varepsilon_2>0$ such that

$$\varepsilon_1\left(1+C+\left\|\int_0^1\Omega(\rho(t_0),h)\ dh\right\|\right)+\varepsilon_2\big(\varepsilon_1+2\|\psi^\nabla(t_0)\|\big)\le\varepsilon.$$

Since ψ and $\phi(\cdot,\psi(t_0))$ are differentiable at t_0, for $\varepsilon_1>0$ there exists a neighborhood $U_1\subseteq U_0$ such that for for each $t\in U_1$, the following hold:

$$\|\psi(t)-\psi(t_0)\|\le\varepsilon_1, \tag{6.4}$$

$$\|\psi(t)-\psi(\rho(t_0))-\nu(\rho(t_0),t)\psi^\nabla(t_0)\|\le\varepsilon_1|\nu(\rho(t_0),t)|, \tag{6.5}$$

and

$$\|\phi(t,\psi(t_0))-\phi(\rho(t_0),\psi(t_0))-\nu(\rho(t_0),t)\ \nabla\phi(t_0,\psi(t_0))\|\le\varepsilon_1|\nu(\rho(t_0),t)|. \tag{6.6}$$

Hence, for $t\in U_1$, using (6.1) and (6.5), we get

$$\begin{aligned}\|\psi(t)-\psi(t_0)\|&=\|\psi(t)-\psi(\rho(t_0))-\nu(\rho(t_0),t)\psi^\nabla(t_0)+\nu(\rho(t_0),t)\psi^\nabla(t_0)\\&\quad+\psi(\rho(t_0))-\psi(t_0)\|\\&\le\|\psi(t)-\psi(\rho(t_0))-\nu(\rho(t_0),t)\psi^\nabla(t_0)\|\\&\quad+|\nu(\rho(t_0),t_0)|\|\psi^\nabla(t_0)\|+\|\psi(\rho(t_0))-\psi(t_0)\|\\&\le\varepsilon_1|\nu(\rho(t_0),t)|+\|\psi^\nabla(t_0)\||\nu(\rho(t_0),t)|+\|\psi^\nabla(t_0)\|\nu^*(t_0)\\&=\big(\varepsilon_1+2\|\psi^\nabla(t_0)\|\big)|\nu(\rho(t_0),t)|.\end{aligned} \tag{6.7}$$

Since ψ is continuous at t_0 and $\mathscr{F}\phi$ is continuous at $(t_0,\psi(t_0))$, keeping in mind (6.2), for any $\varepsilon_2>0$ there exists a neighborhood $U_2\subseteq U$ of t_0 such that for each $t\in U_2$ and $h\in[0,1]$, we have

$$\|\Omega(t,h)-\Omega(t_0,h)\|\le\varepsilon_2. \tag{6.8}$$

Hence, from (6.2), (6.4)-(6.8) for $t\in U_1\cap U_2$, we get

$$\left\|F(t)-F(\rho(t_0))-\nu(\rho(t_0),t)\left(\nabla\phi(t_0,\psi(t_0))+\int_0^1\Omega(\rho(t_0),h)dh\psi^\nabla(t_0)\right)\right\|$$

