Transactions of A. Razmadze
Mathematical Institute
Vol. **1??** (202?), ?–??

VARIATIONAL AND NUMERICAL ANALYSIS OF A DYNAMIC THERMO-VISCOELASTIC CONTACT PROBLEM WITH NORMAL COMPLIANCE FOR A LOCKING MATERIAL

MUSTAPHA BOUALLALA[1,2*] AND EL-HASSAN ESSOUFI[2]

Abstract. This work is devoted to the study of a dynamic frictional contact problem between an ideally locking thermo-viscoelastic material and a thermally conductive foundation. The contact is described by the normal compliance condition, and the friction is modeled by the non-local Coulomb's friction law. The mathematical model of the dynamic process is presented, and the variational formulation is derived. The existence and uniqueness of the weak solution are established. We study the continuous dependence of the solution on the friction coefficient as well as on the initial data of displacement and temperature. Finally, we introduce a fully discrete finite element scheme for the variational problem and derive error estimates for the approximate solution.

1. Introduction

The term 'locking material' refers to a physical phenomenon where the deformation of a material is hindered or halted upon encountering an obstacle or reaching a specific threshold. In this case, the material can no longer be distorted independently of the force. Changes in stress occur that are not accompanied by corresponding changes in deformation.

Recent advances in the modeling of a differential variational-hemivariational inequality and a generalized nonlinear quasi-hemivariational inequality in the context of an optimal control problem can be found in [32, 33]. The application of generalized quasi-variational inequalities, particularly those involving generalized sub-differentials in the Clarke sense, has been observed in various domains. Refer to [9, 33, 34] for further details.

The first studies of variational problems with locking materials initiated, by Prager, are cited in [21–23], and developed by F. Demengel and P. Suquet in [13, 14]. Bourichi et al. addressed a penalty method for an elastic-locking material in a unilateral contact problem with friction, as discussed in [7]. The mathematical model of an elastic locking material with memory was considered in reference [25]. Recently, in [16], the authors have introduced a new model of a contact problem involving an electroelastic-locking material in friction with a conductive foundation.

In [11,18], mathematical models describing the dynamic frictional contact between a thermo-viscoelastic body and a rigid foundation were studied, and more recently in [3]. Elliptic variational and hemivariational inequalities for the displacement field, arising in various types of contact problems with friction for elastic and viscoelastic materials, can be found in [16, 19]. The references [1, 24] provide models for ideally locking materials.

Several papers investigating numerical analysis schemes and their error estimates for the dynamic and quasi-static contact problem, with or without friction, were studied in [4–6, 8, 10].

In this paper, we study the mathematical and numerical analysis of a novel dynamic contact problem for a thermo-viscoelastic ideally locking material with a thermally conductive foundation. Deformation is limited by a positive constant, beyond which the body becomes extremely rigid, causing blockage while the stress and temperature continue to rise. As temperature has an impact on displacement, the locking effect is influenced by it.This model incorporates the subdifferential of the indicator functions, introducing the key novelty of considering the locking effect caused by the temperature intensity.

1991 *Mathematics Subject Classification.* 74D05, 74M15, 47J20, 49J40 , 37C25, 74S05.

Key words and phrases. Thermo-viscoelastic. Locking material. Normal compliance. Coulomb's friction. Variational inequality. Banach fixed point. Finite element method. Error estimate.

*Corresponding author.

Furthermore, the temperature field will be constrained to belong to a specific convex set. In this context, the Fourier law of heat conduction is expressed in the following form:

$$q_{th}(t) \in -\mathscr{K}\nabla\theta(t) - \partial I_B(\nabla\theta(t)) \text{ in } \Omega \times (0,T), \tag{1.1}$$

where B is the closed convex set defined by

$$B = \{\beta_B \in L^2(\Omega),\ \|\beta_B\| \leq M_B\}. \tag{1.2}$$

This new model leads to a complex system of a dynamic quasi-variational nonlinear inequality, and a second parabolic variational nonlinear inequality. The difficulties in solving this problem lie in the coupling of the viscoelastic and the thermal aspects, as well as in the non-linearity of the boundary conditions. In addition, we get a non-linear terms that is hard to deal with in continuous and discrete cases.

The contact is modelled by the penalized normal compliance condition with the parameter of penalization ϵ. The friction is described by Coulomb's law. We derive the existence and uniqueness of the weak solution, the proof is established on the quasi-variational inequalities and Banach fixed point theorem. We prove the dependence of the solution on the friction coefficient and its convergence result. Likewise, we present the discrete problem using the finite element method and a backward Euler finite difference. We establish the convergence of its solution through a proof.

The rest of the paper is structured as follows. Section 2 presents the model for our dynamic frictional contact problem, including the non-linear thermo-viscoelastic-locking constitutive law, equilibrium equation, and boundary conditions that describe the material's behavior. In section 3, we list the notations and assumptions on the problem's data. Additionally, we derive its variational formulation. In section 4, we present the proof of the weak solvability. In section 5, we study a contact problem for thermo-viscoelastic with a small perturbation of the friction coefficient, and we establish a convergence result. In section 6, we analyze a fully discrete scheme, and we derive related error estimates. Finally, in the Appendix, we recall some analysis results.

2. A Mathematical Model of Contact

We consider a locking body which initially occupies an open bounded domain $\Omega \subset \mathbb{R}^d$ for $d = 1, 2, 3$. The boundary Γ of the domain Ω is assumed to be Lipschitz, and is divided into three disjoint measurable parts: Γ_D, Γ_N and Γ_C such that $meas(\Gamma_D) > 0$. Let $T > 0$ be the time interval of interest, $[0,T]$, and let $\rho : \Omega \mapsto \mathbb{R}^+$ represent the mass density of the body.

The body is clamped on $\Gamma_D \times (0,T)$, resulting in a vanishing displacement field in that region. The body is acted upon by a volume force of density f_0 in $\Omega \times (0,T)$, and a volume thermal of density q_0. It is also subject to a surface traction of density f_1 on $\Gamma_N \times (0,T)$. On Γ_C the body may come into contact with a thermally conductive obstacle, the so called foundation. We assume that its temperature is maintained at θ_F. The normalized gap between Γ_C and the conductive foundation is denoted by g.

We use $\mathbb{S}^d$ to denote the space of a second-order symmetric tensor on $\mathbb{R}^d$ while "." and $\|.\|$ will represent the inner product and the Euclidean norm on $\mathbb{R}^d$ and $\mathbb{S}^d$, that is for all $u,\ v \in \mathbb{R}^d$, and for all $\sigma,\ \tau \in \mathbb{S}^d$,

$$u.v = u_i.v_i, \quad \|v\| = (v.v)^{\frac{1}{2}}, \quad \text{and} \quad \sigma.\tau = \sigma_{ij}.\tau_{ij}, \quad \|\tau\| = (\tau.\tau)^{\frac{1}{2}},$$

where the indices i, j run between 1 and d, the index that follows a comma indicates a partial derivative with respect to the corresponding component of the spatial variable, e.g. $u_{i,j} = \frac{\partial u_i}{\partial x_j}$.

We denote by $u : \Omega \times (0,T) \to \mathbb{R}^d$ the displacement field, by $\sigma = (\sigma_{ij}) : \Omega \times (0,T) \to \mathbb{S}^d$, the stress tensor, by $q_{th} = (q_{th_i}) : \Omega \times (0,T) \to \mathbb{R}$ the heat flux vector, and by $\varepsilon(u) = (\varepsilon_{ij}(u)) = \frac{1}{2}(u_{i.j} + u_{j.i})$ the linearized strain tensor.

Here and below, Div denote the divergence operators for tensor and vector valued functions, i.e., $Div(\sigma) = (\sigma_{ij,j})$, and $div(q_{th}) = (q_{th_{i,i}})$ represents the vector valued functions.

Moreover, $\nu = (\nu_i)$ denotes the outward unit normal at Γ, and $u_\nu = u.\nu$, $u_\tau = u - u_\nu\nu$ are the normal and tangential components of u on Γ. Also, $\sigma_\nu = (\sigma\nu).\nu$, $\sigma_\tau = \sigma\nu - \sigma_\nu\nu$ are the normal and tangential stress on Γ, respectively.

Under the previous assumptions, we are in a position to introduce the formulation of the contact

problem.
• **Problem (P)**: Find a displacement field $u : \Omega\times]0,T[\to \mathbb{R}^d$ and a temperature field $\theta : \Omega\times]0,T[\to \mathbb{R}$ such that

$$\sigma(t) \in \mathscr{C}\varepsilon(\dot{u}(t)) + \mathscr{F}\varepsilon(u(t)) - \theta(t)\mathscr{M} + \partial I_A(\varepsilon(u(t))) \text{ in } \Omega\times(0,T), \tag{2.1}$$
$$q_{th}(t) \in -\mathscr{K}\nabla\theta(t) - \partial I_B(\nabla\theta(t)) \text{ in } \Omega\times(0,T), \tag{2.2}$$

$$\rho\ddot{u}(t) = Div\ \sigma(u(t)) + f_0(t) \text{ in } \Omega\times(0,T), \tag{2.3}$$
$$\dot{\theta}(t) + div(q_{th}(t)) = q_0(t) \text{ in } \Omega\times(0,T), \tag{2.4}$$

$$u(t) = 0 \text{ on } \Gamma_D\times(0,T), \tag{2.5}$$
$$\theta(t) = 0 \text{ on } (\Gamma_N\cup\Gamma_D)\times(0,T), \tag{2.6}$$
$$\sigma(t)\nu = f_1(t) \text{ on } \Gamma_N\times(0,T), \tag{2.7}$$
$$u(0,x) = u_0,\ \dot{u}(0,x) = \dot{u}_0,\ \theta(0,x) = \theta_0 \text{ in } \Omega, \tag{2.8}$$

$$\sigma_\nu(u_\nu(t) - g) = -\frac{1}{\epsilon}\left[u_\nu(t) - g\right]^+,\ \epsilon > 0 \text{ on } \Gamma_C\times(0,T), \tag{2.9}$$

$$\left.\begin{array}{l} \|\sigma_\tau(t)\| \le \mu(\|u_\tau(t)\|)|R\sigma_\nu(u(t))|, \\ \dot{u}_\tau(t) \neq 0 \Rightarrow \sigma_\tau(t) = \mu(\|u_\tau(t)\|)|R\sigma_\nu(u(t))|\dfrac{\dot{u}_\tau(t)}{\|\dot{u}_\tau(t)\|} \end{array}\right\} \text{on } \Gamma_C\times(0,T), \tag{2.10}$$

$$\frac{\partial q_{th}(t)}{\partial\nu} = k_c(u_\nu(t) - g)\phi_L(\theta(t) - \theta_F) \text{ on } \Gamma_C\times(0,T). \tag{2.11}$$

Where $I_{P=A,B}$ is the indicator function defined by

$$I_P(\alpha) = \begin{cases} 0, & \text{if } \alpha\in P, \\ +\infty, & \text{otherwise,} \end{cases} \tag{2.12}$$

in the closed and convex set A and B defined, respectively by

$$A = \{\beta_A \in \mathbb{S}^d,\ \|\beta_A\| \le M_A\}, \tag{2.13}$$
$$B = \{\beta_B \in L^2(\Omega),\ \|\beta_B\| \le M_B\}. \tag{2.14}$$

We recall that equations (2.1) and (2.2) represent the thermo-viscoelastic constitutive law of a locking material. In these equations, $\mathscr{F} = (f_{ijkl})$, $\mathscr{C} = (c_{ijkl})$, $\mathscr{M} = (m_{ij})$, and $\mathscr{K} = (k_{ij})$ correspond to the elastic tensor, viscosity tensor, thermal expansion tensor, and thermal conductivity tensor, respectively. Equations (2.3)-(2.4) are, respectively, the equation of motion and the Fourier law of heat conduction. Conditions (2.5) to (2.7) represent the displacement, traction, and thermal boundary conditions. The initial conditions are specified by equation (2.8). Furthermore, equation (2.9) represents the normal compliance contact condition on Γ_C, where $\epsilon > 0$ denotes the penalty parameter. The Coulomb's law of friction is considered in relation (2.10) where μ is the coefficient of friction and R is a regularization operator. The function R can be selected as the convolution product with a regular function ω, which corresponds to a non-local friction law, as described in [12].

$$R\sigma_\nu(x) = \int_{\Gamma_C} \omega(\|x - z\|)\sigma_\nu(z)dz,$$

and

$$\omega(x) = \begin{cases} \varpi.\exp\left(\dfrac{\alpha^2}{\|x\|^2 - \alpha^2}\right) & \text{if } \|x\| \le \alpha, \\ 0 & \text{if } \|x\| > \alpha. \end{cases}$$

The normalization constant ϖ is determined such that $\int\limits_{-\infty}^{+\infty} w(x), dx = 1$, where $w(x)$ is the regular function. Additionally, α represents a length that characterizes the geometry of the material's asperities.
Finally, the equation (2.11) represents the regularized thermal condition, as described in [15], such that

$$\phi_L(s) = \begin{cases} -L & \text{if} \quad s < -L, \\ s & \text{if} \quad -L \leq s \leq L, \\ L & \text{if} \quad s > L, \end{cases} \quad \text{and} \quad \begin{cases} k_c(r) = 0 & \text{if} \quad r < 0, \\ k_c(r) > 0 & \text{if} \quad r \geq 0, \end{cases}$$

where L is a large positive constant.

