UNVEILING BINARY SET FUNCTIONS: APPLICATIONS AND TOPOLOGICAL INSIGHTS

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Abstract. In this paper, our focus is on the introduction, study, and characterization of functions defined on binary sets in binary topological spaces. As part of our applications, we demonstrate how a binary soft set can be seen as a particular case of the functions defined here. Furthermore, we offer a method to enhance decision-making. Additionally, we present an algorithm to determine binary topologies.

1. INTRODUCTION

Two universal set structures hold significant importance in the realm of mathematics, enabling exploration of connections and interactions among distinct mathematical objects or systems. These structures have wide-ranging applications across various fields, including mathematics, physics, computer science, and engineering. Among the commonly encountered structures dealing with two universal sets, fuzzy sets, soft sets, graph [1–3, 7–9] provide a mechanism for information classification, facilitating nuanced and adaptable reasoning and analysis.

Within the investigation of structures involving two universal sets, the concept of binary topology, introduced in [10], assumes a important role. It serves as a fundamental tool for establishing notions of proximity between such structures. The formulation of binary topology is a relatively recent development that has paved the way for further advancements in the field. Noteworthy contributions include the introduction of weaker notions of binary open sets and the subsequent derivation of various characterizations, as demonstrated in studies [4, 5, 11–13]. The significance of structures involving two universal sets cannot be understated, given their broad applicability and their ability to elucidate intricate relationships. By employing binary topology, we gain insights into the proximity and interplay of these structures, contributing to a deeper understanding of their properties and behavior. The concept of classical soft sets extends into a binary context through the mathematical framework of binary soft sets. Initially proposed by Molodtsov in 1999 [9], soft sets provide a more flexible and uncertain representation of data, expanding upon the notion of classical sets. The introduction of binary soft sets further enhances the practicality and computational aspects of soft sets. Extensive research by Acikgoz and Tas [1] has focused on binary soft sets, including the introduction of binary operations and examination of their algebraic properties. In [6] was explored some applications of binary soft sets in decision-making.

In this paper, in Section 2, we compile the relevant preliminaries on binary topological spaces and duly cite them. These concepts will be utilized throughout the paper. Section 3 examines stability properties related to finite unions and intersections of various binary sets and presents a method for endowing finite sets with a binary topology. Moving on to Section 4, we introduce notions of continuity and open functions associated with binary sets, along with relevant characterizations and properties. Given that calculating open and closed sets, and consequently interiors and closures, can be a tedious and challenging task, Section 5 presents an algorithm that provides an efficient and effective way to perform these calculations. Worth noting, all the examples presented in this paper were computed using this algorithm. Lastly, in Section 6, we demonstrate the relationship between soft sets and

binary topological spaces and provide a method to find parameters with certain "similarities" that could eventually contribute to decision-making.

2. Preliminaries

Throughout the entire work, we will use the notation $\mathscr{P}(X)$ to denote the power set of set X, and similarly, $\mathscr{P}(Y)$ will represent the power set of set Y. If (A, B) and (C, D) are elements of $\mathscr{P}(X) \times \mathscr{P}(Y)$, then the following notation will be used:

(1) $(A, B) \subseteq (C, D)$ if $A \subseteq C$ and $B \subseteq D$.
(2) $(A, B) = (C, D)$ if $(A, B) \subseteq (C, D)$ and $(C, D) \subseteq (A, B)$.
(3) $(A, B) \subseteq (X, Y)$ or (A, B) is a subset of (X, Y) if $(A, B) \in \mathscr{P}(X) \times \mathscr{P}(Y)$.
(4) The binary complement of (A, B) is $(X \setminus A, Y \setminus B)$ and it will be denoted as $(X, Y) \setminus (A, B)$.