$$
\begin{aligned}
&= \Big\| \phi(t,\psi(t)) - \phi(\rho(t_0),\psi(\rho(t_0))) - \phi(\rho(t_0),\psi(t_0)) + \phi(\rho(t_0),\psi(t_0)) \\
&\quad - \phi(t,\psi(t_0)) + \phi(t,\psi(t_0)) - \nu(\rho(t_0),t)\ \nabla\phi(t_0,\psi(t_0)) \\
&\quad - \nu(\rho(t_0),t)\int_0^1 \Omega(\rho(t_0),h)dh\psi^{\nabla}(t_0) - \int_0^1 \Omega(\rho(t_0),h)dh(\psi(t) \\
&\quad - \psi(t_0)) + \int_0^1 \Omega(\rho(t_0),h)\ dh\ (\psi(t)-\psi(t_0)) \Big\| \\
&\le \|\phi(t,\psi(t_0)) - \phi(\rho(t_0),\psi(t_0)) - \nu(\rho(t_0),t)\ \nabla\phi(t_0,\psi(t_0))\| \\
&\quad + \left\| \int_0^1 \Omega(\rho(t_0),h)dh(\psi(t)-\psi(t_0)-\nu(\rho(t_0),t)\psi^{\nabla}(t_0)) \right\| \\
&\quad + \Big\| \phi(t,\psi(t)) - \phi(t_0,\psi(t_0)) - (\phi(\rho(t_0),\psi(\rho(t_0))) - \phi(\rho(t_0),\psi(t_0))) \\
&\quad - \int_0^1 \Omega(\rho(t_0),h)dh(\psi(t)-\psi(t_0)) \Big\| \\
&\le \|\phi(t,\psi(t_0)) - \phi(\rho(t_0),\psi(t_0)) - \nu(\rho(t_0),t)\ \nabla\phi(t_0,\psi(t_0))\| \\
&\quad + \left\| \int_0^1 \Omega(\rho(t_0),h)dh \right\| \left\| \psi(t)-\psi(t_0)-\nu(\rho(t_0),t)\psi^{\nabla}(t_0) \right\| \\
&\quad + \left\| \int_0^1 (\Omega(\rho(t),h) - \Omega(\rho(t_0),h))dh(\psi(t)-\psi(t_0)) \right\| \\
&\le \|\phi(t,\psi(t_0)) - \phi(\rho(t_0),\psi(t_0)) - \nu(\rho(t_0),t)\nabla\phi(t_0,\psi(t_0))\| \\
&\quad + \left\| \int_0^1 \Omega(\rho(t_0),h)dh \right\| \left\| \psi(t)-\psi(t_0)-\nu(\rho(t_0),t)\psi^{\nabla}(t_0) \right\| \\
&\quad + \left\| \int_0^1 (\Omega(t,h) - \Omega(t_0,h))dh \right\| \|(\psi(t)-\psi(t_0))\| \\
&\quad + \left\| \int_0^1 (\Omega(t_0,h) - \Omega(\rho(t_0),h))dh \right\| \|(\psi(t)-\psi(t_0))\| \\
&\le \left(\varepsilon_1 \left(1 + C + \left\| \int_0^1 \Omega(\rho(t_0),h)dh \right\| \right) + \varepsilon_2(\varepsilon_1 + 2\|\psi^{\nabla}(t_0)\|) \right) |\nu(\rho(t_0),t)| \\
&\le \varepsilon |\nu(\rho(t_0),t)|.
\end{aligned}
$$

Since $\varepsilon > 0$ is arbitrary and using (6.2), we have

$$
F^{\nabla}(t_0) = \nabla\phi(t_0,\psi(t_0)) + \left(\int_0^1 \mathscr{F}\phi\left(\rho(t_0),\psi(t_0) - h\nu^*(t_0)\psi^{\nabla}(t_0)\right) dh \right) \psi^{\nabla}(t_0).
$$

□

Now we present a generalization of [8, Theorem 1.90].

Theorem 6.2. *Let $(\mathbb{T}, \leq, \nu)$ be a measure chain with backward jump operator ρ and backward graininess function ν^*. Let $t_0 \in \mathbb{T}_\kappa$ be fixed such that $\psi_j \colon \mathbb{T} \to \mathbb{R}$; $j \in \{1, 2, \dots, n\}$ is differentiable at t_0 and $\phi \colon \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}$ defined by $\phi(\cdot, \psi_1(t_0), \psi_2(t_0), \dots, \psi_n(t_0))$ is continuous at t_0. Let $U \subseteq \mathbb{T}$ be a neighborhood of t_0 such that*

(1) $\phi(t, \cdot, \dots, \cdot)$ is continuously differentiable for $t \in U \cup \{\rho(t_0)\}$;
(2) $\nabla\phi(\cdot, \psi_1(\cdot), \dots, \psi_n(\cdot))$ is continuous at t_0;
(3) $\frac{\partial\phi}{\partial\psi_j}(\rho(t_0), \psi_1(\rho(t_0)), \dots, \psi_{j-1}(\rho(t_0)), \cdot, \psi_{j+1}(\rho(t)), \dots, \psi_n(\rho(t)))$ is continuous on the line segment $\{\psi_j(t) + h(\psi_j(\rho(t_0)) - \psi_j(t)) \in \mathbb{R} \colon h \in [0,1]\}$, $j \in 1, 2, \dots, n$, for all $t \in U \cup \{\rho(t_0)\}$;
(4) $\frac{\partial\phi}{\partial\psi_j}$ is continuous at $(t_0, \psi_1(t_0), \psi_2(t_0), \dots, \psi_n(t_0))$.

