

\mathcal{I}_σ -CONVERGENCE IN FUZZY NORMED SPACES

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Abstract. This paper is a study on ideal invariant convergence types and ideal invariant Cauchy sequence in fuzzy normed spaces. In this sense, firstly, we defined the notion of \mathcal{I}_σ -convergence and investigated the relationship between this new notion and invariant convergence in fuzzy normed spaces. Also, we established a relationship between notion \mathcal{I}_σ -convergence and notion q -strongly σ -convergence in fuzzy normed spaces. Then, we defined \mathcal{I}_σ^* -convergence and by using the property (AP), investigated the relationships between \mathcal{I}_σ^* -convergence and \mathcal{I}_σ -convergence in fuzzy normed spaces. Finally, we defined \mathcal{I}_σ -Cauchy sequence and \mathcal{I}_σ^* -Cauchy sequence and examined the some relationships between the new notions in fuzzy normed spaces.

1. SECTION

Zadeh [38] was the first to propose the concept of fuzzy set, which describes imprecise phenomena in the real world. In the sense of fuzzy, several concepts were redefined such as fuzzy number [10], fuzzy calculus [17], fuzzy metric [16, 19, 23], fuzzy norm [2, 5, 14, 20] and fuzzy topology [4, 7, 26].

Banach limit was given as a generalized ordinary limit [3]. Lorentz [25] defined almost convergence with using Banach limit's equality. Later, as a generalization of almost convergence, invariant convergence has taken its place in literature thanks to some important studies [8, 31, 34]. Moreover, statistical invariant convergence types were studied by [27] and [37].

The notion of statistical convergence are based on the studies of Fast [13] and Steinhaus [35]. Some important feature of statistical convergence were given in [6, 15, 32]. Kostyrko et al. [22] introduced \mathcal{I} -convergence, which is the generalization of statistical convergence. \mathcal{I} -Cauchy sequence was investigated by some author [9, 29]. Ideal convergence was studied by Kumar and Kumar [24] on fuzzy numbers sequences, by [12] on double sequences of fuzzy numbers by Hazarika [18] on fuzzy normed spaces and by Kişi and Nuray [21]; Ulusu and Kişi [36] on sequences of sets. Also Nuray et al. [30] and Dündar et al. [11] have taken invariant and ideal convergence together.

Now, let us look at basic notions. The following notions of ideal, filter, \mathcal{I} -convergence and \mathcal{I}^* -convergence are considered as given in [22].

Let $U \neq \emptyset$. A class \mathcal{I} of subsets of U is said to be an ideal on U if and only if

i) $\emptyset \in \mathcal{I}$, ii) $K, M \in \mathcal{I}$ implies $K \cup M \in \mathcal{I}$, iii) $K \in \mathcal{I}$, $M \subset K$ implies $M \in \mathcal{I}$.

\mathcal{I} is called a nontrivial ideal if $\mathcal{I} \neq \emptyset$ and $U \notin \mathcal{I}$. A nontrivial ideal \mathcal{I} in U is called admissible if $\{u\} \in \mathcal{I}$ for each $u \in U$. Then after, we let $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal.

Let $U \neq \emptyset$. A non empty class \mathcal{F} of subsets of U is said to be a filter on U iff

i) $\emptyset \notin \mathcal{F}$, ii) $K, M \in \mathcal{F}$ implies $K \cap M \in \mathcal{F}$, iii) $K \in \mathcal{F}$, $K \subset M$ implies $M \in \mathcal{F}$.

If \mathcal{I} is a nontrivial ideal in U , $U \neq \emptyset$, then the class $\mathcal{F}(\mathcal{I}) = \{F \subset U : (\exists K \in \mathcal{I})(F = U \setminus K)\}$ is a filter on U , called the filter associated with \mathcal{I} .

A sequence $x = (x_n)$ is said to be \mathcal{I} -convergent to L if and only if for every $\varepsilon > 0$ $A(\varepsilon) = \{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\} \in \mathcal{I}$. In this case, we write $\mathcal{I}\text{-}\lim_{n \rightarrow \infty} x_n = L$.

A sequence $x = (x_n)$ is said to be \mathcal{I}^* -convergent to L if and only if there exists a set $M \in \mathcal{F}$ (i.e., $\mathbb{N} \setminus M \in \mathcal{I}$), $M = \{m_1 < m_2 < \dots < m_k < \dots\} \subset \mathbb{N}$ such that $\lim_{k \rightarrow \infty} x_{m_k} = L$.