Definition 1. [10] Let X and Y be any two nonempty sets, a binary topology from X to Y is a binary structure $\mathscr{M} \subset \mathscr{P}(X) \times \mathscr{P}(Y)$ that satisfies the following axioms:

(1) $(\emptyset, \emptyset) \in \mathscr{M}$ and $(X, Y) \in \mathscr{M}$.
(2) If (A_1, B_1) and (A_2, B_2) are elements of $\mathscr{M}$, then

$$(A_1 \cap A_2, B_1 \cap B_2) \in \mathscr{M}.$$

(3) If $\{(A_i, B_i), i \in I\}$ is a family of elements of $\mathscr{M}$, then

$$(\cup_{i \in I} A_i, \cup_{i \in I} B_i) \in \mathscr{M}.$$

If $\mathscr{M}$ is a binary topology from X to Y, then the triple $(X, Y, \mathscr{M})$ is called a binary topological space and the members of $\mathscr{M}$ are called the binary open sets of the binary topological space $(X, Y, \mathscr{M})$. (A, B) is called a binary closed set if $(X \setminus A, Y \setminus B) \in \mathscr{M}$. In [4] and [10] examples of binary topological spaces are provided.

Definition 2. [10] Let $(X, Y, \mathscr{M})$ be a binary topological space, and let (A, B) be a subset of (X, Y). Define the following:

(1) $(A, B)^{1^\diamond} = \cup\{A_\alpha : (A_\alpha, B_\alpha) \in \mathscr{M} \text{ and } (A_\alpha, B_\alpha) \subseteq (A, B)\}$.
(2) $(A, B)^{2^\diamond} = \cup\{B_\alpha : (A_\alpha, B_\alpha) \in \mathscr{M} \text{ and } (A_\alpha, B_\alpha) \subseteq (A, B)\}$.
(3) $(A, B)^{1^*} = \cap\{A_\alpha : (X \setminus A_\alpha, Y \setminus B_\alpha) \in \mathscr{M} \text{ and } (A, B) \subseteq (A_\alpha, B_\alpha)\}$.
(4) $(A, B)^{2^*} = \cap\{B_\alpha : (X \setminus A_\alpha, Y \setminus B_\alpha) \in \mathscr{M} \text{ and } (A, B) \subseteq (A_\alpha, B_\alpha)\}$.

The pair $((A, B)^{1^\diamond}, (A, B)^{2^\diamond})$ is referred to as the binary interior of (A, B) and is denoted by $\mathrm{Int}((A, B))$. And the pair $((A, B)^{1^*}, (A, B)^{2^*})$ is referred to as the binary closure of (A, B) and is denoted by $\mathrm{Cl}(A, B)$.

Properties of the binary interior and binary closure have already been studied in [4] and [10]. Below, we provide some of these properties.

Theorem 2.1. *Let $(X, Y, \mathscr{M})$ be a binary topological space, and let (A, B) be a subset of (C, D) which is itself a subset of (X, Y). Then the following statements hold:*

(1) $\mathrm{Int}(A, B) \in \mathscr{M}$ *and* $\mathrm{Int}(A, B) \subseteq (A, B)$.
(2) $(A, B) \in \mathscr{M}$ *if and only if* $\mathrm{Int}(A, B) = (A, B)$.
(3) $\mathrm{Cl}(A, B)$ *is a binary closed set and* $(A, B) \subseteq \mathrm{Cl}(A, B)$.
(4) (A, B) *is a binary closed set if and only if* $\mathrm{Cl}(A, B) = (A, B)$.
(5) $\mathrm{Int}(A, B) \subseteq \mathrm{Int}(C, D)$ *and* $\mathrm{Cl}(A, B) \subseteq \mathrm{Cl}(C, D)$ *(monotony).*
(6) $\mathrm{Int}(\mathrm{Int}(A, B)) = \mathrm{Int}(A, B)$ *and* $\mathrm{Cl}(\mathrm{Cl}(A, B)) = \mathrm{Cl}(A, B)$.
(7) $(X, Y) \setminus \mathrm{Cl}(A, B) = \mathrm{Int}(X \setminus A, Y \setminus B)$ *and* $(X, Y) \setminus \mathrm{Int}(A, B) = \mathrm{Cl}(X \setminus A, Y \setminus B)$.

As usual, weak notions of binary sets have been defined in terms of binary interior and binary closure. And the relationship between these weak notions of binary open sets was studied in [4].