3. Weak formulation

In order to present the variational formulation of **Problem (P)**, we require some additional notation and preliminary concepts. Let X be a Banach space, we use the classical notations for the spaces $L^p(0,T;X)$ and $W^{k,p}(0,T;X)$ where $1 \leq p \leq \infty$ and $k = 1, ...$; endowed with the norm

$$\|u\|^p_{L^p(0,T;X)} = \int\limits_0^T \|u\|^p_X dt.$$

We introduce the following functional spaces

$$H = L^2(\Omega)^d = \left\{u = (u_i)\ ;\ u_i \in L^2(\Omega)\right\},\ H_1 = H^1(\Omega)^d,$$
$$\mathscr{H} = \left\{\tau = (\tau_{ij}) \mid \tau_{ij} = \tau_{ji} \in L^2(\Omega)\right\},\ \mathscr{H}_1 = \{\sigma \in \mathscr{H} \mid Div\ \sigma \in H\}.$$

These spaces are real Hilbert spaces equipped with inner products corresponding to the associated norms defined as follows:

$$(u,v)_H = \int\limits_\Omega \rho u_i v_i dx,\ (u,v)_{H_1} = (u,v)_H + (\varepsilon(u), \varepsilon(v))_{\mathscr{H}},$$

$$(\sigma,\tau)_{\mathscr{H}} = \int\limits_\Omega \sigma_{ij}\tau_{ij} dx,\ (u,v)_{\mathscr{H}_1} = (\sigma,\tau)_{\mathscr{H}} + (Div\ \sigma, Div\ \tau)_{\mathscr{H}},$$

and the associated norms $\|.\|_H$, $\|.\|_{H_1}$, $\|.\|_{\mathscr{H}}$ and $\|.\|_{\mathscr{H}_1}$, respectively.
Let $H_\Gamma = H^{\frac{1}{2}}(\Gamma)^d$ and $\gamma : H \to H_\Gamma$ be the trace map. For every element $v \in H$, we also use the notation v to denote the trace γv of v on Γ.
Let $H_\Gamma^{'}$ be the dual of H_Γ and $< .,. >$ denote the duality pairing between $H_\Gamma^{'}$ and H_Γ. For every $\sigma \in \mathscr{H}$, $\sigma\nu$ can be defined as the element in $H_\Gamma^{'}$ satisfying

$$\langle \sigma\nu, \gamma v\rangle = (\sigma, \varepsilon(v))_{\mathscr{H}} + (Div\ \sigma, v)_H,\ \forall v \in H_1. \tag{3.1}$$

Moreover, if σ is continuously differentiable on $\overline{\Omega}$, then

$$\langle \sigma\nu, \gamma v\rangle = \int\limits_\Gamma \sigma\nu.v da,\ \forall v \in H_1, \tag{3.2}$$

where da is the surface measure element.
By combining equations (3.1) through (3.2), we obtain the following Green's formula in elasticity

$$(\sigma, \varepsilon(v))_{\mathscr{H}} + (Div\ \sigma, v)_H = \int\limits_\Gamma \sigma\nu.v da. \tag{3.3}$$

Notice that if $q \in H^1(\Omega)$ is a sufficiently regular function, the following Green's type formula holds

$$(q, \nabla\eta)_H + (div\ q, \eta)_{L^2(\Omega)} = \int\limits_\Gamma q.\nu\eta da,\ \forall \eta \in H^1(\Omega). \tag{3.4}$$

Taking into account the boundary conditions (2.5)-(2.9), we introduce the closed subspace of H_1 as defined below.

$$V = \{v \in H_1 \ : \ v = 0 \text{ on } \Gamma_D\},$$
$$Q = \left\{\eta \in H^1(\Omega) \ : \ \eta = 0 \text{ on } \Gamma_D \cup \Gamma_N\right\},$$
$$V_1 = \{v \in V \ : \ v_\nu - g \leq 0 \text{ on } \Gamma_C\},$$

endowed with the inner products and norms defined as follows:

$$(u, v)_V = (\varepsilon(u), \varepsilon(v))_{\mathscr{H}}, \quad \|v\|_V = (v, v)_V^{\frac{1}{2}},$$
$$(\theta, \eta)_Q = (\nabla\theta, \nabla\eta)_H, \quad \|\eta\|_Q = (\eta, \eta)_Q^{\frac{1}{2}}.$$

Since $meas(\Gamma_D) > 0$, the following Korn inequality (see [20])

$$\|\varepsilon(v)\|_{\mathscr{H}} \geq c_t \|v\|_{H_1}, \text{ for all } v \in V, \tag{3.5}$$

holds, where c_t is a non-negative constant depending only on Ω and Γ_D.
The following Frierichs-Poincaré inequality holds on Q

$$\|\nabla\eta\|_H \geq C_F \|\eta\|_Q, \text{ for all } \eta \in Q. \tag{3.6}$$

Furthermore, according to the Sobolev trace theorem, there exist constants $c_0 > 0$ and $c_1 > 0$ that depend on Ω, Γ_C, and Γ_D, such that

$$\|v\|_{[L^2(\Gamma_C)]^d} \leq c_0 \|v\|_V, \quad \|\eta\|_{L^2(\Gamma_C)} \leq c_1 \|\eta\|_Q, \text{ for all } (v, \eta) \in V \times Q. \tag{3.7}$$

Next, utilizing Riesz's representation theorem, we define the elements $f(t) \in V$ and $q(t) \in Q$ for all $v \in V$ and $\eta \in Q$ as follows:

$$(f(t), v)_V = \int_\Omega f_0(t).v dx + \int_{\Gamma_N} f_1(t).v da, \tag{3.8}$$

$$(q(t), \eta)_Q = \int_\Omega q_0(t).\eta dx. \tag{3.9}$$

Also, we define $j : V \times V \to \mathbb{R}$, $\chi : V \times Q \times Q \to \mathbb{R}$ and $\Phi : V \times V \to \mathbb{R}$ by

$$j(u(t), v) = \int_{\Gamma_C} \mu(\|u_\tau(t)\|) |R\sigma_\nu(u(t))| \|v_\tau\| da, \tag{3.10}$$

$$\chi(u(t), \theta(t), \eta) = \int_{\Gamma_C} k_c(u_\nu(t) - g) \phi_L(\theta(t) - \theta_F) \eta da, \tag{3.11}$$

and

$$\Phi(u(t), v) = \frac{1}{\epsilon} \int_{\Gamma_C} [u_\nu(t) - g]^+ v_\nu da = \frac{1}{\epsilon} \left\langle [u_\nu(t) - g]^+, v_\nu \right\rangle_{\Gamma_C}. \tag{3.12}$$

It should be noted that the integrals (3.10) and (3.11) are well-defined due to the conditions (HP_7) and (HP_8) below, respectively. To proceed with the investigation of the mechanical **Problem (P)**, we require the following assumptions:

(HP_1) The elasticity tensor $\mathscr{F} = (f_{ijkl}) : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies the usual properties of symmetry, boundedness, ellipticity and continuity

- i) $f_{ijkl} = f_{jikl} = f_{lkij} \in L^\infty(\Omega)$, for all i, j, k, l;
- ii) $\mathscr{F}(x, \xi).\sigma = \xi.\mathscr{F}(x, \sigma)$, for all $\xi, \sigma \in \mathbb{S}^d$, a.e. $x \in \Omega$;
- iii) $f_{ijkl}(x)\xi_{ij}\xi_{kl} \geq m_{\mathscr{F}} \|\xi\|^2$, with $m_{\mathscr{F}} > 0$, for all $\xi \in \mathbb{S}^d$, a.e. $x \in \Omega$;
- iv) $\|(\mathscr{F}\varepsilon(u), \varepsilon(v))\|_{\mathscr{H}} \leq M_{\mathscr{F}} \|u\|_V \|v\|_V$.

(HP_2) The viscosity tensor $\mathscr{C} = (c_{ijkl}) : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies the following conditions:

- i) $c_{ijkl} = c_{jikl} = c_{lkij} \in L^\infty(\Omega)$;

ii) $\mathscr{C}(x,.)$ is monotone on $\mathbb{S}^d$, i.e.

$$(\mathscr{C}(x,\xi_1)-\mathscr{C}(x,\xi_2)).(\xi_1-\xi_2)\geq 0,\ \text{for all}\ \xi_1,\xi_2\in\mathbb{S}^d,\ \text{a.e.}\ x\in\Omega;$$

iii) $\mathscr{C}(x,.)$ is coercive on $\mathbb{S}^d$, i.e.

$$c_{ijkl}(x)\xi_{ij}\xi_{kl}\geq m_{\mathscr{C}}\|\xi\|^2,\ \text{with}\ m_{\mathscr{C}}>0,\ \text{for all}\ \xi\in\mathbb{S}^d,\ \text{a.e.}\ \in\Omega;$$

iv) There exist $r\in L^\infty(\Omega)$ and $s\in L^2(\Omega)$ such that

$$|\mathscr{C}(x,\xi)|\leq r(x)|\xi|+s(x),\ \forall\xi\in\mathbb{S}^d,\ \text{a.e.}\ x\in\Omega;$$

v) $\mathscr{C}(x,.)$ is continuous on $\mathbb{S}^d$, a.e. $x\in\Omega$

$$\|(\mathscr{C}\varepsilon(u),\varepsilon(v))\|_{\mathscr{H}}\leq M_{\mathscr{C}}\|u\|_V\|v\|_V;$$

vi) $\mathscr{C}(.,\xi)$ is Lebesgue measurable on Ω for all $\xi\in\mathbb{S}^d$.

(HP_3) The thermal conductivity tensor $\mathscr{K}=(k_{ij}):\Omega\times\mathbb{R}^d\to\mathbb{R}^d$ satisfies
i) $k_{ij}=k_{ji}\in L^\infty(\Omega)$;
ii) $k_{ij}(x)\xi_i\xi_j\geq m_{\mathscr{K}}\|\xi\|^2,$ with $m_{\mathscr{K}}>0,$ for all $\xi\in\mathbb{R}^d,\ x\in\Omega$;
iii) $\|(\mathscr{K}\nabla\theta,\nabla\eta)\|_H\leq M_{\mathscr{K}}\|\theta\|_Q\|\eta\|_Q$.

(HP_4) The thermal expansion tensor $\mathscr{M}=(m_{ij}):\Omega\times\mathbb{R}\to\mathbb{R}$ satisfies
i) $m_{ij}=m_{ji}\in L^\infty(\Omega)$;
ii) $\|(\mathscr{M}\theta,\varepsilon(v))\|_{\mathscr{H}}\leq M_{\mathscr{M}}\|\theta\|_Q\|v\|_V$.

(HP_5) i) The fores, the traction, and the thermal flux satisfy

$$f_0\in W^{1,1}(0,T;H),\quad f_1\in W^{1,1}\left(0,T;L^2(\Gamma_N)^d\right),\ q_0\in L^2\left(0,T;L^2(\Omega)\right);$$

ii) The thermal potential, and the gap function satisfy

$$\theta_F\in L^2(0,T;L^2(\Gamma_C)),\ g\in L^2(\Gamma_C),\ g\geq 0.$$

iii) The mapping $R:H'_{\Gamma_C}\to L^\infty(\Gamma_C)$ is linear compact and continuous with

$$c_R=\|R\|_{H'}.$$

iv) The mass density satisfies $\rho\in L^\infty(\Omega)$ and there exists $\rho^*>0$ such that

$$\rho\in L^\infty(\Omega)\text{ and }\rho(x)\geq\rho^*\text{ a.e. }x\in\mathbb{R}.$$

(HP_6) i) The initial data u_0, $\dot{u}_0$ and θ_0 of **Problem (P)** satisfy

$$u_0\in V,\ \dot{u}_0\in D(\partial j),\ \theta_0\in Q,$$

where ∂j denote the sub-differential of j and $D(\partial j)$ represents its domain.

ii) There exists $h\in L^2(\Omega)^d$ for all $v\in V$ such that

$$c(\dot{u}_0,v-\dot{u}_0)+a(u_0,\dot{u}_0)+j(u_0,v)-j(u_0,\dot{u}_0)\geq(h,v-\dot{u}_0).$$

iii) The functional j is proper, convex, and lower semi-continuous on $V\times V$.

(HP_7) The coefficient of friction $\mu:\Gamma_C\times\mathbb{R}^+\to\mathbb{R}^+$ satisfies
i) There exists a positive constant $L_\mu>0$, for all $x,y\in\mathbb{R}^+$,

$$|\mu(.,x)-\mu(.,y)|\leq L_\mu|x-y|\text{ a.e. on }\Gamma_C;$$

ii) There exists a positive constant $M_\mu>0$ such that

$$|\mu(x,u)|\leq M_\mu,\ \text{for all}\ u\in\mathbb{R}^+\text{ and }x\in\Gamma_C;$$

iii) The mapping $x\to\mu(x,u)$ is measurable on Γ_C for all $u\in\mathbb{R}^+$.