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An admissible ideal $\mathcal{I} \subset 2^{\mathbb{N}}$ satisfies the condition (AP) if for every countable family of mutually disjoint sets $\{K_1, K_2, \dots\}$ belonging to \mathcal{I} there exists a countable family of sets $\{M_1, M_2, \dots\}$ such that $K_j \Delta M_j$ is a finite set for $j \in \mathbb{N}$ and $M = \bigcup_{j=1}^{\infty} M_j \in \mathcal{I}$ [22].

A sequence $x = (x_n)$ is said to be \mathcal{I} -Cauchy sequence if for every $\varepsilon > 0$ there exists $N = N(\varepsilon)$ such that $A(\varepsilon) = \{n \in \mathbb{N} : |x_n - x_N| \geq \varepsilon\} \in \mathcal{I}$ [9].

A sequence $x = (x_n)$ is said to be \mathcal{I}^* -Cauchy sequence if there exists a set $M \in \mathcal{F}$ (i.e., $\mathbb{N} \setminus M \in \mathcal{I}$), $M = \{m_1 < m_2 < \dots < m_k < \dots\} \subset \mathbb{N}$ such that the subsequence $x_M = (x_{m_k})$ is an ordinary Cauchy sequence, i.e., $\lim_{k,p \rightarrow \infty} |x_{m_k} - x_{m_p}| = 0$ [29].

Let σ be a mapping of the positive integers into itself. A continuous linear functional ϕ on the space of real bounded sequences (ℓ_{∞}) is an invariant mean (σ -mean) if following axioms are satisfied

- (i) $\phi(x) \geq 0$, when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n ,
- (ii) $\phi(e) = 1$, where $e = (1, 1, 1, \dots)$,
- (iii) $\phi(x_{\sigma(n)}) = \phi(x)$, for all $x \in \ell_{\infty}$.

The mapping σ is one-to-one and satisfied the condition $\sigma^m(n) \neq n$ for any positive integers n and m , where $\sigma^m(n)$ denotes the m -th iterate of the mapping σ at n . Invariant mean, ϕ , is a generalization of the limit functional on the space of convergent sequences. The sequence is called invariant convergent when its invariant means have the same value. If $\sigma(n) = n + 1$, the σ -mean is Banach limit and invariant convergent is almost convergent [34].

A bounded sequence $x = (x_n)$ is said to be invariant convergent to the number ℓ if and only if $\lim_{m \rightarrow \infty} t_{mn} = \ell$ uniformly in n , where $t_{mn} = \frac{x_{\sigma(n)} + x_{\sigma^2(n)} + \dots + x_{\sigma^m(n)}}{m}$ [34].

Let $K \subseteq \mathbb{N}$,

$$s_m = \min_n |K \cap \{\sigma(n), \sigma^2(n), \dots, \sigma^m(n)\}| \quad \text{and} \quad S_m = \max_n |K \cap \{\sigma(n), \sigma^2(n), \dots, \sigma^m(n)\}|.$$

If the limits $\underline{\mathcal{V}}(K) = \lim_{m \rightarrow \infty} \frac{s_m}{m}$ and $\overline{\mathcal{V}}(K) = \lim_{m \rightarrow \infty} \frac{S_m}{m}$ exist, then they are called lower and upper σ -uniform density of K , respectively. If $\underline{\mathcal{V}}(K) = \overline{\mathcal{V}}(K)$, then $\mathcal{V}(K) = \underline{\mathcal{V}}(K) = \overline{\mathcal{V}}(K)$ is called the σ -uniform density of K [30].

In the case $\sigma(n) = n + 1$, \mathcal{V} density become the definition of uniform density u in [1].

Denote by \mathcal{I}_{σ} the class of all $K \subset \mathbb{N}$ with $\mathcal{V}(K) = 0$.

Let $\mathcal{I}_{\sigma} \subset 2^{\mathbb{N}}$ be an admissible ideal. A sequence $x = (x_k)$ is said to be \mathcal{I}_{σ} -convergent to the number L if for every $\varepsilon > 0$ $A_{\varepsilon} = \{k : |x_k - L| \geq \varepsilon\} \in \mathcal{I}_{\sigma}$; i.e., $V(A_{\varepsilon}) = 0$. In this case, we write $\mathcal{I}_{\sigma} - \lim_k = L$.