Definition 3. Let $(X, Y, \mathscr{M})$ be a binary topological space, and let (A, B) be a subset of (X, Y). Then (A, B) is said to be:

(1) Binary regular closed set [11], if $(A, B) = \mathrm{Cl}(\mathrm{Int}(A, B))$.

(2) Binary semi open set [12], if $(A, B) \subseteq \text{Cl}(\text{Int}(A, B))$.
(3) Binary α^*-set [4], if $\text{Int}(A, B) = \text{Int}(\text{Cl}(\text{Int}(A, B)))$.
(4) Binary t-set [4], if $\text{Int}(A, B) = \text{Int}(\text{Cl}(A, B))$.
(5) Binary s-set [4], if $\text{Int}(A, B) = \text{Cl}(\text{Int}(A, B))$.
(6) Binary β^*-set [4], if $\text{Int}(A, B) = \text{Cl}(\text{Int}(\text{Cl}(A, B)))$.
(7) Binary $\mathscr{A}$-set [4], if $(A, B) = (U \cap C, V \cap D)$, where $(U, V) \in \mathscr{M}$ and (C, D) is a binary regular closed set.
(8) Binary $\mathscr{B}$-set [4], if $(A, B) = (U \cap C, V \cap D)$, where $(U, V) \in \mathscr{M}$ and (C, D) is a binary t- set.
(9) Binary $\mathscr{C}$-set [4], if $(A, B) = (U \cap C, V \cap D)$, where $(U, V) \in \mathscr{M}$ and (C, D) is a binary α^*-set.
(10) Binary locally closed set [4], if $(A, B) = (U \cap C, V \cap D)$, where $(U, V) \in \mathscr{M}$ and (C, D) is a binary closed set.
(11) Binary β-set [4], if $(A, B) = (U \cap C, V \cap D)$, where $(U, V) \in \mathscr{M}$ and (C, D) is a binary β^*-set.
(12) Binary $\mathscr{S}$-set [4], if $(A, B) = (U \cap C, V \cap D)$, where $(U, V) \in \mathscr{M}$ and (C, D) is a binary s-set.

3. Some properties of binary sets

In this section, we will present stability properties related to the union and intersection of established binary set notions. Additionally, we will introduce a method to generate a binary topology on finite sets from a given collection.

Theorem 3.1. *Let $(X, Y, \mathscr{M})$ be a binary topological space, and let (A, B) and (C, D) be subsets of (X, Y). Then the following hold:*

(1) *If (A, B) and (C, D) are binary α^*-sets, then $(A \cap C, B \cap D)$ is a binary α^*-set.*
(2) *If (A, B) and (C, D) are binary t-sets, then $(A \cap C, B \cap D)$ is a binary t-set.*
(3) *If (A, B) and (C, D) are binary $\mathscr{B}$-sets, then $(A \cap C, B \cap D)$ is a binary $\mathscr{B}$-set.*
(4) *If (A, B) and (C, D) are binary $\mathscr{C}$-sets, then $(A \cap C, B \cap D)$ is a binary $\mathscr{C}$-set.*
(5) *If (A, B) and (C, D) are binary locally closed sets, then $(A \cap C, B \cap D)$ is a binary locally closed set.*

Proof. We will only demonstrate the first item, since the demonstrations of the other items are similar to this one. Let (A, B) and (C, D) be a binary α^*-sets, then $\text{Int}(A, B) = \text{Int}(\text{Cl}(\text{Int}(A, B)))$ and $\text{Int}(C, D) = \text{Int}(\text{Cl}(\text{Int}(C, D)))$. For monotony and by hypothesis

$$\text{Int}(\text{Cl}(\text{Int}(A \cap C, B \cap D))) \subseteq \text{Int}(\text{Cl}(\text{Int}(A, B))) = \text{Int}(A, B) \subseteq (A, B)$$

and

$$\text{Int}(\text{Cl}(\text{Int}(A \cap C, B \cap D))) \subseteq \text{Int}(\text{Cl}(\text{Int}(C, D))) = \text{Int}(C, D) \subseteq (C, D),$$

hence $\text{Int}(\text{Cl}(\text{Int}(A \cap C, B \cap D))) \subseteq (A \cap C, B \cap D)$. Therefore, $\text{Int}(\text{Cl}(\text{Int}(A \cap C, B \cap D))) \subseteq \text{Int}(A \cap C, B \cap D)$. Note that always is true that $\text{Int}(A \cap C, B \cap D) \subseteq \text{Int}(\text{Cl}(\text{Int}(A \cap C, B \cap D)))$. So, $(A \cap C, B \cap D)$ is a binary α^*-set. $\square$