(HP_8) The coefficient of heat exchange $k_c:\Gamma_C\times\mathbb{R}\to\mathbb{R}^+$ satisfy
i) There exists $M_{k_c}>0$ such that $|k_c(x,u)|<M_{k_c}$ for all $u\in\mathbb{R},x\in\Gamma_C$ $x\mapsto k_c(x,u)$ is measurable on Γ_C for all $x\in\mathbb{R},\ k_c(x,u)=0$ for all $x\in\Gamma_C$ and $u\leq 0$;
ii) There exists $L_{k_c}>0$ such that $|k_c(x,u_1)-k_c(x,u_2)|\leq L_{k_c}|u_1-u_2|$, for all $u_1,u_2\in\mathbb{R}$.

Using $(HP_6)(iv)$ the inclusion mapping of $(V, \|.\|_V)$ into $(H, \|.\|_H)$ is continuous and dense. Then, we identify H and H' and we write $V \subset H \equiv H' \subset V'$.
Since Q is dense in $L^2(\Omega)$, we write $Q \subset L^2(\Omega) \subset Q'$.
We denote by $a : V \times V \to \mathbb{R}$, $c : V \times V \to \mathbb{R}$, $d : Q \times Q \to \mathbb{R}$ and $m : Q \times V \to \mathbb{R}$ are the bilinear operators given by

$$a(u,v) = (\mathscr{F}\varepsilon(u), \varepsilon(v))_{\mathscr{H}}\,, \quad c(u,v) = (\mathscr{C}\varepsilon(u), \varepsilon(v))_{\mathscr{H}}\,.$$
$$d(\theta,\eta) = (\mathscr{K}\nabla\theta, \nabla\eta)_H\,, \quad m(\theta,v) = (\mathscr{M}\theta, \varepsilon(v))_{\mathscr{H}}\,.$$

From the constitutive laws (2.1)-(2.2), we have

$$\sigma(t) = \mathscr{C}\varepsilon(\dot{u}(t)) + \mathscr{F}\varepsilon(u(t)) - \theta(t)\mathscr{M} + \mathscr{A}\,(u(t))\,, \tag{3.13}$$
$$q_{th}(t) = -\mathscr{K}\nabla\theta(t) - \mathscr{B}\,(\theta(t))\,, \tag{3.14}$$

where

$$\mathscr{A}u(t) \in \partial I_A(\varepsilon(u(t))) \text{ and } \mathscr{B}\theta(t) \in \partial I_B(\nabla\theta(t)) \text{ in } \Omega. \tag{3.15}$$

For almost all $t \in]0,T[$, let us define the following subsets

$$V_2 = \{u(t) \in V\ ;\ \sup\{\|\varepsilon(u(t))\|_{\mathscr{H}}, \|\varepsilon(\dot{u}(t))\|_{\mathscr{H}}\} \leq M_A \text{ a.e. in } \Omega\}\,, \tag{3.16}$$

and

$$Q_1 = \{\theta(t) \in Q\ ;\ \|\nabla\theta(t)\|_H \leq M_B \text{ a.e. in } \Omega\}\,. \tag{3.17}$$

For all $u, v \in V_2$ and $\theta, \eta \in Q_1$, we obtain

$$\langle \mathscr{A}u(t), \varepsilon(v) - \varepsilon(\dot{u}(t))\rangle \leq I_A(\varepsilon(v)) - I_A(\varepsilon(\dot{u}(t))) \leq 0, \tag{3.18}$$
$$\langle \mathscr{B}\theta(t), \nabla\eta - \nabla\theta(t)\rangle \leq I_B(\nabla\eta) - I_B(\nabla\theta(t)) \leq 0. \tag{3.19}$$

Then,

$$(\sigma(t), \varepsilon(v) - \varepsilon(\dot{u}(t)))_{\mathscr{H}} \leq (\mathscr{C}\varepsilon(\dot{u}(t)) + \mathscr{F}\varepsilon(u(t)) - \theta(t)\mathscr{M}, \varepsilon(v) - \varepsilon(\dot{u}(t)))_{\mathscr{H}}\,, \tag{3.20}$$

$$(-q_{th}(t), \nabla\eta - \nabla\theta(t))_H \leq (\mathscr{K}\nabla\theta(t), \nabla\eta - \nabla\theta(t))_H\,. \tag{3.21}$$

Next, we assume that $\{u, \sigma, \theta, q_{th}\}$ are regular functions satisfying (2.1)-(2.11) and let $v \in V_1 \cap V_2$, $\eta \in Q_1$. Using (3.3)-(3.4), we have

$$\begin{aligned} &-(Div\ \sigma(t), \varepsilon(v) - \varepsilon(\dot{u}(t)))_{\mathscr{H}} + \int_\Gamma (\sigma\nu.\varepsilon(v) - \varepsilon(\dot{u}(t)))\,da \\ &\leq (\mathscr{C}\varepsilon(\dot{u}(t)) + \mathscr{F}\varepsilon(u(t)) - \theta(t)\mathscr{M}, \varepsilon(v) - \varepsilon(\dot{u}(t)))_{\mathscr{H}}\,, \end{aligned} \tag{3.22}$$

$$(div\ q_{th}(t), \nabla\eta - \nabla\theta(t))_H - \int_\Gamma (q_{th}\nu.\nabla\eta - \nabla\theta(t))\,da \leq (\mathscr{K}\nabla\theta(t), \nabla\eta - \nabla\theta(t))_H\,. \tag{3.23}$$

By the condition (2.5)-(2.7) and (3.8)-(3.9) we deduce that

$$\begin{aligned} &(-\rho\ddot{u}(t) - \mathscr{C}\varepsilon(\dot{u}(t)) - \mathscr{F}\varepsilon(u(t)) + \theta(t)\mathscr{M}, \varepsilon(v) - \varepsilon(\dot{u}(t)))_{\mathscr{H}} \\ &+ \int_{\Gamma_C} (\sigma\nu.\varepsilon(v) - \varepsilon(\dot{u}(t)))\,da \leq -(f(t), \varepsilon(v) - \varepsilon(\dot{u}(t)))_V\,, \end{aligned} \tag{3.24}$$

$$\left(-\dot{\theta}(t) - \mathscr{K}\nabla\theta(t), \nabla\eta - \nabla\theta(t)\right)_H - \int_{\Gamma_C} (q_{th}.\nu, \nabla\eta - \nabla\theta(t))\,da \leq -(q(t), \nabla\eta - \nabla\theta(t))_H\,. \tag{3.25}$$

Taking into consideration equations (3.20)-(3.21), (2.9)-(2.11), (3.10)-(3.12), as well as the inner product in H, we obtain the following weak formulation

• **Problem (PV)**: Find a displacement field $u \in (V_1 \cap V_2) \times]0;T[$ and a temperature field $\theta \in Q_1 \times]0,T[$, for all $v \in V_1 \cap V_2$ and $\eta \in Q_1$ such that:

$$\begin{aligned} &(\ddot{u}(t), v-\dot{u}(t))_H + c\,(\dot{u}(t), v-\dot{u}(t)) + a\,(u(t), v-\dot{u}(t)) - m\,(\theta(t), v-\dot{u}(t)) \\ &+ \Phi\,(u(t), v-\dot{u}(t)) + j(u(t), v) - j(u(t), \dot{u}(t)) \geq (f(t), v-\dot{u}(t))_V\,, \end{aligned} \tag{3.26}$$

$$\left(\dot{\theta}(t), \eta-\theta(t)\right)_{Q'\times Q} + d\,(\theta(t), \eta-\theta(t)) + \chi\,(u(t), \theta(t), \eta-\theta(t)) \geq (q(t), \eta-\theta(t))_Q\,, \tag{3.27}$$

$$u(0) = u_0,\ \dot{u}(0) = \dot{u}_0,\ \theta(0) = \theta_0. \tag{3.28}$$

4. Existence and uniqueness of the weak solution

The existence and uniqueness of the weak solution to **Problem (PV)** are established as follows

Theorem 4.1. *Assuming that (HP_1)-(HP_8) hold, there exists a unique solution to **Problem (PV)** that satisfies*

$$u \in W^{1,\infty}(0,T;V) \cap W^{2,\infty}(0,T;H),\ \theta \in W^{1,2}(0,T;Q) \cap L^2(0,T;Q). \tag{4.1}$$

The proof of Theorem 4.1 will be carried out in several steps and is based on arguments involving quasi-variational inequalities and the Banach fixed point theorem.
In the first step, let $\lambda \in W^{1,1}(0,T;H)$ and $\gamma \in L^2(0,T;L^2(\Omega))$ be given, and consider the following intermediate variational problems.

• **Problem** (PV_λ): Find a displacement field $u_\lambda \in (V_1 \cap V_2) \times]0,T[$ for all $v \in V_1 \cap V_2$ such that:

$$\begin{aligned} &(\ddot{u}_\lambda(t), v-\dot{u}_\lambda(t))_H + c\,(\dot{u}_\lambda(t), v-\dot{u}_\lambda(t)) + a\,(u_\lambda(t), v-\dot{u}_\lambda(t)) \\ &+ (\lambda(t), v-\dot{u}_\lambda(t))_V + j\,(u_\lambda(t), v) - j\,(u_\lambda(t), \dot{u}_\lambda(t)) \geq (f(t), v-\dot{u}_\lambda(t))_V\,, \end{aligned} \tag{4.2}$$

$$u_\lambda(0) = u_0,\ \dot{u}_\lambda(0) = \dot{u}_0. \tag{4.3}$$

• **Problem** (PV_γ): Find a temperature field $\theta_\gamma \in Q_1 \times]0,T[$ for all $\eta \in Q_1$ such that:

$$\left(\dot{\theta}_\gamma(t), \eta-\theta_\gamma(t)\right)_{Q'\times Q} + d\,(\theta_\gamma(t), \eta-\theta_\gamma(t)) + (\gamma(t), \eta-\theta_\gamma(t))_Q \geq (q(t), \eta-\theta_\gamma(t))_Q\,, \tag{4.4}$$

$$\theta_\gamma(0) = \theta_0. \tag{4.5}$$

In the second step, we present and prove the solvability result for the intermediate problems

Lemma 1. *There exists a unique solution to **Problem** (PV_λ) that satisfies.*

$$u_\lambda \in W^{1,\infty}(0,T;V) \cap W^{2,\infty}(0,T;H).$$

Proof. Using Riesz's representation theorem, we can define the following operator

$$(f_\lambda(t), v)_V = (f(t), v)_V - (\lambda(t), v)_V\,. \tag{4.6}$$

Then, **Problem** (PV_λ) can be written as follows

$$\begin{aligned} &(\ddot{u}_\lambda(t), v-\dot{u}_\lambda(t))_H + c\,(\dot{u}_\lambda(t), v-\dot{u}_\lambda(t)) + a\,(u_\lambda(t), v-\dot{u}_\lambda(t)) \\ &+ j\,(u_\lambda(t), v) - j\,(u_\lambda(t), \dot{u}_\lambda(t)) \geq (f_\lambda(t), v-\dot{u}_\lambda(t))_V\,, \end{aligned} \tag{4.7}$$

$$u_\lambda(0) = u_0,\ \dot{u}_\lambda(0) = \dot{u}_0. \tag{4.8}$$

Keeping in mind $(HP_5)(i)$, (3.8), we have $f \in W^{1,1}(0,T,H)$ and from the regularity $\lambda \in W^{1,1}(0,T;V)$, we can conclude that $f_\lambda \in W^{1,1}(0,T;H)$.
By the assumption (HP_1), the operator a is continuous symmetric bilinear and satisfies the following coerciveness condition

$$\langle cu, u\rangle_{V'\times V} + \alpha\|u\|^2 \geq w\|u\|^2,\ \forall u \in V,$$

with $\alpha = 0$ and $w = m_{\mathscr{C}}$.
Next, we define the set-valued operator $M : V \to V'$ by $M = c + \partial j$.
From (HP_2), we have

$$(\mathscr{C}(\varepsilon(u_1)) - \mathscr{C}(\varepsilon(u_2)), \varepsilon(u_1) - \varepsilon(u_2))_{\mathscr{H}} \geq 0,\ \forall u_1, u_2 \in V,$$

then, the operator c is monotone.
Since c is continuous and in accordance with $(HP_2)(iv)$, we can deduce that c is bounded.
Using $(HP_6)(iii)$, the operator j is proper, convex and lower semi-continuous on $V\times V$, which implies that ∂j is maximal monotone
Consequently, since c is monotone, bounded and continuous from in $V\times V$, we conclude [2, P. 39] that $M = c+\partial j$ is maximal monotone. Moreover, by $(HP_6)(i)-(ii)$, the initial data u_0 and $\dot{u}_0$ satisfy the condition

$$\{au_0 + c\dot{u}_0\}\cap H \neq \varnothing.$$

It follows now from Theorem 6.2 that there exists a unique function satisfying $u_\lambda \in W^{1,\infty}(0,T;V)\cap W^{2,\infty}(0,T;H)$. □