Let $\mathcal{I}_{\sigma} \subset 2^{\mathbb{N}}$ be an admissible ideal. A sequence $x = (x_k)$ is said to be \mathcal{I}_{σ}^* -convergent to the number L if there exists a set $M = \{m_1 < m_2 < \dots\} \in \mathcal{F}(\mathcal{I}_{\sigma})$ such that $\lim_{k \rightarrow \infty} x_{m_k} = L$. In this case, we write $\mathcal{I}_{\sigma}^* - \lim_k = L$ [30].

Now, the concepts and definitions given below are available in references [19, 28].

A fuzzy number μ is a fuzzy set provided that

- (i) μ is normal, i.e., there exists a $t_0 \in \mathbb{R}$ such that $\mu(t_0) = 1$;
- (ii) μ is fuzzy convex, i.e., $\mu(ct + (1 - c)s) \geq \min\{\mu(t), \mu(s)\}$ for any $t, s \in \mathbb{R}$ and $0 \leq c \leq 1$,
- (iii) μ is upper semi-continuous,
- (iv) $cl\{t \in \mathbb{R} : \mu(t) > 0\}$ is a compact set.

The set of all fuzzy number is denoted by $L(\mathbb{R})$. \mathbb{R} is included by $L(\mathbb{R})$ since each $r \in \mathbb{R}$ can be written as fuzzy real number \tilde{r} with $\tilde{r}(k) = 1$ if $k = r$ and $\tilde{r}(k) = 0$ if $k \neq r$.

For $\mu \in L(\mathbb{R})$, the α -level set of μ is defined by

$$[\mu]_{\alpha} = \begin{cases} \{t \in \mathbb{R} : \mu(t) \geq \alpha\}, & \alpha \in (0, 1], \\ cl\{t \in \mathbb{R} : \mu(t) > \alpha\}, & \alpha = 0. \end{cases}$$

The α -level set of a fuzzy number denoted by $[\mu]_{\alpha} = [\mu_{\alpha}^{-}, \mu_{\alpha}^{+}]$ is a non-empty, bounded and closed interval for each $\alpha \in [0, 1]$. Also $\mu_{\alpha}^{-} = -\infty$ and $\mu_{\alpha}^{+} = \infty$ are admissible.

If $\mu \in L(\mathbb{R})$ and $\mu(t) = 0$ for $t < 0$, then μ is called a non-negative fuzzy number. Let $L^*(\mathbb{R})$ denote the set of all non-negative fuzzy number. Clearly, $\tilde{0} \in L^*(\mathbb{R})$.

The partial ordering \preceq on $L(\mathbb{R})$ is defined by $\mu \preceq \eta$ iff $\mu_\alpha^- \leq \eta_\alpha^-$ and $\mu_\alpha^+ \leq \eta_\alpha^+$, for all $\alpha \in [0, 1]$ and all $\mu, \eta \in L(\mathbb{R})$.

Let $\mu, \eta \in L(\mathbb{R})$. Arithmetic equations are defined by

- (i) $(\mu \oplus \eta)(t) = \sup_{s \in \mathbb{R}} \{\mu(s) \wedge \eta(t-s)\}$, $t \in \mathbb{R}$,
- (ii) $(\mu \odot \eta)(t) = \sup_{s \in \mathbb{R}, s \neq 0} \{\mu(s) \wedge \eta(t/s)\}$, $t \in \mathbb{R}$,
- (iii) For $c \in \mathbb{R}^+$, $c\mu$ is defined as $c\mu(t) = \mu(t/c)$ and $0\mu(t) = \tilde{0}$, $t \in \mathbb{R}$.

Let $\mu, \eta \in L(\mathbb{R})$ and $[\mu]_\alpha = [\mu_\alpha^-, \mu_\alpha^+]$, $[\eta]_\alpha = [\eta_\alpha^-, \eta_\alpha^+]$. Arithmetic equations in terms of α -level sets are defined by

- (i) $[\mu \oplus \eta]_\alpha = [\mu_\alpha^- + \eta_\alpha^-, \mu_\alpha^+ + \eta_\alpha^+]$, (ii) $[\mu \odot \eta]_\alpha = [\mu_\alpha^- \cdot \eta_\alpha^-, \mu_\alpha^+ \cdot \eta_\alpha^+]$, $\mu, \eta \in L^*(\mathbb{R})$,
- (iii) $[c\mu]_\alpha = c[\mu]_\alpha = \begin{cases} [c\mu_\alpha^-, c\mu_\alpha^+], & c \geq 0, \\ [c\mu_\alpha^+, c\mu_\alpha^-], & c < 0. \end{cases}$

The supremum metric on $L(\mathbb{R})$ is defined by $D(\mu, \eta) = \sup_{0 \leq \alpha \leq 1} \max\{|\mu_\alpha^- - \eta_\alpha^-|, |\mu_\alpha^+ - \eta_\alpha^+|\}$. $D(\mu, \tilde{0}) = \sup_{0 \leq \alpha \leq 1} \max\{|\mu_\alpha^-|, |\mu_\alpha^+|\} = \max\{|\mu_0^-|, |\mu_0^+|\}$. $D(\mu, \tilde{0}) = \mu_\alpha^+$ if $\mu \in L^*(\mathbb{R})$.