Then $F \colon \mathbb{T} \to \mathbb{R}$, $F(t) = (t, \psi_1(t), \psi_2(t), \dots, \psi_n(t))$ is nabla differentiable at t_0 and

$$\begin{aligned}
F^\nabla(t_0) &= \nabla\phi(t_0, \psi_1(t_0), \psi_2(t_0) \dots, \psi_n(t_0)) \\
&\quad + \left(\int_0^1 \frac{\partial\phi}{\partial\psi_1}(\rho(t_0), \psi_1(t_0) - h\nu^*(t_0)\psi_1^\nabla(t_0), \psi_2(t_0), \dots, \psi_n(t_0))dh \right) \psi_1^\nabla(t_0) \\
&\quad + \left(\int_0^1 \frac{\partial\phi}{\partial\psi_2}(\rho(t_0), \psi_1(\rho(t_0)), \psi_2(t_0) - h\nu^*(t_0)\psi_2^\nabla(t_0), , \dots, \psi_n(t_0))dh \right) \psi_2^\nabla(t_0) \\
&\quad + \cdots \\
&\quad + \left(\int_0^1 \frac{\partial\phi}{\partial\psi_n}(\rho(t_0), \psi_1(\rho(t_0)), \dots, \psi_{n-1}(\rho(t_0)), \psi_n(t_0)) - h\nu^*(t_0)\psi_n^\nabla(t_0)dh \right) \psi_n^\nabla(t_0).
\end{aligned}$$

Proof. For $\delta > 0$, small enough, let $s \in (t_0 - \delta, t_0 + \delta) \cap \mathbb{T}$ be such that $s \neq \rho(t_0)$ and $\rho(t_0) < s$ for $\rho(t_0) < t_0$. Then

$$\begin{aligned}
&F(\rho(t_0)) - F(s) \\
&= \phi(\rho(t_0), \psi_1(\rho(t_0)), \psi_2(\rho(t_0)), \dots, \psi_n(\rho(t_0))) - \phi(s, \psi_1(s), \psi_2(s), \dots, \psi_n(s)) \\
&= \phi(\rho(t_0), \psi_1(s), \psi_2(s), \dots, \psi_n(s)) - \phi(s, \psi_1(s), \psi_2(s), \dots, \psi_n(s)) \\
&\quad + \phi(\rho(t_0), \psi_1(\rho(t_0)), \psi_2(s), \dots, \psi_n(s)) - \phi(\rho(t_0), \psi_1(s), \psi_2(s), \dots, \psi_n(s)) \\
&\quad + \phi(\rho(t_0), \psi_1(\rho(t_0)), \psi_2(\rho(t_0)), \dots, \psi_n(s)) - \phi(\rho(t_0), \psi_1(\rho(t_0)), \psi_2(s), \dots, \psi_n(s)) \\
&\quad + \dots \\
&\quad + \phi(\rho(t_0), \psi_1(\rho(t_0)), \psi_2(\rho(t_0)), \dots, \psi_n(\rho(t_0))) - \phi(\rho(t_0), \psi_1(\rho(t_0)), \psi_2(\rho(t_0)), \dots, \psi_n(s)) \\
&= \phi(\rho(t_0), \psi_1(s), \psi_2(s), \dots, \psi_n(s) - \phi(s, \psi_1(s), \psi_2(s), \dots, \psi_n(s)) \\
&\quad + \left(\int_0^1 \frac{\partial\phi}{\partial\psi_1}(\rho(t_0), \psi_1(s) + h[\psi_1(\rho(t_0)) - \psi_1(s)], \psi_2(s), \dots, \psi_n(s))dh \right) \\
&\quad\quad \times (\psi_1(\rho((t_0)) - \psi_1(s))) \\
&\quad + \left(\int_0^1 \frac{\partial\phi}{\partial\psi_2}(\rho(t_0), \psi_1(\rho(t_0)), \psi_2(s) + h[\psi_2(\rho(t_0)) - \psi_2(s)], \dots, \psi_n(\rho(t_0)))dh \right) \\
&\quad\quad \times (\psi_2(\rho((t_0)) - \psi_2(s))) \\
&\quad + \dots \\
&\quad + \left(\int_0^1 \frac{\partial\phi}{\partial\psi_n}(\rho(t_0), \psi_1(\rho(t_0)), \psi_2(\rho(t_0)), \dots, \psi_n(s) + h[\psi_n(\rho(t_0)) - \psi_n(s)])dh \right) \\
&\quad\quad \times (\psi_n(\rho((t_0)) - \psi_n(s))).
\end{aligned}$$