Lemma 2. *There exists a unique solution to* ***Problem*** (PV_γ) *that satisfies the given conditions.*

$$\theta_\gamma \in W^{1,2}(0,T;L^2(\Omega))\cap L^2(0,T;Q).$$

Proof. By the Riesz representation theorem, there exists an operator $q_\gamma(t)\in Q$ such that

$$(q_\gamma(t),\eta)_Q = (q(t),\eta)_Q - (\gamma(t),\eta)_Q\,,\ \forall \eta\in Q_1. \tag{4.9}$$

Then, problem (4.4)-(4.5) can be written as follows

$$\left(\dot{\theta}_\gamma(t),\eta-\theta_\gamma(t)\right)_{Q'\times Q} + d\left(\theta_\gamma(t),\eta-\theta_\gamma(t)\right) \geq \left(q_\gamma(t),\eta-\theta_\gamma(t)\right)_Q, \tag{4.10}$$

for all $\eta\in Q_1$. From (3.9), (4.9), and the regularity of q_0, we can observe that $q_\gamma(t)\in L^2(0,T;L^2(\Omega))$. The hypothesis (HP_3) assures that the operator d is a continuous, symmetric bilinear and satisfies the following condition

$$d(\theta,\theta)+c_0\|\theta\|^2_{L^2(\Omega)} \geq \alpha\|\theta\|^2_Q,\ \forall\theta\in Q,$$

with $c_0=0$ and $\alpha = m_{\mathscr{K}}$.
Now, utilizing Theorem 6.3, we can conclude the result. □

From now on, the constant denoted by C may differ line to line.
In the last step, for $(\lambda,\gamma)\in W^{1,1}(0,T;H)\times L^2(0,T;L^2(\Omega))$, we define the mapping

$$\Lambda(\lambda,\gamma)(t) = (\Lambda_1(\lambda,\gamma)(t),\Lambda_2(\lambda,\gamma)(t))\in V\times Q, \tag{4.11}$$

given by

$$(\Lambda_1(\lambda,\gamma)(t),v) = \Phi\left(u_\lambda(t),v\right) - m\left(\theta_\gamma(t),v\right), \tag{4.12}$$

$$(\Lambda_2(\lambda,\gamma)(t),\eta) = \chi\left(u_\lambda(t),\theta_\gamma(t),\eta\right), \tag{4.13}$$

for all $v\in V_1\cap V_2$ and $\eta\in Q_1$. We have the following.

Lemma 3. *For* $(\lambda,\gamma)\in W^{1,1}(0,T;H)\times L^2(0,T;L^2(\Omega))$*, the operator* $\Lambda:[0,T]\to V\times Q$ *is continuous. Moreover, there exists a unique* $(\lambda^*,\gamma^*)\in(\lambda,\gamma)\in W^{1,1}(0,T;H)\times L^2(0,T;L^2(\Omega))$ *such that* $\Lambda(\lambda^*,\gamma^*) = (\lambda^*,\gamma^*)$.

Proof. Let $(\lambda,\gamma)\in W^{1,1}(0,T;H)\times L^2(0,T;L^2(\Omega))$ and $t_1,t_2\in]0,T[$.
We use (3.12), (4.12), (HP_4) and the inequality $|[x]^+-[y]^+|\leq|x-y|$, yield

$$\begin{aligned}&\|\Lambda_1(\lambda,\gamma)(t_1)-\Lambda_1(\lambda,\gamma)(t_2)\|_{V\times Q}\\ &\leq \sup\left(M_{\mathscr{M}},\frac{c_0^2}{\epsilon}\right)\left(\|\theta_\gamma(t_1)-\theta_\gamma(t_2)\|_Q + \|u_\lambda(t_1)-u_\lambda(t_2)\|_V\right).\end{aligned} \tag{4.14}$$

By the regularities of u_λ and θ_γ, we conclude that $\Lambda_1(\lambda,\gamma)\in C([0,T];V)$.
On the other hand by (3.11), (4.13), and (HP_8), it follows that

$$\|\Lambda_2(\lambda,\gamma)(t_1)-\Lambda_2(\lambda,\gamma)(t_2)\|_{V\times Q} \leq M_{k_c}c_1^2L\,\|\theta_\gamma(t_1)-\theta_\gamma(t_2)\|_Q\,. \tag{4.15}$$

Then, $\Lambda_2(\lambda,\gamma)\in C([0,T];Q)$. Consequently, $\Lambda(\lambda,\gamma)\in C([0,T];V\times Q)$.
Let now $(\lambda_1,\gamma_1),(\lambda_2,\gamma_2)\in W^{1,1}(0,T;H)\times L^2(0,T;L^2(\Omega))$ and $t\in[0,T]$.

Using arguments similar to (4.14) and (4.15), we can deduce the existence of a positive constant C such that

$$\|\Lambda(\lambda_1,\gamma_1)(t)-\Lambda(\lambda_2,\gamma_2)(t)\|_{V\times Q}\leq C\left(\|\theta_{\gamma_1}(t)-\theta_{\gamma_2}(t)\|_Q+\|u_{\lambda_1}(t)-u_{\lambda_2}(t)\|_V\right). \tag{4.16}$$

Notice that $u_{\lambda_i}(t)=\int_0^t \dot{u}_{\lambda_i}(s)ds+u_{\lambda_i}(0)$, for $i=1,2$, imply

$$\|u_{\lambda_1}(t)-u_{\lambda_2}(t)\|_V^2\leq C\int_0^t \|\dot{u}_{\lambda_1}(s)-\dot{u}_{\lambda_2}(s)\|_V^2\,ds. \tag{4.17}$$

From (4.2), we write

$$\begin{aligned}&(\ddot{u}_{\lambda_1}(t)-\ddot{u}_{\lambda_2}(t),\dot{u}_{\lambda_1}(t)-\dot{u}_{\lambda_2}(t))+c\left(\dot{u}_{\lambda_1}(t)-\dot{u}_{\lambda_2}(t),\dot{u}_{\lambda_1}(t)-\dot{u}_{\lambda_2}(t)\right)\\&+a\left(u_{\lambda_1}(t)-u_{\lambda_2}(t),\dot{u}_{\lambda_1}(t)-\dot{u}_{\lambda_2}(t)\right)+(\lambda_1(t)-\lambda_2(t),\dot{u}_{\lambda_1}(t)-\dot{u}_{\lambda_2}(t))\\&+j\left(u_{\lambda_1}(t),\dot{u}_{\lambda_1}(t)\right)-j\left(u_{\lambda_1}(t),\dot{u}_{\lambda_2}(t)\right)-j\left(u_{\lambda_2}(t),\dot{u}_{\lambda_1}(t)\right)+j\left(u_{\lambda_2}(t),\dot{u}_{\lambda_2}(t)\right)\leq 0.\end{aligned} \tag{4.18}$$

We first bound the terms of j

$$\begin{aligned}&\left|j\left(u_{\lambda_1}(t),\dot{u}_{\lambda_1}(t)\right)-j\left(u_{\lambda_1}(t),\dot{u}_{\lambda_2}(t)\right)-j\left(u_{\lambda_2}(t),\dot{u}_{\lambda_1}(t)\right)+j\left(u_{\lambda_2}(t),\dot{u}_{\lambda_2}(t)\right)\right|\\&\leq c_0^2L_\mu c_R\left\|u_{\lambda_1}(t)-u_{\lambda_2}(t)\right\|_V\left\|\dot{u}_{\lambda_1}(t)-\dot{u}_{\lambda_2}(t)\right\|_V.\end{aligned} \tag{4.19}$$

By integrating (4.18) over $[0,t]$ and combining it with (4.19), (HP_1), and (HP_2), we can deduce that.

$$\begin{aligned}m_{\mathscr{C}}\int_0^t \|\dot{u}_{\lambda_1}(s)-\dot{u}_{\lambda_2}(s)\|_V^2\,ds\leq&\left(M_{\mathscr{F}}+c_0^2L_\mu c_B\right)\int_0^t \|u_{\lambda_1}(s)-u_{\lambda_2}(s)\|_V\|\dot{u}_{\lambda_1}(s)-\dot{u}_{\lambda_2}(s)\|_V\,ds\\+\frac{1}{2}\|\dot{u}_{\lambda_1}(t)-\dot{u}_{\lambda_2}(t)\|_V-&\int_0^t(\lambda_1(s)-\lambda_2(s),\dot{u}_{\lambda_1}(s)-\dot{u}_{\lambda_2}(s))\,ds.\end{aligned} \tag{4.20}$$

Using the inequality $xy\leq\alpha x^2+\frac{1}{4\alpha}y^2$, $(\alpha>0)$, (4.17) and Gronwall's inequality, it follows that

$$\|u_{\lambda_1}(t)-u_{\lambda_2}(t)\|_V^2\leq C\int_0^t \|\lambda_1(s)-\lambda_2(s)\|_V^2\,ds. \tag{4.21}$$

Moreover, by (4.4), we conclude that

$$\begin{aligned}&\left(\dot{\theta}_{\gamma_1}(t)-\dot{\theta}_{\gamma_2}(t),\theta_{\gamma_1}(t)-\theta_{\gamma_2}(t)\right)_{Q'\times Q}+d\left(\theta_{\gamma_1}(t)-\theta_{\gamma_2}(t),\theta_{\gamma_1}(t)-\theta_{\gamma_2}(t)\right)\\&+\left(\gamma_1(t)-\gamma_2(t),\theta_{\gamma_1}(t)-\theta_{\gamma_2}(t)\right)_Q\leq 0.\end{aligned} \tag{4.22}$$

Similarly, from (HP_3), we obtain

$$\|\theta_{\gamma_1}(t)-\theta_{\gamma_2}(t)\|_Q^2\leq C\int_0^t \|\gamma_1(s)-\gamma_2(s)\|_Q^2\,ds. \tag{4.23}$$

Combining (4.16) through (4.21) and (4.24), it follows that

$$\|\Lambda(\lambda_1,\gamma_1)(t)-\Lambda(\lambda_2,\gamma_2)(t)\|_{V\times Q}^2\leq C\int_0^t \|(\lambda_1,\gamma_1)(s)-(\lambda_2,\gamma_2)(s)\|_{V\times Q}^2. \tag{4.24}$$

Finally, using the result presented in [26, P. 41-45], we deduce that the operator Λ possesses a unique fixed point. □

We are now ready to prove Theorem 4.1.

• **Existence:** Let $(\lambda^*,\gamma^*)\in W^{1,1}(0,T;H)\times L^2(0,T;L^2(\Omega))$ be the fixed point of Λ, denote by $u^*_\lambda,\ \theta^*_\gamma$ the solutions of problems (4.2)-(4.3) and (4.4)-(4.5) respectively. For $(\lambda,\gamma)=(\lambda^*,\gamma^*)$, the definition (4.11) of Λ, we deduce that the pair $(u^*_\lambda,\theta^*_\gamma)$ is a solution of **Problem (PV)**.

• **Uniqueness:** The uniqueness of the solution follows from the uniqueness of the fixed point of the operator Λ.

5. Continuous dependence Result

In this section, we consider $\{\mu_\delta\}_{\delta\geq 1}$ as a family of perturbations of the friction coefficient μ. We aim to study the dependence of the solution of **Problem (PV)** on perturbation of μ, with respect to initial condition $(u_0,\dot u_0,\theta_0)$.

We introduce the following variational problem involving a perturbed friction coefficient μ_δ, and denote its solution as (u_δ,θ_δ), with respect to the initial data $(u_{\delta 0},\theta_{\delta 0})$.

• **Problem** (PV_δ): Find a displacement field $u_\delta\in(V_1\cap V_2)\times]0;T[$ and a temperature field $\theta_\delta\in Q_1\times]0,T[$, for all $v\in V_1\cap V_2$ and $\eta\in Q_1$ such that:

$$\begin{aligned}&(\ddot u_\delta(t),v-\dot u_\delta(t))_H+c\left(\dot u_\delta(t),v-\dot u_\delta(t)\right)+a\left(u_\delta(t),v-\dot u_\delta(t)\right)-m\left(\theta_\delta(t),v-\dot u_\delta(t)\right)\\&+\Phi\left(u_\delta(t),v-\dot u_\delta(t)\right)+j(u_\delta(t),v)-j(u_\delta(t),\dot u_\delta(t))\geq(f(t),v-\dot u_\delta(t))_V,\end{aligned}\tag{5.1}$$

$$\left(\dot\theta_\delta(t),\eta-\theta_\delta(t)\right)_{Q'\times Q}+d\left(\theta_\delta(t),\eta-\theta_\delta(t)\right)+\chi\left(u_\delta(t),\theta_\delta(t),\eta-\theta_\delta(t)\right)\geq(q(t),\eta-\theta_\delta(t))_Q,\tag{5.2}$$

$$u_\delta(0)=u_{\delta 0},\ \dot u_\delta(0)=\dot u_{\delta 0},\ \theta_\delta(0)=\theta_{\delta 0}.\tag{5.3}$$

Problem (PV_δ) has a unique solution, and the arguments of the proof are similar to that used in Section 4.