A sequence (μ_n) in $L(\mathbb{R})$ is called convergent to $\mu \in L(\mathbb{R})$ denoted by $(D) - \lim_{n \rightarrow \infty} \mu_n = \mu$ if $\lim_{n \rightarrow \infty} D(\mu_n, \mu) = 0$, i.e., for all given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n > N$, $D(\mu_n, \mu) < \varepsilon$.

Felbin defined the following fuzzy norm [14], which is the basis of this study, using Kaleva and Seikkala's metric [19].

Let U be a vector space over \mathbb{R} , $\|\cdot\| : U \rightarrow L^*(\mathbb{R})$ and the mappings $L, R : [0, 1] \times [0, 1] \rightarrow [0, 1]$ be symmetric, nondecreasing in both arguments and satisfy $L(0, 0) = 0$ and $R(1, 1) = 1$. The quadruple $(U, \|\cdot\|, L, R)$ is called fuzzy normed linear space (FNS) and $\|\cdot\|$ is a fuzzy norm if the following axioms are satisfied

- (i) $\|u\| = \tilde{0}$ iff $u = \theta$, (ii) $\|cu\| = |c| \|u\|$ for $u \in U$, $c \in \mathbb{R}$,
- (iii) For all $u, v \in U$,
 - (a) $\|u + v\|(s + t) \geq L(\|u\|(s), \|v\|(t))$, whenever $s \leq \|u\|_1^-$, $t \leq \|v\|_1^-$ and $s + t \leq \|u + v\|_1^-$,
 - (b) $\|u + v\|(s + t) \leq R(\|u\|(s), \|v\|(t))$, whenever $s \geq \|u\|_1^-$, $t \geq \|v\|_1^-$ and $s + t \geq \|u + v\|_1^-$.

When $L = \min$ and $R = \max$ are taken in above (iii), triangle inequalities become $\|u + v\|_\alpha^- \leq \|u\|_\alpha^- + \|v\|_\alpha^-$ and $\|u + v\|_\alpha^+ \leq \|u\|_\alpha^+ + \|v\|_\alpha^+$ for all $\alpha \in (0, 1]$ and $u, v \in U$. And, $\|u\|_\alpha^-$ and $\|u\|_\alpha^+$ have the properties as ordinary norms on U . From now on, let $(U, \|\cdot\|)$ be a fuzzy normed linear space.

Example. Let $(U, \|\cdot\|_C)$ be an ordinary normed linear space. Then, a fuzzy norm $\|\cdot\|$ on U can be obtained as;

$$\|u\|(t) = \begin{cases} 0, & \text{if } 0 \leq t \leq a \|u\|_C \text{ or } t \geq b \|u\|_C, \\ \frac{t}{(1-a)\|u\|_C} - \frac{a}{1-a}, & \text{if } a \|u\|_C \leq t \leq \|u\|_C, \\ \frac{t}{(b-1)\|u\|_C} + \frac{b}{b-1}, & \text{if } \|u\|_C \leq t \leq b \|u\|_C, \end{cases}$$

where $\|u\|_C$ is the ordinary norm of u ($\neq \theta$), $0 < a < 1$ and $1 < b < \infty$. For $u = \theta$, define $\|u\| = \tilde{0}$. Hence, $(U, \|\cdot\|)$ is a fuzzy normed linear space.

Let us consider the topological structure of an FNS $(U, \|\cdot\|)$. For any $\varepsilon > 0$, $\alpha \in [0, 1]$ and $u \in U$, the (ε, α) -neighborhood of u is the set $\mathcal{N}_u(\varepsilon, \alpha) := \{v \in U : \|u - v\|_\alpha^+ < \varepsilon\}$.