If $\rho(t_0) < t_0$, then by the mean value theorem there exist $\xi_1, \xi_2 \in (\rho(t_0), s] = [t_0, s]$ such that

$$\begin{aligned}&\nabla\phi(\xi_1, \psi_1(s), \psi_2(s) \dots, \psi_n(s))(s - \rho(t_0))\\ &\quad \le \phi(s, \psi_1(s), \psi_2(s) \dots, \psi_n(s)) - \phi(\rho(t_0), \psi_1(s), \psi_2(s) \dots, \psi_n(s))\\ &\quad \le \nabla\phi(\xi_2, \psi_1(s), \psi_2(s) \dots, \psi_n(s))(s - \rho(t_0)),\end{aligned}$$

and

$$\begin{aligned}&\nabla\phi(t_0, \psi_1(s), \psi_2(s) \dots, \psi_n(s))\\ &\quad = \lim_{s\to t_0} \nabla\phi(\xi_1, \psi_1(s), \psi_2(s) \dots, \psi_n(s))\\ &\quad = \lim_{s\to t_0} \frac{\phi(s, \psi_1(s), \psi_2(s) \dots, \psi_n(s)) - \phi(\rho(t_0), \psi_1(s), \psi_2(s) \dots, \psi_n(s))}{s - \rho(t_0)}\\ &\quad \le \lim_{s\to t_0} \nabla\phi(\xi_2, \psi_1(s), \psi_2(s) \dots, \psi_n(s))\\ &\quad = \nabla\phi(t_0, \psi_1(s), \psi_2(s) \dots, \psi_n(s)).\end{aligned}$$

If $\rho(t_0) = t_0$, then, by the mean value theorem, there exist ξ_1, ξ_2 between s and t_0 such that

$$\begin{aligned}&\nabla\phi(\xi_1, \psi_1(s), \psi_2(s) \dots, \psi_n(s))(s - t_0)\\ &\quad \le \phi(s, \psi_1(s), \psi_2(s) \dots, \psi_n(s)) - \phi(t_0, \psi_1(s), \psi_2(s) \dots, \psi_n(s))\\ &\quad \le \nabla\phi(\xi_2, \psi_1(s), \psi_2(s) \dots, \psi_n(s))(s - t_0).\end{aligned}$$

In this case, for $s < t_0$, we have

$$\begin{aligned}\nabla\phi(t_0, \psi_1(s), \psi_2(s) \dots, \psi_n(s)) &= \lim_{s\to t_0^-} \nabla\phi(\xi_1, \psi_1(s), \psi_2(s) \dots, \psi_n(s))\\ &\ge \lim_{s\to t_0^-} \frac{\phi(s, \psi_1(s), \psi_2(s) \dots, \psi_n(s)) - \phi(t_0, \psi_1(s), \psi_2(s) \dots, \psi_n(s))}{s - t_0}\\ &\ge \lim_{s\to t_0^-} \nabla\phi(\xi_2, \psi_1(s), \psi_2(s) \dots, \psi_n(s))\\ &= \nabla\phi(t_0, \psi_1(s), \psi_2(s) \dots, \psi_n(s)),\end{aligned}$$

and for $s > t_0$, we have

$$\begin{aligned}\nabla\phi(t_0, \psi_1(s), \psi_2(s) \dots, \psi_n(s)) &= \lim_{s\to t_0^+} \nabla\phi(\xi_1, \psi_1(s), \psi_2(s) \dots, \psi_n(s))\\ &\le \lim_{s\to t_0^+} \frac{\phi(s, \psi_1(s), \psi_2(s) \dots, \psi_n(s)) - \phi(t_0, \psi_1(s), \psi_2(s) \dots, \psi_n(s))}{s - t_0}\\ &\le \lim_{s\to t_0^+} \nabla\phi(\xi_2, \psi_1(s), \psi_2(s) \dots, \psi_n(s))\\ &= \nabla\phi(t_0, \psi_1(s), \psi_2(s) \dots, \psi_n(s)).\end{aligned}$$