Note that the previous result establishes that the collections of binary α^*-sets, binary t-sets, binary $\mathscr{B}$-sets, binary $\mathscr{C}$-sets and binary locally closed satisfy condition (2) of Definition 1

Theorem 3.2. *Let $(X, Y, \mathscr{M})$ be a binary topological space, and let (A, B) and (C, D) be subsets of (X, Y). Then the following hold:*

(1) *If (A, B) and (C, D) are binary semi-open sets, then $(A \cup C, B \cup D)$ is a binary semi-open set.*
(2) *If (A, B) and (C, D) are binary regular closed sets, then $(A \cup C, B \cup D)$ is a binary regular closed set.*

Proof. 1. Let (A, B) and (C, D) be a binary semi-open sets, then $(A, B) \subseteq \text{Cl}(\text{Int}(A, B))$ and $(C, D) \subseteq \text{Cl}(\text{Int}(C, D))$. For monotony and by hypothesis $(A, B) \subseteq \text{Cl}(\text{Int}(A, B)) \subseteq \text{Cl}(\text{Int}(A \cup C, B \cup D))$ and $(C, D) \subseteq \text{Cl}(\text{Int}(C, D)) \subseteq \text{Cl}(\text{Int}(A \cup C, B \cup D))$, hence $(A \cup C, B \cup D) \subseteq \text{Cl}(\text{Int}(A \cup C, B \cup D))$. Therefore, $(A \cup C, B \cup D)$ is a binary semi-open set.
2. Let (A, B) and (C, D) be a binary regular closed sets, then $(A, B) = \text{Cl}(\text{Int}(A, B))$ and $(C, D) = \text{Cl}(\text{Int}(C, D))$, by hypothesis $(A, B) = \text{Cl}(\text{Int}(A, B)) \subseteq \text{Cl}(\text{Int}(A \cup C, B \cup D))$ and $(C, D) = \text{Cl}(\text{Int}(C, D)) \subseteq \text{Cl}(\text{Int}(A \cup C, B \cup D))$ hence $(A \cup C, B \cup D) \subseteq \text{Cl}(\text{Int}(A \cup C, B \cup D))$. Now, always is true that

$\mathrm{Cl}(\mathrm{Int}(A \cup C, B \cup D)) \subseteq \mathrm{Cl}(A \cup C, B \cup D)$. Note that (A, B) and (C, D) are binary closed sets and therefore $(A \cup C, B \cup D)$ is a binary closed set, as result $(A \cup C, B \cup D) = \mathrm{Cl}(A \cup C, B \cup D)$. The above allows us to conclude that $\mathrm{Cl}(\mathrm{Int}(A \cup C, B \cup D)) \subseteq (A \cup C, B \cup D)$. By being able to demonstrate this, we can conclude that $(A \cup C, B \cup D)$ is a binary regular closed set. $\qquad\square$

Just as in general topology, we can obtain a binary topology from a smaller collection that satisfies a certain condition. Below, we present the condition that this collection must satisfy and how to construct the binary topology. Let $\Delta \subseteq \mathscr{P}(X) \times \mathscr{P}(Y)$ be a collection such that for all $(x, y) \in X \times Y$ there exist $(A, B), (C, D) \in \Delta$ such that $(\{x\}, \{y_1\}) \subseteq (A, B)$ and $(\{x_1\}, \{y\}) \subseteq (C, D)$ for some $x_1 \in X$ and some $y_1 \in Y$; consider the collection

$$\mathscr{B} = \left\{ (C, D) : (C, D) = \left(\bigcap_{i=1}^{n} A_i, \bigcap_{i=1}^{n} B_i \right); \ where \ (A_i, B_i) \in \Delta \right\}.$$

Then the collection

$$\mathscr{M} = \left\{ (U, V) : (U, V) = \left(\bigcup_{i=1}^{n} C_i, \bigcup_{i=1}^{n} D_i \right); \ where \ (C_i, D_i) \in \mathscr{B} \right\} \cup \{(\emptyset, \emptyset)\}$$

is a binary topology from X to Y for X, Y finite sets. We will refer to Δ as a binary subbase and to $\mathscr{B}$ as a binary base for $\mathscr{M}$, respectively.