Now, we present the following convergence result

Theorem 5.1. *Assume the conditions hold*

$$\begin{aligned}&\left\|\mu\left(\|u_\tau\|\right)-\mu_\delta\left(\|u_{\delta\tau}\|\right)\right\|_{L^2(0,T,L^2(\Gamma_C))}\to 0,\ \text{as }\delta\to+\infty\text{ and}\\&\left\{\|u_0-u_{\delta 0}\|_V+\|\dot u_0-\dot u_{\delta 0}\|_H+\|\theta_0-\theta_{\delta 0}\|_Q\right\}\to 0,\ \text{as }\delta\to+\infty.\end{aligned}\tag{5.4}$$

Then, the solution of ***Problem*** (PV_δ) *converge to the solution of* ***Problem (PV)****, in accordance with the following estimate.*

$$\begin{aligned}&\Big\{\|u(t)-u_\delta(t)\|_V^2+\|\dot u(t)-\dot u_\delta(t)\|_H^2+\|\theta(t)-\theta_\delta(t)\|_Q^2\\&+\|\theta(t)-\theta_\delta(t)\|^2_{L^2(0,T,Q)}\Big\}\to 0,\ \text{as }\delta\to+\infty.\end{aligned}\tag{5.5}$$

Proof. By substituting $v=\dot u_\delta(t)$ into (3.26) and $v=\dot u(t)$ into (5.1), we deduce that

$$\begin{aligned}&(\ddot u(t)-\ddot u_\delta(t),\dot u(t)-\dot u_\delta(t))_H+c\left(\dot u(t)-\dot u_\delta(t),\dot u(t)-\dot u_\delta(t)\right)+a\left(u(t)-u_\delta(t),\dot u(t)-\dot u_\delta(t)\right)\\&-m\left(\theta(t)-\theta_\delta(t),\dot u(t)-\dot u_\delta(t)\right)+\Phi\left(u(t)-u_\delta(t),\dot u(t)-\dot u_\delta(t)\right)\\&\leq j\left(u(t),\dot u_\delta(t)\right)-j\left(u(t),\dot u(t)\right)-j\left(u_\delta(t),\dot u(t)\right)+j\left(u_\delta(t),\dot u_\delta(t)\right).\end{aligned}\tag{5.6}$$

On another word, referring to (3.7), $(HP_5)(iii)$ and (HP_7), there exists a positive constant C that depends on c_R, M_μ, and c_0 such that

$$\begin{aligned}&\left|j\left(u(t),\dot u_\delta(t)\right)-j\left(u(t),\dot u(t)\right)-j\left(u_\delta(t),\dot u(t)\right)+j\left(u_\delta(t),\dot u_\delta(t)\right)\right|\\&\leq C\left(\|u(t)-u_\delta(t)\|_V^2+\|\dot u(t)-\dot u_\delta(t)\|_V^2+\left\|\mu\left(\|u_\tau\|\right)-\mu_\delta\left(\|u_{\delta\tau}\|\right)\right\|^2_{L^2(0,T,L^2(\Gamma_C))}\right).\end{aligned}\tag{5.7}$$

By employing hypotheses (HP_1)-(HP_3) along with Young's inequality in conjunction with (5.7), we can conclude that

$$\begin{aligned}&\|\dot u(t)-\dot u_\delta(t)\|_V^2+\frac{1}{2}\frac{d}{dt}\|\dot u(t)-\dot u_\delta(t)\|_H^2\leq C\Big\{\|u(t)-u_\delta(t)\|_V^2+\|\dot u(t)-\dot u_\delta(t)\|_V^2\\&+\|\theta(t)-\theta_\delta(t)\|_Q^2+\left\|\mu\left(\|u_\tau\|\right)-\mu_\delta\left(\|u_{\delta\tau}\|\right)\right\|_{L^2(0,T,L^2(\Gamma_C))}\Big\}.\end{aligned}\tag{5.8}$$

Similarly to (4.17), we observe that

$$\|u(t)-u_\delta(t)\|_V^2 \le C\left\{\int_0^t \|\dot u(s)-\dot u_\delta(s)\|_V^2\,ds + \|u_0-u_{\delta 0}\|_V^2\right\}. \tag{5.9}$$

We integrate equality (5.8) over the interval $[0,t]$ and apply Grownwall's inequality, yielding:

$$\begin{aligned}&\|u(t)-u_\delta(t)\|_V^2 + \|\dot u(t)-\dot u_\delta(t)\|_H^2 \le C\Big\{\|\theta(t)-\theta_\delta(t)\|_Q^2 + \|u_0-u_{\delta 0}\|_V^2\\ &+\|\dot u_0-\dot u_{\delta 0}\|_H^2 + \|\mu(\|u_\tau\|)-\mu_\delta(\|u_{\delta\tau}\|)\|_{L^2(0,T,L^2(\Gamma_C))}\Big\}.\end{aligned} \tag{5.10}$$

Take $\eta=\theta_\delta(t)$ in (3.27) and $\eta=\theta(t)$ in (5.2), we have

$$\begin{aligned}&\left(\dot\theta(t)-\dot\theta_\delta(t),\theta(t)-\theta_\delta(t)\right)_{Q'\times Q} + d\left(\theta(t)-\theta_\delta(t),\theta(t)-\theta_\delta(t)\right)\\ &\le \chi\left(u(t),\theta(t),\theta_\delta(t)-\theta(t)\right)-\chi\left(u_\delta(t),\theta_\delta(t),\theta_\delta(t)-\theta(t)\right).\end{aligned} \tag{5.11}$$

We deduce from (3.11) and (HP_8) that

$$\begin{aligned}&\left|\chi\left(u(t),\theta(t),\theta_\delta(t)-\theta(t)\right)-\chi\left(u_\delta(t),\theta_\delta(t),\theta_\delta(t)-\theta(t)\right)\right|\\ &\le M_{k_c}Lc_0c_1\|u(t)-u_\delta(t)\|_V\|\theta(t)-\theta_\delta(t)\|_Q.\end{aligned} \tag{5.12}$$

Using the property of operator d combined with (5.11)-(5.12) yield

$$\|\theta(t)-\theta_\delta(t)\|_Q^2 + \frac{1}{2}\frac{d}{dt}\|\theta(t)-\theta_\delta(t)\|_Q^2 \le C\|u(t)-u_\delta(t)\|_V^2. \tag{5.13}$$

We integrate the two sides of (5.13) and by Gronwall inequality, there exists a constant $C>0$ such that

$$\|\theta(t)-\theta_\delta(t)\|_Q^2 + \|\theta(t)-\theta_\delta(t)\|_{L^2(0,T;Q)}^2 \le C\left\{\int_0^t \|u(s)-u_\delta(s)\|_V^2 + \|\theta_0-\theta_{\delta 0}\|_Q^2\right\}. \tag{5.14}$$

Finally, by combining (5.10) with (5.14) and performing some algebraic manipulations, we can deduce the existence of a positive constant C such that

$$\begin{aligned}&\|u(t)-u_\delta(t)\|_V^2 + \|\dot u(t)-\dot u_\delta(t)\|_H^2 + \|\theta(t)-\theta_\delta(t)\|_Q^2 + \|\theta(t)-\theta_\delta(t)\|_{L^2(0,T;Q)}^2\\ &\le C\Big\{\|u_0-u_{\delta 0}\|_V^2 + \|\dot u_0-\dot u_{\delta 0}\|_H^2 + \|\theta_0-\theta_{\delta 0}\|_Q^2 + \|\mu(\|u_\tau\|)-\mu_\delta(\|u_{\delta\tau}\|)\|_{L^2(0,T,L^2(\Gamma_C))}\Big\}.\end{aligned} \tag{5.15}$$

Hence, by the hypothesis stated in relation (5.4), we can derive the result presented in Theorem 5.1. □

6. Fully discrete approximation

In this section, we introduce a fully discrete numerical scheme for the solution of **Problem (PV)** and we derive the error estimate. We use the finite element space V^h and Q^h, and we introduce a partition of the time interval $[0;T]$: $0=t_0<t_1<\ldots<t_N=T$. Let $k_n>0$ represent the time step size defined by $k_n=t_n-t_{n-1}$ for $n=1,\ldots,N$. We permit nonuniform partitioning of the time interval and denote $k=\max\limits_n k_n$ as the maximum step size.

For a continuous function $u(t)$ taking values in a function space, we denote $u_n=u(t_n)$ for $0\le n\le N$. We represent the difference as $\delta u_n=\dfrac{1}{k}(u_n-u_{n-1})$, where k is the step size.

Let $\left\{\mathscr{T}^h\right\}$ be a regular family of triangular finite element partition of $\overline\Omega$ are compatible with the boundary decomposition $\Gamma=\overline\Gamma_C\cup\overline\Gamma_D\cup\overline\Gamma_N$. We then define a finite element space $V^h\subset V$ and $Q^h\subset Q$ for the approximates of the displacement field u and the temperature θ defined by

$$V^h=\left\{v^h\in\left[C(\overline\Omega)\right]^d;\ v^h_{|Tr}\in[\mathbb{P}_1(Tr)]^d;\ \forall Tr\in\mathscr{T}^h;\ v^h=0\text{ on }\overline\Gamma_D\right\},$$

$$Q^h=\left\{\eta^h\in\left[C(\overline\Omega)\right];\ \eta^h_{|Tr}\in[\mathbb{P}_1(Tr)];\ \forall Tr\in\mathscr{T}^h;\ \eta^h=0\text{ on }\overline\Gamma_D\cup\overline\Gamma_N\right\}.$$

Also, we have $V_1^h = V_1 \cap V^h$, $V_2^h = V_2 \cap V^h$ and $Q_1^h = Q_1 \cap Q^h$.
To again simplify the notations, we introduce the following notations for $n = 1, ..., N$

$$w_n^{hk} = \delta u_n^{hk},\ u_n^{hk} = \sum_{j=1}^{n} k_j w_j^{hk} + u_0^h \text{ and } \theta_n^{hk} = \sum_{j=1}^{n} k_j \delta\theta_j^{hk} + \theta_0^h.$$

The fully discrete approximation method is based on an backward Euler scheme, it has the following form

• **Problem** (PV_{hk}): Find a displacement field $\left\{u_n^{hk}\right\} \subset V_1^h \cap V_2^h$ and a temperature $\left\{\theta_n^{hk}\right\} \subset Q_1^h$, for all $v^h \in V_1^h \cap V_2^h$, $\eta^h \in Q_1^h$ and $n = 1, ..., N$ such that

$$\begin{aligned}&\left(\delta w_n^{hk}, v^h - w_n^{hk}\right)_H + c\left(w_n^{hk}, v^h - w_n^{hk}\right) + a\left(u_n^{hk}, v^h - w_n^{hk}\right) - m\left(\theta_{n-1}^{hk}, v^h - w_n^{hk}\right)\\ &+ \Phi\left(u_{n-1}^{hk}, v^h - w_n^{hk}\right) + j\left(u_n^{hk}, v^h\right) - j\left(u_n^{hk}, w_n^{hk}\right) \geq \left(f_n, v^h - w_n^{hk}\right)_V,\end{aligned} \tag{6.1}$$

$$\left(\delta\theta_n^{hk}, \eta^h - \theta_n^{hk}\right)_{L^2(\Omega)} + d\left(\theta_n^{hk}, \eta^h - \theta_n^{hk}\right) + \chi\left(u_{n-1}^{hk}, \theta_{n-1}^{hk}, \eta^h - \theta_n^{hk}\right) \geq \left(q_n, \eta^h - \theta_n^{hk}\right)_Q, \tag{6.2}$$

$$u_0^{hk} = u_0^h,\ w_0^{hk} = w_0^h,\ \theta_n^{hk} = \theta_0^h, \tag{6.3}$$

with $u_0^h \in V^h$, $w_0^h \in V^h$ and $\theta_0^h \in Q^h$ are approximations of u_0, w_0 and θ_0 respectively.

Remark 1. The choice of u_{n-1}^{hk} and θ_{n-1}^{hk} in (6.2) is motivated by the fixed point method employed in the previous section.

Applying a discrete analogue of Theorem 4.1, we observe that given $u_{n-1}^{hk} \in V_1^h \cap V_2^h$ and $\theta_{n-1}^{hk} \in Q_1^h$, **Problem** (PV_{hk}) possesses a unique solution $u_n^{hk} \in V_1^h \cap V_2^h$ and $\theta_n^{hk} \in Q_1^h$.
Now, we derive the following convergence result in the fully discrete solution.