Moreover

$$\begin{aligned}&\lim_{s\to t_0} \Bigg(\Bigg(\int_0^1 \frac{\partial\phi}{\partial\psi_j}(\rho(t_0), \psi_1(\rho(t_0)), \dots, \psi_{j-1}(\rho(t_0)), \psi_j(s) + h[\psi_j(\rho(t_0)) - \psi_j(s)],\\ &\psi_{j+1}(s), \dots, \psi_n(s))dh \Bigg) \times \frac{\psi_j(\rho(t_0)) - \psi_j(s)}{\rho(t_0) - s} \Bigg)\\ &= \lim_{s\to t_0} \Bigg(\int_0^1 \frac{\partial\phi}{\partial\psi_j}(\rho(t_0), \psi_1(\rho(t_0)), \dots, \psi_{j-1}(\rho(t_0)), \psi_j(s) + h[\psi_j(\rho(t_0)) - \psi_j(s)],\\ &\psi_{j+1}(s), \dots, \psi_n(s))dh \Bigg) \times \frac{\psi_j(\rho(t_0)) - \psi_j(s)}{\rho(t_0) - s}\end{aligned}$$

$$= \Bigg(\int_0^1 \frac{\partial}{\partial \psi_j} \phi(\rho(t_0), \psi_1(\rho(t_0)), \dots, \psi_{j-1}(\rho(t_0)), \psi_j(t_0) + h[\psi_j(\rho(t_0)) - \psi_j(t_0)],$$

$$\psi_{j+1}(t_0), \dots, \psi_n(t_0)) dh \Bigg) \times \psi_j^{\nabla}(t_0), \quad j \in 1, \dots, n.$$

Therefore

$$\begin{aligned}
&\lim_{s \to t_0} \frac{F(\rho(t_0)) - F(s)}{\rho(t_0) - s} \\
&= \lim_{s \to t_0} \frac{\phi(\rho(t_0), \psi_1(s), \psi_2(s), \dots, \psi_n(s)) - \phi(s, \psi_1(s), \psi_2(s), \dots, \psi_n(s))}{\rho(t_0) - s} \\
&\quad + \lim_{s \to t_0} \Bigg(\Bigg(\int_0^1 \frac{\partial \phi}{\partial \psi_1} (\rho(t_0), \psi_1(s) + h[\psi_1(\rho(t_0)) - \psi_1(s)], \psi_2(s), \dots, \psi_n(s)) dh \Bigg) \\
&\quad \times \frac{\psi_1(\rho(t_0)) - \psi_1(s)}{\rho(t_0) - s} \Bigg) \\
&\quad + \lim_{s \to t_0} \Bigg(\Bigg(\int_0^1 \frac{\partial \phi}{\partial \psi_2} (\rho(t_0), \psi_1(\rho(t_0)), \psi_2(s) + h[\psi_2(\rho(t_0)) - \psi_2(s)], \dots, \psi_n(\rho(t_0))) dh \Bigg) \\
&\quad \times \frac{\psi_2(\rho(t_0)) - \psi_2(s)}{\rho(t_0) - s} \Bigg) \\
&\quad + \dots \\
&\quad + \lim_{s \to t_0} \Bigg(\Bigg(\int_0^1 \frac{\partial \phi}{\partial \psi_n} (\rho(t_0), \psi_1(\rho(t_0)), \psi_2(\rho(t_0)), \dots, \psi_n(s) + h[\psi_n(\rho(t_0)) - \psi_n(s)]) dh \Bigg) \\
&\quad \times \frac{\psi_2(\rho(t_0)) - \psi_2(s)}{\rho(t_0) - s} \Bigg),
\end{aligned}$$