A sequence $u = (u_n)$ in U is said to be convergent to $\ell \in U$ with respect to the fuzzy norm and we denote by $u_n \xrightarrow{FN} \ell$, provided that $(D) - \lim_{n \rightarrow \infty} \|u_n - \ell\| = \tilde{0}$, i.e., for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $D(\|u_n - \ell\|, \tilde{0}) < \varepsilon$. This means that for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n > N$, $\sup_{\alpha \in [0, 1]} \|u_n - \ell\|_\alpha^+ = \|u_n - \ell\|_0^+ < \varepsilon$. In terms of neighborhoods, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $u_n \in \mathcal{N}_\ell(\varepsilon, 0)$.

A sequence $u = (u_n)$ in U is said to be invariant convergent to ℓ with respect to fuzzy norm and denoted by $u_n \xrightarrow{\sigma-FN} \ell$ if $(D) - \lim_{m \rightarrow \infty} \|t_{mn} - \ell\| = \tilde{0}$, uniformly in n . Namely, since $D(\|t_{mn} - \ell\|, \tilde{0}) = \sup_{\alpha \in [0, 1]} \|t_{mn} - \ell\|_\alpha^+ = \|t_{mn} - \ell\|_0^+$, $\lim_{m \rightarrow \infty} \|t_{mn} - \ell\|_0^+ = 0$, uniformly in n . So, for every $\varepsilon > 0$, there exists

$m_0 \in \mathbb{N}$ such that for all $m > m_0$ and every $n \in \mathbb{N}$, $\|t_{mn} - \ell\|_0^+ < \varepsilon$. In terms of neighborhood, for every $\varepsilon > 0$, there exists $m_0 \in \mathbb{N}$ such that for all $m > m_0$, $t_{mn} \in \mathcal{N}_\ell(\varepsilon, 0)$, uniformly in n [37].

The sequence $u = (u_n)$ in U is said to be q -strongly invariant convergent to ℓ with respect to fuzzy norm ($0 < q < \infty$) if

$$\lim_{m \rightarrow \infty} \frac{1}{m} \sum_{k=1}^m (D(\|u_{\sigma^k(n)} - \ell\|, \tilde{0}))^q = \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{k=1}^m (\|u_{\sigma^k(n)} - \ell\|_0^+)^q = 0,$$

uniformly in n . This convergence is denoted by $u_n \xrightarrow{[\sigma-FN]_q} \ell$.

2. MAIN RESULTS

Definition 1. The sequence $u = (u_n)$ in U is said to be \mathcal{I}_σ -convergent to ℓ with respect to fuzzy norm if for every $\varepsilon > 0$,

$$K(\varepsilon) = \{n \in \mathbb{N} : \|u_n - \ell\|_0^+ \geq \varepsilon\} \in \mathcal{I}_\sigma,$$

that is, $\mathcal{V}(K(\varepsilon)) = 0$. This convergence can be written as

$$\mathcal{I}_\sigma - \lim_{n \rightarrow \infty} \|u_n - \ell\|_0^+ = 0$$

and denoted by $u_n \xrightarrow{\mathcal{I}_\sigma-FN} \ell$.

Theorem 2.1. Let $u = (u_n)$ be a bounded sequence in U . If u is \mathcal{I}_σ -convergent to ℓ with respect to fuzzy norm, then u is invariant convergent to ℓ with respect to fuzzy norm.

Proof. Assume that the bounded sequence $u = (u_n)$ is \mathcal{I}_σ -convergent to ℓ with respect to fuzzy norm. We can write

$$\begin{aligned} \left\| \frac{1}{m} \sum_{k=1}^m u_{\sigma^k(n)} - \ell \right\|_0^+ &\leq \frac{1}{m} \sum_{k=1}^m \|u_{\sigma^k(n)} - \ell\|_0^+ \\ &= \frac{1}{m} \sum_{\substack{k=1 \\ \|u_{\sigma^k(n)} - \ell\|_0^+ \geq \varepsilon}}^m \|u_{\sigma^k(n)} - \ell\|_0^+ + \frac{1}{m} \sum_{\substack{k=1 \\ \|u_{\sigma^k(n)} - \ell\|_0^+ < \varepsilon}}^m \|u_{\sigma^k(n)} - \ell\|_0^+ \\ &\leq \frac{1}{m} M \max_n \left| \left\{ 1 \leq k \leq m : \|u_{\sigma^k(n)} - \ell\|_0^+ \geq \varepsilon \right\} \right| + \varepsilon \\ &= M \frac{S_m}{m} + \varepsilon \end{aligned}$$

for some $M > 0$, because of boundedness. Since u is \mathcal{I}_σ -convergent to ℓ with respect to fuzzy norm, i.e.,

$$\lim_{m \rightarrow \infty} \frac{S_m}{m} = 0,$$

we have

$$\lim_{m \rightarrow \infty} \|t_{mn} - \ell\|_0^+ = \lim_{m \rightarrow \infty} \left\| \frac{1}{m} \sum_{k=1}^m u_{\sigma^k(n)} - \ell \right\|_0^+ = 0,$$

uniformly in n . Hence u is invariant convergent to ℓ with respect to fuzzy norm. \square