Theorem 6.1. *Assume the initial values $u_0^h \in V_1^h \cap V_2^h$, $\theta_0^h \in Q_1^h$ and the assumptions $(HP_1) - (HP_8)$. Then, if*

$$\left\|u_0 - u_0^h\right\|_V \to 0,\ \left\|w_0 - w_0^h\right\|_V \to 0,\ \left\|\theta_0 - \theta_0^h\right\|_Q \to 0 \text{ as } h \to 0, \tag{6.4}$$

*the fully discrete solution of **Problem** (PV_{hk}) converges:*

$$\max_{1 \leq n \leq N} \left\{\left\|w_n - w_n^{hk}\right\|_V^2 + \left\|u_n - u_n^{hk}\right\|_V^2 + \left\|\theta_n - \theta_n^{hk}\right\|_Q^2\right\} \to 0,\ \text{as } h,\ k \to 0. \tag{6.5}$$

Proof. Rewriting (6.1) in the form

$$\begin{aligned}&\left(\delta w_n^{hk}, w_n - w_n^{hk}\right)_H + c\left(w_n^{hk}, w_n - w_n^{hk}\right) + a\left(u_n^{hk}, w_n - w_n^{hk}\right) - \left(\theta_{n-1}^{hk}, w_n - w_n^{hk}\right)\\ &+ \Phi\left(u_{n-1}^{hk}, w_n - w_n^{hk}\right) + j\left(u_n^{hk}, v_n^h\right) - j\left(u_n^{hk}, w_n^{hk}\right) \geq \left(\delta w_n^{hk}, w_n - v_n^h\right)_H + c\left(w_n^{hk}, w_n - v_n^h\right)\\ &+ a\left(u_n^{hk}, w_n - v_n^h\right) - \left(\theta_{n-1}^{hk}, w_n - v_n^h\right) + \Phi\left(u_{n-1}^{hk}, w_n - v_n^h\right) + \left(f_n, v_n^h - w_n^{hk}\right)_V.\end{aligned} \tag{6.6}$$

Substituting v with $w_n^{hk} \in V^h$ at time $t = t_n$ in equation (3.26), we obtain:

$$\begin{aligned}&\left(\dot{w}_n, w_n^{hk} - w_n\right)_H + c\left(w_n, w_n^{hk} - w_n\right) + a\left(u_n, w_n^{hk} - w_n\right) - m\left(\theta_n, w_n^{hk} - w_n\right)\\ &+ \Phi\left(u_n, w_n^{hk} - w_n\right) + j\left(u_n, w_n^{hk}\right) - j\left(u_n, w_n\right) \geq \left(f_n, w_n^{hk} - w_n\right)_V.\end{aligned} \tag{6.7}$$

By combining relations (6.6) and (6.7), we obtain

$$\begin{aligned}&\left(\dot{w}_n - \delta w_n^{hk}, w_n - w_n^{hk}\right)_H + c\left(w_n - w_n^{hk}, w_n - w_n^{hk}\right) + a\left(u_n - u_n^{hk}, w_n - w_n^{hk}\right)\\ &- m\left(\theta_n - \theta_{n-1}^{hk}, w_n - w_n^{hk}\right) + \Phi\left(u_n - u_{n-1}^{hk}, w_n - w_n^{hk}\right)\\ &- j\left(u_n, w_n^{hk}\right) + j\left(u_n, w_n\right) - j\left(u_n^{hk}, v_n^h\right) + j\left(u_n^{hk}, w_n^{hk}\right)\\ &\leq \left(\delta w_n^{hk}, v_n^h - w_n\right)_H + c\left(w_n^{hk}, v_n^h - w_n\right) + a\left(u_n^{hk}, v_n^h - w_n\right)\\ &- m\left(\theta_{n-1}^{hk}, v_n^h - w_n\right) + \Phi\left(u_{n-1}^{hk}, v_n^h - w_n\right) + \left(f_n, w_n - v_n^h\right)_V.\end{aligned} \tag{6.8}$$

Writing

$$\begin{aligned}&\left(\dot{w}_n - \delta w_n^{hk}, w_n - w_n^{hk}\right)_H + \left(\delta w_n^{hk}, w_n - v_n^h\right)_H = \left(\delta w_n - \delta w_n^{hk}, w_n - w_n^{hk}\right)_H\\ &+ \left(\delta w_n - \delta w_n^{hk}, v_n^h - w_n\right)_H + \left(\dot{w}_n - \delta w_n, v_n^h - w_n^{hk}\right)_H + \left(\dot{w}_n, w_n - v_n^h\right)_H.\end{aligned} \tag{6.9}$$

After performing some algebraic manipulations, we deduce that

$$\begin{aligned}&\left(\delta w_n-\delta w_n^{hk},w_n-w_n^{hk}\right)_H+c\left(w_n-w_n^{hk},w_n-w_n^{hk}\right)\leq\left(\delta w_n-\delta w_n^{hk},w_n-v_n^h\right)_H\\&+c\left(w_n^{hk}-w_n,v_n^h-w_n\right)+a\left(u_n^{hk}-u_n,v_n^h-w_n\right)-m\left(\theta_{n-1}^{hk}-\theta_n,v_n^h-w_n\right)\\&+\Phi\left(u_{n-1}^{hk}-u_n,v_n^h-w_n^{hk}\right)+\left(\delta w_n-\dot{w}_n,v_n^h-w_n^{hk}\right)+R\left(w_n,v_n^h\right)+R_j,\end{aligned}\tag{6.10}$$

where

$$\begin{aligned}R\left(w_n,v_n^h\right)&=\left(\dot{w}_n,v_n^h-w_n\right)_H+c\left(w_n,v_n^h-w_n\right)+a\left(u_n,v_n^h-w_n\right)-m\left(\theta_n,v_n^h-w_n\right)\\&+\Phi\left(u_n,v_n^h-w_n\right)+j\left(u_n,v_n^h\right)-j\left(u_n,w_n\right)-\left(f_n,w_n-v_n^h\right)_V,\end{aligned}\tag{6.11}$$

and

$$R_j=j\left(u_n,w_n^{hk}\right)-j\left(u_n,v_n^h\right)+j\left(u_n^{hk},v_n^h\right)-j\left(u_n^{hk},w_n^{hk}\right).\tag{6.12}$$

From the relation

$$\delta w_n-\delta w_n^{hk}=\frac{1}{k}\left(w_n-w_{n-1}\right)-\frac{1}{k}\left(w_n^{hk}-w_{n-1}^{hk}\right),\tag{6.13}$$

and after some calculation, it follows that

$$\left\|w_n-w_n^{hk}\right\|_H^2-\left\|w_{n-1}-w_{n-1}^{hk}\right\|_H^2\leq 2k\left(\delta w_n-\delta w_n^{hk},w_n-w_n^{hk}\right)_H.\tag{6.14}$$

We conclude from (HP_1)-(HP_3) that

$$c\left(w_n-w_n^{hk},w_n-w_n^{hk}\right)\geq m_{\mathscr{C}}\left\|w_n-w_n^{hk}\right\|_V^2,\tag{6.15}$$

$$\left|c\left(w_n-w_n^{hk},w_n-v_n^h\right)\right|\leq M_{\mathscr{C}}\left\|w_n-w_n^{hk}\right\|_V\left\|w_n-v_n^h\right\|_V,\tag{6.16}$$

$$\left|a\left(u_n-u_{n-1}^{hk},w_n^{hk}-v_n^h\right)\right|\leq M_{\mathscr{F}}\left\|u_n-u_{n-1}^{hk}\right\|_V\left\|w_n^{hk}-v_n^h\right\|_V,\tag{6.17}$$

and

$$\left|m\left(\theta_n-\theta_{n-1}^{hk},w_n^{hk}-v_n^h\right)\right|\leq M_{\mathscr{K}}\left\|\theta_n-\theta_{n-1}^{hk}\right\|_Q\left\|w_n^{hk}-v_n^h\right\|_V.\tag{6.18}$$

Similar to (4.14), we observe that

$$\left\|\Phi\left(u_n-u_{n-1}^{hk},w_n^{hk}-v_n^h\right)\right\|\leq\frac{c_0^2}{\epsilon}\left\|u_n-u_{n-1}^{hk}\right\|_V\left\|w_n^{hk}-v_n^h\right\|_V.\tag{6.19}$$

Applying (3.7), (3.10), (6.12), $(HP_5)(iii)$ and (HP_7), we have

$$|R_2|\leq M_\mu c_R c_0^2\left\|u_n-u_n^{hk}\right\|_V\left\|w_n^{hk}-v_n^h\right\|_V.\tag{6.20}$$

By combining (6.10), (6.14)-(6.20), and utilizing the given inequality, we can derive

$$\left\|w_n^{hk}-v_n^h\right\|_V\leq\left\|w_n^{hk}-w_n\right\|_V+\left\|w_n-v_n^h\right\|_V,\tag{6.21}$$

there exists a positive constant c such that

$$\begin{aligned}&\left\|w_n-w_n^{hk}\right\|_H^2-\left\|w_{n-1}-w_{n-1}^{hk}\right\|_H^2+ck\left\|w_n-w_n^{hk}\right\|_V^2\\&\leq ck\Big\{\left\|\dot{w}_n-\delta w_n\right\|_H^2+\left\|u_n-u_n^{hk}\right\|_V^2+\left\|u_n-u_{n-1}^{hk}\right\|_V^2\\&+\left\|\theta_n-\theta_{n-1}^{hk}\right\|_Q^2+\left\|w_n-v_n^h\right\|_V^2+\left\|w_n-v_n^h\right\|_H^2+R\left(w_n,v_n^h\right)\Big\}\\&+2k\left(\delta w_n-\delta w_n^{hk},w_n-v_n^h\right)_H.\end{aligned}\tag{6.22}$$

Substituting $\eta=\theta_n^{hk}\in Q^h$ at time $t=t_n$ into (3.27), we have

$$\left(\dot{\theta}_n,\theta_n^{hk}-\theta_n\right)_{L^2(\Omega)}+d\left(\theta_n,\theta_n^{hk}-\theta_n\right)+\chi\left(u_n,\theta_n,\theta_n^{hk}-\theta_n\right)\geq\left(q_n,\theta_n^{hk}-\theta_n\right)_Q.\tag{6.23}$$

By rewriting (6.2) in the following form

$$\begin{aligned}&\left(\delta\theta_n^{hk},\theta_n-\theta_n^{hk}\right)_{L^2(\Omega)}+d\left(\theta_n^{hk},\theta_n-\theta_n^{hk}\right)+\chi\left(u_{n-1}^{hk},\theta_{n-1}^{hk},\eta_n^h-\theta_n^{hk}\right)\\&\geq\left(\delta\theta_n^{hk},\theta_n-\eta_n^h\right)_{L^2(\Omega)}+d\left(\theta_n^{hk},\theta_n-\eta_n^h\right)+\left(q_n,\eta_n^h-\theta_n^{hk}\right)_Q.\end{aligned}\tag{6.24}$$

Adding (6.23) and (6.24), we have

$$\begin{aligned}&\left(\dot{\theta}_n-\delta\theta_n^{hk},\theta_n-\theta_n^{hk}\right)_{L^2(\Omega)}+\left(\delta\theta_n^{hk},\theta_n-\eta_n^h\right)_{L^2(\Omega)}+d\left(\theta_n-\theta_n^{hk},\theta_n-\theta_n^{hk}\right)\\&\leq\chi\left(u_n,\theta_n,\theta_n^{hk}-\theta_n\right)+\chi\left(u_{n-1}^{hk}n\theta_{n-1}^{hk},\eta_n^h-\theta_n^{hk}\right)+d\left(\theta_n^{hk},\eta_n^h-\theta_n\right)+\left(q_n,\theta_n-\eta_n^h\right)_Q.\end{aligned}\tag{6.25}$$

Using the inequality

$$\begin{aligned}&\left(\dot{\theta}_n-\delta\theta_n^{hk},\theta_n-\theta_n^{hk}\right)_{L^2(\Omega)}+\left(\delta\theta_n^{hk},\theta_n-\eta_n^h\right)_{L^2(\Omega)}=\left(\delta\theta_n-\delta\theta_n^{hk},\theta_n-\theta_n^{hk}\right)_{L^2(\Omega)}\\&+\left(\delta\theta_n-\delta\theta_n^{hk},\eta_n^h-\theta_n\right)_{L^2(\Omega)}+\left(\dot{\theta}_n-\delta\theta_n,\eta_n^h-\theta_n^{hk}\right)_{L^2(\Omega)}+\left(\dot{\theta}_n,\theta_n-\eta_n^h\right)_{L^2(\Omega)}.\end{aligned}\tag{6.26}$$

we deduce that

$$\begin{aligned}&\left(\delta\theta_n-\delta\theta_n^{hk},\theta_n-\theta_n^{hk}\right)_{L^2(\Omega)}+d\left(\theta_n-\theta_n^{hk},\theta_n-\theta_n^{hk}\right)\leq d\left(\theta_n^{hk}-\theta_n,\eta_n^h-\theta_n\right)\\&+R_\chi+R\left(\theta_n,\eta_n^h\right)+\left(\delta\theta_n-\delta\theta_n^{hk},\theta_n-\eta_n^h\right)_{L^2(\Omega)}+\left(\delta\theta_n-\dot{\theta}_n,\eta_n^h-\theta_n^{hk}\right)_{L^2(\Omega)},\end{aligned}\tag{6.27}$$

where

$$R\left(\theta_n,\eta_n^h\right)=\left(\dot{\theta}_n,\eta_n^h-\theta_n\right)_{L^2(\Omega)}+d\left(\theta_n,\eta_n^h-\theta_n\right)+\chi\left(u_n,\theta_n,\eta_n^h-\theta_n\right)+\left(q_n,\theta_n-\eta_n^h\right)_Q,\tag{6.28}$$

and

$$R_\chi=\chi\left(u_n,\theta_n,\theta_n^{hk}-\theta_n\right)+\chi\left(u_{n-1}^{hk},\theta_{n-1}^{hk},\eta_n^h-\theta_n^{hk}\right)-\chi\left(u_n,\theta_n,\eta_n^h-\theta_n\right).\tag{6.29}$$