which yields

$$\begin{aligned}
&F^{\nabla}(t_0) \\
&= \nabla \phi(t_0, \psi_1(t_0), \psi_2(t_0) \dots, \psi_n(t_0)) \\
&\quad + \Bigg(\int_0^1 \frac{\partial \phi}{\partial \psi_1} (\rho(t_0), \psi_1(t_0) - h\nu^*(t_0)\psi_1^{\nabla}(t_0), \psi_2(t_0), \dots, \psi_n(t_0)) dh \Bigg) \psi_1^{\nabla}(t_0) \\
&\quad + \Bigg(\int_0^1 \frac{\partial \phi}{\partial \psi_2} (\rho(t_0), \psi_1(\rho(t_0)), \psi_2(t_0) - h\nu^*(t_0)\psi_2^{\nabla}(t_0), , \dots, \psi_n(t_0)) dh \Bigg) \psi_2^{\nabla}(t_0) \\
&\quad + \cdots \\
&\quad + \Bigg(\int_0^1 \frac{\partial \phi}{\partial \psi_n} (\rho(t_0), \psi_1(\rho(t_0)), \dots, \psi_{n-1}(\rho(t_0)), \psi_n(t_0)) - h\nu^*(t_0)\psi_n^{\nabla}(t_0) dh \Bigg) \psi_n^{\nabla}(t_0).
\end{aligned}$$

□

Below is an illustration of Theorem 6.1.

Example. In accordance with the prerequisites of Theorem 6.1, take $X = Y = \mathbb{R}$ and $\mathbb{T} = k\mathbb{Z} = \{kz\colon z \in \mathbb{Z}\}$, $k > 0$. Let $\phi : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ be defined by $\phi(t, x) = x^2$ and $\psi : \mathbb{T} \to \mathbb{R}$ be defined by $\psi(t) = t$. Since ϕ is continuous and every $t_0 \in k\mathbb{Z}$ is left-scattered, we have $\psi^{\nabla}(t_0) = \frac{\psi(t_0) - \psi(\rho(t_0))}{\nu^*(t_0)} = 1$.

On one hand $F(t) = (\phi \circ \psi)(t) = \phi(t, \psi(t)) = \phi(t,t) = t^2$. Therefore

$$\begin{aligned} F^{\nabla}(t_0) &= \frac{F(t_0) - F(\rho(t_0))}{\nu^*(t_0)} \\ &= \frac{F(t_0) - F(t_0 - k)}{k} \\ &= 2t_0 - k. \end{aligned} \tag{6.9}$$

On the other hand

$$\begin{aligned} &\nabla\phi(t_0, \psi(t_0)) + \left(\int_0^1 \mathscr{F}\phi(\rho(t_0), \psi(t_0) - h\nu^*(t_0)\psi^{\nabla}(t_0))dh \right) \psi^{\nabla}(t_0) \\ &= \nabla\phi(t_0, t_0) + \left(\int_0^1 \mathscr{F}\phi\big(t_0 - k, t_0 - h \times k \times 1\big)dh \right) \times 1 \\ &= 0 + 2\int_0^1 (t_0 - hk)\ dh \\ &= -2\left[\frac{(t_0 - hk)^2}{2k}\right]_{h=0}^{h=1} \\ &= 2t_0 - k. \end{aligned} \tag{6.10}$$

This illustration verifies Theorem 6.1.

We now give an illustration of Theorem 6.2.

Example. let $\mathbb{T} = 2^{\mathbb{Z}} \cup \{0\}$. Define $\phi\colon \mathbb{T} \times \mathbb{R}^2 \to \mathbb{R}$ by $\phi(t, \psi_1, \psi_2) = t + \psi_1 + \psi_2$, where $\psi_1(t) = t$ and $\psi_2(t) = t^3$. Let $t_0 \in \mathbb{T}_\kappa = 2^{\mathbb{Z}} \cup \{0\}$ and U be a neighborhood of t_0. Since $\mathscr{F}\phi(t, \psi_1, \psi_2) = \begin{pmatrix}1 & 1\end{pmatrix}$ exists and is a constant matrix, it follows that $\phi(t, \cdot, \cdot)$ is continuously differentiable for $t \in U \cup \{\rho(t_0)\}$. Next, $\nabla\phi(t, \psi_1(t), \psi_2(t)) = 1$ and hence is continuous at t_0. Also, since $\frac{\partial\phi}{\partial\psi_1}(\rho(t_0), \cdot, \psi_2(t_0)) = 1$ and $\frac{\partial\phi}{\partial\psi_2}(\rho(t_0), \psi_1(\rho(t_0)), \cdot) = 1$, they are continuous on the line segment $\{\psi_j(t) + h(\psi_j(\rho(t_0)) - \psi_j(t)) \in \mathbb{R}\colon h \in [0,1]\}$, $j \in 1, 2$, for all $t \in U \cup \{\rho(t_0)\}$. Further, $\frac{\partial\phi}{\partial\psi_j}$ are also continuous at $(t_0, \psi_1(t_0), \psi_2(t_0))$ for $j = 1, 2$. Let $F(t) = \phi(t, \psi_1(t), \psi_2(t)) = 2t + t^3$. Then, for $t_0 \in 2^{\mathbb{Z}}$, we have