The converse of previous theorem does not hold. For example the sequence $u = (u_n)$, defined by

$$u_n = \begin{cases} 1, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd,} \end{cases}$$

is invariant convergent to $\frac{1}{2}$, but it is not \mathcal{I}_σ -convergent with respect to fuzzy norm when $\sigma(n) = n+1$.

Theorem 2.2. Let $u = (u_n)$ be a sequence in U and $0 < q < \infty$.

- (i) If $u_n \xrightarrow{[\sigma-FN]_q} \ell$ then $u_n \xrightarrow{\mathcal{I}_\sigma-FN} \ell$.
- (ii) If u is bounded and $u_n \xrightarrow{\mathcal{I}_\sigma-FN} \ell$ then $u_n \xrightarrow{[\sigma-FN]_q} \ell$.
- (iii) u is bounded then $u_n \xrightarrow{\mathcal{I}_\sigma-FN} \ell$ if and only if $u_n \xrightarrow{[\sigma-FN]_q} \ell$.

Proof. (i) Suppose that the sequence $u = (u_n)$ is q -strongly σ -convergent to ℓ with respect to fuzzy norm, $0 < q < \infty$ and $\varepsilon > 0$. Then, we can write

$$\begin{aligned} \frac{1}{m} \sum_{k=1}^m \left(\|u_{\sigma^k(n)} - \ell\|_0^+ \right)^q &\geq \frac{1}{m} \sum_{\substack{k=1 \\ \|u_{\sigma^k(n)} - \ell\|_0^+ \geq \varepsilon}}^m \left(\|u_{\sigma^k(n)} - \ell\|_0^+ \right)^q \\ &\geq \frac{1}{m} \varepsilon^q \left| \left\{ 1 \leq k \leq m : \|u_{\sigma^k(n)} - \ell\|_0^+ \geq \varepsilon \right\} \right| \\ &\geq \varepsilon^q \frac{1}{m} \max_n \left| \left\{ 1 \leq k \leq m : \|u_{\sigma^k(n)} - \ell\|_0^+ \geq \varepsilon \right\} \right| \\ &= \varepsilon^q \frac{S_m}{m}, \end{aligned}$$

for every $n \in \mathbb{N}$. Since

$$\lim_{m \rightarrow \infty} \frac{1}{m} \sum_{k=1}^m \left(\|u_{\sigma^k(n)} - \ell\|_0^+ \right)^q = 0,$$

uniformly in n , we obtain that

$$\lim_{m \rightarrow \infty} \frac{S_m}{m} = 0.$$

So, u is \mathcal{I}_σ -convergent to ℓ with respect to fuzzy norm.

(ii) Assume that the sequence $u = (u_n)$ is bounded and \mathcal{I}_σ -convergent to ℓ with respect to fuzzy norm. So, there exists $M > 0$ such that for every $m, n \in \mathbb{N}$,

$$\|u_{\sigma^m(n)} - \ell\|_0^+ < M$$

and we have

$$\mathcal{V} \left(\left\{ m \in \mathbb{N} : \|u_{\sigma^m(n)} - \ell\|_0^+ \geq \varepsilon \right\} \right) = 0.$$

Let $0 < q < \infty$ and $\varepsilon > 0$. We know that

$$\begin{aligned} \frac{1}{m} \sum_{k=1}^m \left(\|u_{\sigma^k(n)} - \ell\|_0^+ \right)^q &= \frac{1}{m} \sum_{\substack{k=1 \\ \|u_{\sigma^k(n)} - \ell\|_0^+ \geq \varepsilon}}^m \left(\|u_{\sigma^k(n)} - \ell\|_0^+ \right)^q \\ &\quad + \frac{1}{m} \sum_{\substack{k=1 \\ \|u_{\sigma^k(n)} - \ell\|_0^+ < \varepsilon}}^m \left(\|u_{\sigma^k(n)} - \ell\|_0^+ \right)^q \\ &\leq \frac{1}{m} M \max_n \left| \left\{ 1 \leq k \leq m : \|u_{\sigma^k(n)} - \ell\|_0^+ \geq \varepsilon \right\} \right| + \varepsilon^q \\ &= M \frac{S_m}{m} + \varepsilon^q, \end{aligned}$$

for every $n \in \mathbb{N}$. Since

$$\lim_{m \rightarrow \infty} \frac{S_m}{m} = 0$$

we obtain that

$$\lim_{m \rightarrow \infty} \frac{1}{m} \sum_{k=1}^m \left(\|u_{\sigma^k(n)} - \ell\|_0^+ \right)^q = 0,$$

uniformly in n . Therefore, u is q -strongly invariant convergent to ℓ with respect to fuzzy norm.