Since

$$\delta\theta_n-\delta\theta_n^{hk}=\frac{1}{k}\left(\theta_n-\theta_n^{hk}\right)-\frac{1}{k}\left(\theta_{n-1}-\theta_{n-1}^{hk}\right).\tag{6.30}$$

Similar to (6.14), we have that

$$\left\|\theta_n-\theta_n^{hk}\right\|_{L^2(\Omega)}^2-\left\|\theta_{n-1}-\theta_{n-1}^{hk}\right\|_{L^2(\Omega)}^2\leq 2k\left(\delta\theta_n-\delta\theta_n^{hk},\theta_n-\theta_n^{hk}\right)_{L^2(\Omega)}.\tag{6.31}$$

Using (3.7), (3.11) and (HP_8), we obtain

$$|R_\chi|\leq c\left\{\left\|\theta_n-\theta_{n-1}^{hk}\right\|_Q^2+\left\|u_n-u_{n-1}^{hk}\right\|_V^2+\left\|\eta_n^h-\theta_n\right\|_Q^2\right\}.\tag{6.32}$$

Now, we combine (6.27), (6.31)-(6.32) and (6.22), using the properties of operator d and this inequality

$$\left\|\theta_n^{hk}-\eta_n^h\right\|_Q\leq\left\|\theta_n^{hk}-\theta_n\right\|_Q+\left\|\theta_n-\eta_n^h\right\|_Q,\tag{6.33}$$

there exists a positive constant c such that

$$\begin{aligned}&\left\|w_n-w_n^{hk}\right\|_H^2+\left\|\theta_n-\theta_n^{hk}\right\|_{L^2(\Omega)}^2-\left\|w_{n-1}-w_{n-1}^{hk}\right\|_H^2+\left\|\theta_{n-1}-\theta_{n-1}^{hk}\right\|_{L^2(\Omega)}^2\\&+ck\left\{\left\|w_n-w_n^{hk}\right\|_V^2+\left\|\theta_n-\theta_n^{hk}\right\|_Q^2\right\}\\&\leq ck\left\{\left\|\dot{w}_n-\delta w_n\right\|_H^2+\left\|u_n-u_n^{hk}\right\|_V^2+\left\|u_n-u_{n-1}^{hk}\right\|_V^2+\left\|w_n-v_n^h\right\|_V^2\right.\\&\left.+\left\|\dot{\theta}_n-\delta\theta_n\right\|_{L^2(\Omega)}^2+\left\|\theta_n-\theta_{n-1}^{hk}\right\|_Q^2+\left\|\theta_n-\eta_n^h\right\|_Q^2+R\left(w_n,v_n^h\right)+R\left(\theta_n,\eta_n^h\right)\right\}\\&+2k\left\{\left(\delta w_n-\delta w_n^{hk},w_n-v_n^h\right)_H+\left(\delta\theta_n-\delta\theta_n^{hk},\theta_n-\eta_n^h\right)_{L^2(\Omega)}\right\}.\end{aligned}\tag{6.34}$$

By replacing n with j and summing over j from 1 to n, we obtain

$$\begin{aligned}
&\left\|w_n - w_n^{hk}\right\|_H^2 + \left\|\theta_n - \theta_n^{hk}\right\|_{L^2(\Omega)}^2 + ck\sum_{j=1}^{n}\left\|w_j - w_j^{hk}\right\|_V^2 + ck\sum_{j=1}^{n}\left\|\theta_j - \theta_j^{hk}\right\|_Q^2 \\
&\leq \left\|w_0 - w_0^{hk}\right\|_H^2 + \left\|\theta_0 - \theta_0^{hk}\right\|_{L^2(\Omega)}^2 \\
&+ ck\sum_{j=1}^{n}\Big\{\left\|\dot{w}_j - \delta w_j\right\|_H^2 + \left\|u_j - u_j^{hk}\right\|_V^2 + \left\|u_j - u_{j-1}^{hk}\right\|_V^2 + \left\|w_j - v_j^h\right\|_H^2 \\
&+ \left\|\dot{\theta}_j - \delta\theta_j\right\|_{L^2(\Omega)}^2 + \left\|\theta_j - \theta_{j-1}^{hk}\right\|_Q^2 + \left\|\theta_j - \eta_j^h\right\|_Q^2 + R\left(w_j, v_j^h\right) + R\left(\theta_j, \eta_j^h\right)\Big\} \\
&+ 2k\sum_{j=1}^{n}\Big\{\left(\delta w_j - \delta w_j^{hk}, w_j - v_j^h\right)_H + \left(\delta\theta_j - \delta\theta_j^{hk}, \theta_j - \eta_j^h\right)_{L^2(\Omega)}\Big\}.
\end{aligned} \tag{6.35}$$

Estimate the term $\sum_{j=1}^{n} k\left(\delta w_j - \delta w_j^{hk}, w_j - v_j^h\right)_H$

$$\begin{aligned}
&k\sum_{j=1}^{n}\left(\delta w_j - \delta w_j^{hk}, w_j - v_j^h\right)_H = \sum_{j=1}^{n}\left(\left(w_j - w_j^{hk}\right) - \left(w_{j-1} - w_{j-1}^{hk}\right), w_j - v_j^h\right)_H \\
&+ \left(w_n - w_n^{hk}, w_n - v_j^h\right)_H - \left(w_0 - w_0^{hk}, w_1 - v_1^h\right)_H \\
&+ \sum_{j=1}^{n-1}\left(w_j - w_j^{hk}, w_j - v_j^h - \left(w_{j+1} - v_{j+1}^h\right)\right)_H \\
&\leq c\Big\{\left\|w_n - w_n^{hk}\right\|_H^2 + \left\|w_n - v_n^h\right\|_H^2 + \left\|w_0 - w_0^h\right\|_H^2 + \left\|w_1 - v_1^h\right\|_H^2\Big\} \\
&+ 4\sum_{j=1}^{n-1}k\left\|w_j - w_j^{hk}\right\|_H^2 + \sum_{j=1}^{n-1}\frac{1}{k}\left\|w_j - v_j^h - \left(w_{j+1} - v_{j+1}^h\right)\right\|_H^2.
\end{aligned} \tag{6.36}$$

Using the same approach, we can conclude that

$$\begin{aligned}
&k\sum_{j=1}^{n}\left(\delta\theta_j - \delta\theta_j^{hk}, \theta_j - \eta_j^h\right)_{L^2(\Omega)} \leq \\
&+ c\Big\{\left\|\theta_n - \theta_n^{hk}\right\|_{L^2(\Omega)}^2 + \left\|\theta_n - \eta_n^h\right\|_{L^2(\Omega)}^2 + \left\|\theta_0 - \theta_0^h\right\|_{L^2(\Omega)}^2 + \left\|\theta_1 - \eta_1^h\right\|_{L^2(\Omega)}^2\Big\} \\
&+ \sum_{j=1}^{n-1}\left\|\theta_j - \theta_j^{hk}\right\|_{L^2(\Omega)}\left\|\left(\theta_j - \eta_j^h\right) - \left(\theta_{j+1} - \eta_{j+1}^h\right)\right\|_{L^2(\Omega)} \\
&\leq c\Big\{\left\|\theta_n - \theta_n^{hk}\right\|_{L^2(\Omega)}^2 + \left\|\theta_n - \eta_n^h\right\|_{L^2(\Omega)}^2 + \left\|\theta_0 - \theta_0^h\right\|_{L^2(\Omega)}^2 + \left\|\theta_1 - \eta_1^h\right\|_{L^2(\Omega)}^2\Big\} \\
&+ k\sum_{j=1}^{n-1}\left\|\theta_j - \theta_j^{hk}\right\|_{L^2(\Omega)}^2 + \frac{1}{k}\sum_{j=1}^{n-1}\left\|\left(\theta_j - \eta_j^h\right) - \left(\theta_{j+1} - \eta_{j+1}^h\right)\right\|_{L^2(\Omega)}^2.
\end{aligned} \tag{6.37}$$

We would like to recall the following classical inequality

$$\left\|u_j - u_j^{hk}\right\|_V \leq \left\|u_0 - u_0^h\right\|_V + \sum_{l=1}^{j}k\left\|w_l - w_l^{hk}\right\|_V + I_1, \tag{6.38}$$

where

$$I_1 = \left\|\int_0^{t_j} w(s)ds - \sum_{l=1}^{j}kw_l\right\|_V \leq k\|u\|_{H^2(0,T;V)}. \tag{6.39}$$

Then

$$\left\|u_j - u_j^{hk}\right\|_V^2 \leq c\left\{\left\|u_0 - u_0^h\right\|_V^2 + j\sum_{l=1}^{j} k^2\left\|w_l - w_l^{hk}\right\|_V^2 + k^2\|u\|_{H^2(0,T;V)}\right\}. \tag{6.40}$$

By employing the inequality for $j \leq n \leq N$ and $Nk = T$, we can deduce that

$$\sum_{j=1}^{n} k\left\|u_j - u_j^{hk}\right\|_V^2 \leq cT\left(\left\|u_0 - u_0^h\right\|_V^2 + k^2\|u\|_{H^2(0,T;V)}\right) + T\sum_{j=1}^{n} k\sum_{l=1}^{j}\left\|w_l - w_l^{hk}\right\|_V^2. \tag{6.41}$$

Similarly, we have

$$\sum_{j=1}^{n} k\left\|\theta_j - \theta_{j-1}^{hk}\right\|_Q^2 \leq cT\left(\left\|\theta_0 - \theta_0^h\right\|_Q^2 + k^2\|\theta\|_{H^1(0,T;Q)}\right) + T\sum_{j=1}^{n} k\sum_{l=1}^{j}\left\|\delta\theta_l - \delta\theta_l^{hk}\right\|_Q^2, \tag{6.42}$$

$$\sum_{j=1}^{n} k\left\|u_j - u_{j-1}^{hk}\right\|_V^2 \leq cT\left(\left\|u_0 - u_0^h\right\|_V^2 + k^2\|u\|_{H^2(0,T;V)}\right) + T\sum_{j=1}^{n-1} k\sum_{l=1}^{j}\left\|w_l - w_l^{hk}\right\|_V^2, \tag{6.43}$$

and

$$\sum_{j=1}^{n} k\left\|\theta_j - \theta_{j-1}^{hk}\right\|_Q^2 \leq cT\left(\left\|\theta_0 - \theta_0^h\right\|_Q^2 + k^2\|\theta\|_{H^1(0,T;Q)}\right) + T\sum_{j=1}^{n-1} k\sum_{l=1}^{j}\left\|\delta\theta_l - \delta\theta_l^{hk}\right\|_Q^2. \tag{6.44}$$

Let us denote by

$$e_n = \left\|w_n - w_n^{hk}\right\|_H^2 + \left\|\theta_n - \theta_n^{hk}\right\|_{L^2(\Omega)}^2 + ck\sum_{j=1}^{n}\left(\left\|w_j - w_j^{hk}\right\|_V^2 + \left\|\theta_j - \theta_j^{hk}\right\|_Q^2\right), \tag{6.45}$$

and

$$\begin{aligned} g_n = {} & \left\|w_0 - w_0^{hk}\right\|_H^2 + \left\|u_0 - u_0^{hk}\right\|_V^2 + \left\|w_1 - v_1^h\right\|_H^2 + \left\|w_n - v_n^h\right\|_H^2 \\ & + \left\|\theta_0 - \theta_0^{hk}\right\|_{L^2(\Omega)}^2 + \left\|\theta_0 - \theta_0^{hk}\right\|_Q^2 + \left\|\theta_1 - \eta_1^h\right\|_{L^2(\Omega)}^2 + \left\|\theta_n - \eta_n^h\right\|_{L^2(\Omega)}^2. \\ & + k\sum_{j=1}^{n}\left(\|\dot{w}_j - \delta w_j\|_H^2 + \left\|w_j - v_j^h\right\|_V^2 + R\left(w_j, v_j^h\right)\right) \\ & + \sum_{j=1}^{n}\left(\left\|\dot{\theta}_j - \delta\theta_j\right\|_{L^2(\Omega)}^2 + \left\|\theta_j - \eta_j^h\right\|_Q^2 + R\left(\theta_j, \eta_j^h\right)\right) \\ & + \frac{1}{k}\sum_{j=1}^{n-1}\left(\left\|\left(w_j - v_j^n\right) - \left(w_{j+1} - v_{j+1}^h\right)\right\|_H^2 + \left\|\left(\theta_j - \eta_j^n\right) - \left(\theta_{j+1} - \eta_{j+1}^h\right)\right\|_{L^2(\Omega)}^2\right). \end{aligned} \tag{6.46}$$

Now, Keep the assumptions stated in Theorem 4.1 under the regularity conditions

$$\begin{aligned} & u \in C^1\left(0,T;H^2\left(\Omega;\mathbb{R}^d\right)\right) \cap H^3\left(0,T;H\right), \\ & \dot{u}_{|\Gamma_C} \in C\left(0,T;H^2\left(\Gamma_C,\mathbb{R}^d\right)\right), \\ & \theta \in C\left(0,T;H^2(\Omega)\right) \cap H^2\left(0,T;L^2(\Omega)\right),\ \dot{\theta} \in L^2\left(0,T;H^1(\Omega)\right). \end{aligned} \tag{6.47}$$