4. Some types of binary continuous functions.

In this section, we introduced the notion of binary continuous functions using any class $\mathscr{W}$ of binary sets. Let X, Y, and Z be nonempty sets, $f : Z \to \mathscr{P}(X) \times \mathscr{P}(Y)$ be a function, and (A, B) be a subset of (X, Y). Then, $f^{-1}(A, B) = \{z \in Z : f(z) = (C, D) \subseteq (A, B)\}$, and $f^{-1}(A, B)$ will be called preimage of (A, B).

Definition 4. Let $(X, Y, \mathscr{M})$ be a binary topological space, (Z, τ) be a topological space, and $\mathscr{W}$ be a class of subsets of (X, Y). A function $f : Z \to \mathscr{P}(X) \times \mathscr{P}(Y)$ is called binary $\mathscr{W}$-continuous if $f^{-1}(A, B) \in \tau$ for all $(A, B) \in \mathscr{W}$.

Remark 1. Depending on the class $\mathscr{W}$ considered we will obtain specifics kind of continuous function, for instance, if $\mathscr{W}$ is the collection of all binary open set then f is said to be binary continuous.

Now, we are going to prove a pointwise characterization of binary $\mathscr{W}$-continuous functions.

Theorem 4.1. *Let $(X, Y, \mathscr{M})$ be a binary topological space, and (Z, τ) be a topological space. A function $f : Z \to \mathscr{P}(X) \times \mathscr{P}(Y)$ is a binary $\mathscr{W}$-continuous function if and only if for every $z \in Z$ and for every $(A, B) \in \mathscr{W}$ with $f(z) \subseteq (A, B)$, there exists an open set $U \subseteq Z$ such that $f(U) \subseteq (A, B)$.*

Proof. Let $f : Z \to \mathscr{P}(X) \times \mathscr{P}(Y)$ be a binary $\mathscr{W}$-continuous function, $(A, B) \in \mathscr{W}$ and $z \in Z$ such that $f(z) = (C, D) \subseteq (A, B)$, then $U = f^{-1}(A, B)$ is an open in Z, $z \in U$ and $f(U) \subset (A, B)$. Now, let $(A, B) \in \mathscr{W}$ and $z \in f^{-1}(A, B)$, then $f(z) = (C, D) \subseteq (A, B)$, by hypotheses there exist an open set $U_z \subseteq Z$ such that $f(U_z) \subseteq (A, B)$ and therefore $U_z \subseteq f^{-1}(A, B)$. So, $f^{-1}(A, B) = \bigcup_{z \in f^{-1}(A, B)} U_z$. Hence, $f^{-1}(A, B)$ is open in Z because of it is union of open sets in Z. $\qquad\square$

Theorem 4.2. *Let $(X, Y, \mathscr{M})$ be a binary topological space, Z be a non empty set and $f : Z \to \mathscr{P}(X) \times \mathscr{P}(Y)$ be a function, then the collection $\mu = \left\{ f^{-1}(A, B) : (A, B) \in \mathscr{M} \right\}$ is stable under finite intersections. Moreover, if $\mathscr{W}$ is a class of binary sets that satisfies condition (2) of the Definition 1, then the collection $\beta = \left\{ f^{-1}(A, B) : (A, B) \in \mathscr{W} \right\}$ is stable under finite intersections.*

Proof. Let $\{(A_i, B_i)\}_{i=1}^{n}$ be a finite collection of elements of $\mathscr{M}$, then

$$f^{-1}\left(\left(\bigcap_{i=1}^{n} A_i, \bigcap_{i=1}^{n} B_i \right) \right) = \bigcap_{i=1}^{n} f^{-1}(A_i, B_i).$$

And the result follows because of $(\bigcap_{i=1}^{n} A_i, \bigcap_{i=1}^{n} B_i) \in \mathscr{M}$. $\qquad\square$

It would be natural to think that the previously defined collection is stable for arbitrary unions; however, this is generally not true, as shown in the following example.