(iii) This is a consequence of (i) and (ii). □

Definition 2. The sequence $u = (u_n)$ in U is said to be \mathcal{I}_σ^* -convergent to ℓ with respect to fuzzy norm, if there exists a set $F = \{n_1 < n_2 < \dots < n_k < \dots\} \in \mathcal{F}(\mathcal{I}_\sigma)$ (i.e., $\exists H = \mathbb{N} \setminus F \in \mathcal{I}_\sigma$) such that

$$\lim_{k \rightarrow \infty} \|u_{n_k} - \ell\|_0^+ = 0.$$

This convergence is denoted by $u_n \xrightarrow{\mathcal{I}_\sigma^* - FN} \ell$.

Theorem 2.3. Let \mathcal{I}_σ be an admissible ideal and $u = (u_n)$ be a sequence in U . If $u_n \xrightarrow{\mathcal{I}_\sigma^* - FN} \ell$, then $u_n \xrightarrow{\mathcal{I}_\sigma - FN} \ell$.

Proof. Assume that the sequence $u = (u_n)$ is \mathcal{I}_σ^* -convergent to ℓ . Then, there exists a set $F = \{n_1 < n_2 < \dots < n_k < \dots\} \in \mathcal{F}(\mathcal{I}_\sigma)$ (i.e., $\exists H = \mathbb{N} \setminus F \in \mathcal{I}_\sigma$) such that

$$\lim_{k \rightarrow \infty} \|u_{n_k} - \ell\|_0^+ = 0.$$

Namely, for every $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that for every $k > k_0$,

$$\|u_{n_k} - \ell\|_0^+ < \varepsilon.$$

Then,

$$K(\varepsilon) = \{n \in \mathbb{N} : \|u_n - \ell\|_0^+ \geq \varepsilon\} \subseteq H \cup \{n_1, n_2, \dots, n_{k_0}\}.$$

Since \mathcal{I}_σ is admissible ideal,

$$H \cup \{n_1, n_2, \dots, n_{k_0}\} \in \mathcal{I}_\sigma.$$

Therefore, by definition of ideal $K(\varepsilon) \in \mathcal{I}_\sigma$ and so u is \mathcal{I}_σ -convergent to ℓ with respect to fuzzy norm. \square

To be hold the converse of the theorem, it is needed (AP) condition.

Theorem 2.4. Let \mathcal{I}_σ be an admissible ideal with the property (AP) and $u = (u_n)$ be a sequence in U . Then, $u_n \xrightarrow{\mathcal{I}_\sigma^* - FN} \ell$ if and only if $u_n \xrightarrow{\mathcal{I}_\sigma - FN} \ell$.

Proof. Let $u = (u_n)$ be a sequence in U . We have that if $u_n \xrightarrow{\mathcal{I}_\sigma^* - FN} \ell$, then $u_n \xrightarrow{\mathcal{I}_\sigma - FN} \ell$, by Theorem 2.3.

Now, assume that the admissible ideal \mathcal{I}_σ satisfies condition (AP) and u is \mathcal{I}_σ -convergent to ℓ with respect to fuzzy norm. Then for every $\varepsilon > 0$,

$$K(\varepsilon) = \{n \in \mathbb{N} : \|u_n - \ell\|_0^+ \geq \varepsilon\} \in \mathcal{I}_\sigma.$$