Let $v_j^h \in V^h$ and $\eta_j^h \in Q^h$ be the finite element interpolates of u_j and θ_j, respectively. It is important to note [29, 30] the following approximation properties

$$\begin{aligned} & \max_{1\leq n\leq N}\left\|w_n - v_n^h\right\|_V \leq ch\|w\|_{C(0,T;H^2(\Omega)^d)}, \\ & \max_{1\leq n\leq N}\left\|\theta_n - \eta_n^h\right\|_Q \leq ch\|\theta\|_{C(0,T;H^2(\Omega))}, \end{aligned} \tag{6.48}$$

$$\begin{aligned}
&\left\|w_0 - w_0^h\right\|_V \le ch \left\|w_0\right\|_{H^2(\Omega,\mathbb{R}^d)}, \\
&\left\|u_0 - u_0^h\right\|_H \le ch \left\|u_0\right\|_{H^1(\Omega,\mathbb{R}^d)}, \\
&\left\|\theta_0 - \theta_0^h\right\|_{L^2(\Omega)} \le ch \left\|\theta_0\right\|_{L^2(\Omega)},
\end{aligned} \tag{6.49}$$

and

$$\begin{aligned}
&k\sum_{j=1}^{n}\left(\left\|\dot{w}_j - \delta w_j\right\|_H + \left\|\dot{\theta}_j - \delta\theta_j\right\|_{L^2(\Omega)}\right) \le ck^2\left\|u\right\|_{H^2(0,T;L^2(\Omega))} + ck^2\left\|\theta\right\|^2_{H^2(0,T;L^2(\Omega)}, \\
&\frac{1}{k}\sum_{j=1}^{n-1}\left(\left\|\left(w_j - v_j^n\right) - \left(w_{j+1} - v_{j+1}^h\right)\right\|_H^2 + \left\|\left(\theta_j - \eta_j^n\right) - \left(\theta_{j+1} - \eta_{j+1}^h\right)\right\|^2_{L^2(\Omega)}\right) \\
&\le ch^2\left\|u\right\|^2_{H^2(0,T;V)} + ch^2\left\|\theta\right\|_{H^2(0,T;Q)}.
\end{aligned} \tag{6.50}$$

Next, by following a similar proof technique as presented in [29, 31], we can obtain

$$\left|R\left(w_j, v_j^h\right)\right| \le c\left\|w_n - v_n^h\right\|_{L^2(\Gamma_C)^d} \le ch^2\|w_n\|_{C(0,T;H^2(\Omega)^d)}. \tag{6.51}$$

and

$$\left|R\left(\theta_j, \eta_j^h\right)\right| \le ch^2\|\theta_n\|_{C(0,T;H^2(\Omega))}. \tag{6.52}$$

Finally, by combining relations (6.42)-(6.46), (6.48)-(6.52) and applying the discrete Gronwall inequality [27], we can conclude that

$$\max_{1\le n\le N}\left\{\left\|w_n - w_n^{hk}\right\|_H + \left\|u_n - u_n^{hk}\right\|_V + \left\|\theta_n - \theta_n^{hk}\right\|_{L^2(\Omega)}\right\} \le c(h+k). \tag{6.53}$$

□

Appendix

In this section, we recall some results of existence and uniqueness result concerning evolution problems, which can be found in the references [2, 27]

Definition 1. Let X be a topological vector space, $f : X \to \overline{\mathbb{R}}$ is a function.
The subdifferential of f at x on the space X defined by

$$\partial f(x) = \left\{z \in X' \mid \langle z, y - x\rangle \le f(y) - f(x),\ \forall y \in X\right\},$$

where X' the topological space of X.

Theorem 6.2. *Let V and H be two real Hilbert spaces such that $V \subset H$ and the inclusion mapping of V into H is continuous and densely defined. We suppose that V is endowed with the norm $\|.\|$ induced by the inner product $(.,.)$ and H is endowed with the norm $|.|$. We denote by V' the dual space of V, by $<.,.>_{V',V}$ the duality pairing between an element of V and an element of V', and H is identified with its own dual H'. We assume that M is a maximal monotone set in $V \times V'$ and A is a linear, continuous and symmetric operator from V to V' satisfying the following coerciveness condition:*

$$\langle Au, u\rangle_{V'\times V} + \alpha\|u\|^2 \ge \omega\|u\|^2,\ \forall u \in V, \tag{6.54}$$

where $\alpha \in \mathbb{R}$ and $\omega > 0$. Let g be in $W^{1,1}(0,T;H)$ and u_0, v_0 be given with

$$u_0 \in V,\ v_0 \in D(M),\ \{Au_0 + Mv_0\} \cap H \ne \phi. \tag{6.55}$$

Then there exists a unique solution u to the following problem:

$$\begin{cases}\dfrac{d^2u}{dt^2} + Au + M\left(\dfrac{du}{dt}\right) \ni g(t) \ a.e. \ on\ (0,T), \\ u(0) = u_0,\ \dfrac{du}{dt}(0) = v_0.\end{cases} \tag{6.56}$$

which satisfies $u \in W^{1,\infty}(0,T;V) \cap W^{2,\infty}(0,T;H)$.

Theorem 6.3. *Let $V \subset H \subset V'$ be a Gelfand triple. Let K be a nonempty, closed, and convex set of V. Assume that $a(.,.) : V \times V \to \mathbb{R}$ is a continuous and symmetric bilinear from such that for some constants $\alpha > 0$ and c_0,*

$$a(v,v) + c_0\|v\|_H^2 \geq \alpha\|v\|_V^2,\ \forall v \in V. \tag{6.57}$$

Then, for every $u_0 \in K$ and $f \in L^2(0,T;H)$, there exists a unique function $u \in H^1(0,T;H) \cap L^2(0,T;V)$ such that $u(0) = u_0$, $u(t) \in K$ for all $t \in [0,T]$, and for almost all $t \in (0,T)$,

$$\langle \dot{u}(t), v - u(t)\rangle_{V'\times V} + a\left(u(t), v - u(t)\right) \geq \left(f(t), v - u(t)\right)_H,\ \forall v \in K. \tag{6.58}$$

References

1. M. Barboteu, W. Han, and S. Migórski, *On numerical approximation of a variational-hemivariational inequality modeling contact problems for locking materials.* Computers & Mathematics with Applications **77** (2019), no. 11, 2894-2905.
2. V. Barbu, *Nonlinear semigroups and differential equations in banach spacese*, (1976).
3. M. Bouallala and El-H. Essoufi, *Analysis results for dynamic contact problem with friction in thermo-viscoelasticity*, Methods Funct. Anal. Topology **26** (2020), no. 4, 317-326.
4. M. Bouallala, El-H. Essoufi, M. Alaoui, *Error Estimates and Analysis Results for Signorinis Problem in Thermo-Electro-Viscoelasticity*, General Letters in Mathematics **2** (2017), no. 2, 25-41.
5. M. Bouallala, El-H. Essoufi, M. Alaoui, *Numerical Analysis of The Penalty Method For Unilateral Contact Problem With Tresca's Friction In Thermo-Electro-Visco-Elasticity.* Eurasian Journal of Mathematical and Computer Applications **8**, no. 3 (2020) 12-32.
6. M. Bouallala, El-H. Essoufi, M. Alaoui, *Variational and Numerical Analysis for Frictional Contact Problem with Normal Compliance in Thermo-Electro-Viscoelasticity.* International Journal of Differential Equations, **14** (2019) 1-14.
7. S. Bourichi, and El-H. Essoufi, *Penalty Method for an Unilateral Contact Problem with Coulomb's Friction for Locking Materials.* Int. J. Math. Model. Comput **6** (2016), no. 1, 61-81.
8. M. Campo, J.R. Fernndez, and W. Han, M. Sofonea, *A dynamic viscoelastic contact problem with normal compliance and damage.* Finite Elements in Analysis and Design **47** (2005), 124.
9. J. Cen, AA. Khan, D. Motreanu and S. Zeng, *Inverse problems for generalized quasi-variational inequalities with application to elliptic mixed boundary value systems.* Inverse Problems **38**, no. 6 (2022), 065006
10. O.Chau, J.R. Fernandez, W. Han, and M. Sofonea, *Variational and numerical analysis of a dynamic frictionless contact problem with adhesion.* Journal of Computational and Applied Mathematics **156** (2003), 127-157.
11. O. Chau, R. Oujja, M. Rochdi, *A mathematical analysis of a dynamical frictional contact model in thermoviscoelasticity.* Discrete Contin. Dyn. Syst. Ser. S. **1** (2008) 61-70.
12. P.G. Ciarlet, *The Finite Element Method for Elliptic Problems, in: P.G. Ciarlet, J.L. Lions (Eds.)*, Handbook of Numerical Analysis, vol. II (Part 1), North Holland, Amsterdam, 1991, pp. 17352.
13. F. Demengel, *Displacements bounded deformation and measures stress*, published in annals of superior school of Pise, 1972.
14. F. Demengel, and P. Suquet, *On locking materials.* Acta Applicandae Mathematica, **6** (1986), no. 2, 185-211.
15. G. Duvaut, *Free boundary problem connected with thermoelasticity and unilateral contact.* Free boundary problems **2** (1979), 217-236.
16. E-H. Essoufi, and A. Zafrar, *Dual methods for frictional contact problem with electroelastic-locking materials.* Optimization. Taylor & Francis **70** (2021), no. 7, 1581-1608.
17. W. Han and M. Sofonea, *Evolutionary variational inequalities arising in viscoelastic contact problems.* SIAM Journal on Numerical Analysis **38** (2001), no.1 , 556-579.
18. W. Han and M. Sofonea, *Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity,* Studies in Advance Mathematics, American Mathematical Society-International Press **30** (2002).
19. S. Migórski, and J. Ogorzały, *A variational-hemivariational inequality in contact problem for locking materials and nonmonotone slip dependent friction.* Acta Mathematica Scientia **37** (2017), no. 6, 1639-1652.
20. J. Necas, and I. Hlavácek, *Mathematical theory of elastic and elasto-plastic bodies: an introduction.* Elsevier, 2017.
21. W. Prager, *Elastic solids of limited compressibility.* roceedings of the 9th International Congress of Applied Mechanics (1958), no. 5, 205-211.
22. W. Prager, *On elastic, perfectly locking materials.* Applied Mechanics, (1966), 538-544.
23. W. Prager, *On ideal locking materials.* Transactions of the Society of Rheology **1** (1957), no.1 , 169-175.
24. M. Sofonea, *A nonsmooth static frictionless contact problem with locking materials.* Analysis and Applications **16** (2018), no. 6, 851-689.
25. M. Sofonea, *History-Dependent Inequalities for Contact Problems with Locking Materials.* Journal of Elasticity **134** (2019), no. 2, 127-148.
26. M. Sofonea, and S. Migorski, *Variational-hemivariational inequalities with applications.* CRC Press, 2017.
27. M.Sofonea, H.Weimin and M. Shillor, *Analysis and approximation of contact problems with adhesion or damage.* bookinfo, TBiMC, CRC Press, 2005.

28. P. Szafraniec P, *Dynamic nonsmooth frictional contact problems with damage in thermoviscoelasticity*. Math Mech Solids **21** (2016), 525-538.
29. W. Han and M. Sofonea, *Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity*. American Mathematical Society and International Press, 2002.
30. W. Han, M. Shillor and M. Sofonea, *Variational and numerical analysis of a quasistatic viscoelastic problem with normal compliance, friction and damage*. J Comput Appl Math **137** (2001), no.2 , 377-398.
31. H. Xuan, X. Cheng, W. Han, Q. Xiao, *Numerical Analysis of a Dynamic Contact Problem with History-Dependent Operators*, SNumer. Math. Theor. Meth. Appl **13** (2020), no.3 , 569-594.
32. S. Zeng, S. Migórski, Z. Liu, *Well-posedness, optimal control, and sensitivity analysis for a class of differential variational-hemivariational inequalities*. SIAM J. Control Optim **31** (2021), no.4 , 2829-2862.
33. S. Zeng, S. Migórski and AA. Khan, *Nonlinear quasi-hemivariational inequalities: existence and optimal control*, SIAM J. Control Optim **59** (2021), no.2 , 1246-1274.
34. S. Zeng, Y. Bai, L. Gasiński, P. Winkert, *Existence results for double phase implicit obstacle problems involving multivalued operators* Calculus of Variations and Partial Differential Equations **59** (2020), 1-18.
35. S. Zeng, S. P. Nikolaos, D. R. Viceniu, *Nonsmooth dynamical systems: From the existence of solutions to optimal and feedback control* Bulletin des Sciences Mathmatiques **176** (2022), 103131.

(Received 05.02.2023)

[1]Cadi Ayyad University, Polydisciplinary faculty, Modeling and Combinatorics Laboratory, Department of Mathematics and Computer Science B.P. 4162,Safi, Morocco

[2]Faculty of Science and Technology, Hassan 1st University Settat Laboratory Mathemat- ics, Computer Science and Engineering Sciences (MISI), 26000 Settat, Morocco

E-mail address: `bouallalamustaphaan@gmail.com`

E-mail address: `e.h.essoufi@gmail.com`