$$F^{\nabla}(t_0) = 2 + \frac{7}{4}t_0^2, \tag{6.11}$$

$$\begin{aligned} \psi_1^{\nabla}(t_0) &= 1, \\ \psi_2^{\nabla}(t_0) &= \frac{7}{4}t_0^2, \end{aligned}$$

and

$$\begin{aligned} &\nabla\phi(t_0, \psi_1(t_0), \psi_2(t_0)) + \left(\int_0^1 \frac{\partial\phi}{\partial\psi_1}(\rho(t_0), \psi_1(t_0) - h\nu^*(t_0)\psi_1^{\nabla}(t_0), \psi_2(t_0))dh \right) \psi_1^{\nabla}(t_0) \\ &\quad + \left(\int_0^1 \frac{\partial\phi}{\partial\psi_2}(\rho(t_0), \psi_1(\rho(t_0)), \psi_2(t_0) - h\nu^*(t_0)\psi_2^{\nabla}(t_0))dh \right) \psi_2^{\nabla}(t_0) \\ &= 1 + 1 + t_0^2\left(\frac{7}{4}\right) \\ &= 2 + \frac{7}{4}t_0^2. \end{aligned} \tag{6.12}$$

Also, for $t_0 = 0$, we have

$$\begin{aligned} F^{\nabla}(0) &= \lim_{s\to 0} \frac{2s+s^3}{s} \\ &= 2, \end{aligned} \tag{6.13}$$

and

$$\begin{aligned} &\nabla\phi(t_0,\psi_1(t_0),\psi_2(t_0)) + \left(\int_0^1 \frac{\partial\phi}{\partial\psi_1}(\rho(t_0),\psi_1(t_0)-h\nu^*(t_0)\psi_1^{\nabla}(t_0),\psi_2(t_0))dh\right)\psi_1^{\nabla}(t_0) \\ &\quad + \left(\int_0^1 \frac{\partial\phi}{\partial\psi_2}(\rho(t_0),\psi_1(\rho(t_0)),\psi_2(t_0)-h\nu^*(t_0)\psi_2^{\nabla}(t_0))dh\right)\psi_2^{\nabla}(t_0) \\ &\quad = 1+1+0 \\ &\quad = 2. \end{aligned} \tag{6.14}$$

As (6.11) agrees with (6.12), and (6.13) agrees with (6.14), we see that Theorem 6.2 is verified.

7. Conclusion

In this paper, we have discussed different versions of the chain rule for nabla derivatives. Starting with the preliminaries, some results required to establish the fundamental rules are built up. Suitable examples are provided to understand the theory of nabla differentiation. We first establish three versions of the chain rule on measure chains – an extension of chain rules for the delta derivatives on time scales. To give a better understanding, the newly established chain rules are suitably illustrated on time scales. We then generalize these chain rules in a more general setup by including functions that are Fréchet differentiable. Two striking consequences of these chain rules have been deduced – namely, the derivative of the inverse and the substitution rule for nabla integrals on measure chains.

Acknowledgment

The authors express their gratitude to the reviewers for providing valuable suggestions to enhance the quality of paper. Jervin Zen Lobo is thankful to the Department of Science & Technology and Waste Management, Government of Goa, for providing financial grant-in-aid support with Order No. 6-85-2022/S&T&WM/1113.

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(Received ??.??.20??)

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