Take the following sets

$$K_1 = \{n \in \mathbb{N} : \|u_n - \ell\|_0^+ \geq 1\} \text{ and } K_i = \left\{n \in \mathbb{N} : \frac{1}{i-1} > \|u_n - \ell\|_0^+ \geq \frac{1}{i}\right\},$$

for $i \geq 2$. Obviously $K_i \cap K_j = \emptyset$ for $i \neq j$. By condition (AP), for the sets $\{K_1, K_2, \dots\}$ belonged to \mathcal{I}_σ , there exists sets $\{M_1, M_2, \dots\}$ such that $K_j \Delta M_j$ is a finite set for any $j \in \mathbb{N}$ and

$$M = \bigcup_{j=1}^{\infty} M_j \in \mathcal{I}_\sigma.$$

We need to prove that for $F = \mathbb{N} \setminus M$ and all $n \in F$,

$$\lim_{n \rightarrow \infty} \|u_n - \ell\|_0^+ = 0.$$

For any $\eta > 0$, choose $t \in \mathbb{N}$ such that $\frac{1}{t+1} < \eta$. Then,

$$\{n \in \mathbb{N} : \|u_n - \ell\|_0^+ \geq \eta\} \subset \left\{n \in \mathbb{N} : \|u_n - \ell\|_0^+ \geq \frac{1}{t+1}\right\} = \bigcup_{j=1}^{t+1} K_j.$$

Since $K_j \Delta M_j$ is finite set for $j = 1, 2, \dots, t + 1$, there exists $n_0 \in \mathbb{N}$ such that

$$\left(\bigcup_{j=1}^{t+1} M_j \right) \cap \{n \in \mathbb{N} : n > n_0\} = \left(\bigcup_{j=1}^{t+1} K_j \right) \cap \{n \in \mathbb{N} : n > n_0\}.$$

Namely, when n exceeds some fixed n_0 ,

$$\bigcup_{j=1}^{t+1} M_j = \bigcup_{j=1}^{t+1} K_j.$$

If $n > n_0$ and $n \in F$ (i.e., $n \notin M = \mathbb{N} \setminus F$), then

$$n \notin \bigcup_{j=1}^{t+1} M_j \quad \text{and so} \quad n \notin \bigcup_{j=1}^{t+1} K_j.$$

Then, we have $\|u_n - \ell\|_0^+ < \frac{1}{t+1} < \eta$. Therefore, we conclude that

$$\lim_{\substack{n \rightarrow \infty \\ n \in F}} \|u_n - \ell\|_0^+ = 0,$$

i.e., u is \mathcal{I}_σ^* -convergent to ℓ . □

Definition 3. Let \mathcal{I}_σ be an admissible ideal. The sequence $u = (u_n)$ in U is said to be \mathcal{I}_σ -Cauchy sequence with respect to fuzzy norm if for every $\varepsilon > 0$, there exists N such that

$$K(\varepsilon) = \{n \in \mathbb{N} : \|u_n - u_N\|_0^+ \geq \varepsilon\} \in \mathcal{I}_\sigma,$$

that is, $\mathcal{V}(K(\varepsilon)) = 0$.

Definition 4. Let \mathcal{I}_σ be an admissible ideal. The sequence $u = (u_n)$ in U is said to be \mathcal{I}_σ^* -Cauchy sequence with respect to fuzzy norm if there exists a set $F = \{n_1 < n_2 < \dots < n_k < \dots\} \in \mathcal{F}(\mathcal{I}_\sigma)$ such that

$$\lim_{k, l \rightarrow \infty} \|u_{n_k} - u_{n_l}\|_0^+ = 0.$$

Following theorems show relationships between \mathcal{I}_σ -convergence and \mathcal{I}_σ -Cauchy sequence, between \mathcal{I}_σ -Cauchy sequence and \mathcal{I}_σ^* -Cauchy sequence in fuzzy normed spaces. These theorems can be proved like in [9, 29], these theorems are given without the proof.

Theorem 2.5. Let \mathcal{I}_σ be an admissible ideal. If the sequence $u = (u_n)$ in U is \mathcal{I}_σ -convergent to any value, then u is \mathcal{I}_σ -Cauchy sequence with respect to fuzzy norm.

Theorem 2.6. Let \mathcal{I}_σ be an admissible ideal. If $u = (u_n)$ in U is \mathcal{I}_σ^* -Cauchy sequence with respect to fuzzy norm, then u is \mathcal{I}_σ -Cauchy sequence with respect to fuzzy norm.

Theorem 2.7. Let \mathcal{I}_σ be an admissible ideal with the property (AP). If $u = (u_n)$ in U is \mathcal{I}_σ -Cauchy sequence with respect to fuzzy norm, then u is \mathcal{I}_σ^* -Cauchy sequence with respect to fuzzy norm.

3. CONCLUSION

In this paper, the definitions of ideal invariant convergences in fuzzy normed spaces are given and their properties and relations with each other are analyzed. As a new field of study, fuzzy norm space and ideal invariant convergence are expected to inspire many future studies.

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