Example. Take $X = \{1, 2\}$, $Y = \{5, 6\}$ and

$$\mathscr{M} = \{(\emptyset, \emptyset), (X, Y), (\{1\}, \{6\}), (\{2\}, Y), (\emptyset, \{6\})\}.$$

and $Z = \{p, q, r, s\}$. Define $f : Z \to \mathscr{P}(X) \times \mathscr{P}(Y)$, as follow: $f(p) = (\{1\}, \{6\})$, $f(q) = (\{2\}, \{5\})$, $f(r) = (\{2\}, \{6\})$, $f(s) = (\{1\}, \{5\})$. Note that: $f^{-1}(\{1\}, \{6\}) \cup f^{-1}(\{2\}, Y) = \{p, q, r\}$ and $f^{-1}(\{1\} \cup \{2\}, \{6\} \cup Y) = \{p, q, r, s\}$. Therefore, $f^{-1}(\{1\}, \{6\}) \cup f^{-1}(\{2\}, Y) \neq f^{-1}(\{1\} \cup \{2\}, \{6\} \cup Y)$.

However, as a direct consequence of the previous result, it follows that if $(X, Y, \mathscr{M})$ is a binary topological space, Z is a non empty set and $f : Z \to \mathscr{P}(X) \times \mathscr{P}(Y)$ is a function, then the collection $\beta = \{f^{-1}(A, B) : (A, B) \in \mathscr{M}\}$ is a basis for a topology on Z. If on Z we consider the topology generated by the previously described base β, then f becomes a binary continuous function.

Below, open functions in binary spaces are defined, and a characterization of these functions is provided.

Definition 5. Let $(X, Y, \mathscr{M})$ be a binary topological space, (Z, τ) be a topological space, and $\mathscr{W}$ be a class of subsets of (X, Y). A function $f : Z \to \mathscr{P}(X) \times \mathscr{P}(Y)$ is called binary $\mathscr{W}$-open if $f(U) \in \mathscr{W}$ for all $U \in \tau$.

Theorem 4.3. *Let $(X, Y, \mathscr{M})$ be a binary topological space, (Z, τ) be a topological space. A function $f : Z \to \mathscr{P}(X) \times \mathscr{P}(Y)$ is binary $\mathscr{M}$-open if and only if $f(\operatorname{Int}(G)) \subseteq \operatorname{Int}(f(G))$ for all $G \subseteq Z$.*

Proof. Let $f : Z \to \mathscr{P}(X) \times \mathscr{P}(Y)$ be a binary $\mathscr{W}$-open function and let $G \subseteq Z$. Since $\operatorname{Int}(G) \subseteq G$, then by monotony of direct image follows that $f(\operatorname{Int}(G)) \subseteq f(G)$ and therefore

$$f(\operatorname{Int}(G)) = \operatorname{Int}(f(\operatorname{Int}(G))) \subseteq \operatorname{Int}(f(G)).$$

Reciprocally, consider $G \in \tau$, then $f(G) = f(\operatorname{Int}(G)) \subseteq \operatorname{Int}(f(G))$, this implies that $f(G) \in \mathscr{M}$. $\square$

The following result states that the composition of a continuous function and a binary $\mathscr{W}$-open function is a $\mathscr{W}$-open function.

Theorem 4.4. *Let $(X, Y, \mathscr{M})$ be a binary topological space, (Z, τ) and (W, Γ) be topological spaces. Consider functions $f : Z \to \mathscr{P}(X) \times \mathscr{P}(Y)$ and $g : W \to Z$. If $f \circ g$ is a binary $\mathscr{W}$-open function and g is a continuous and surjective function, then f is also a binary $\mathscr{W}$-open function.*

Proof. Let G an open set in Z, since $g : W \to Z$ is a continuous function $g^{-1}(G)$ is open in W. Due to $f \circ g$ is a binary $\mathscr{W}$-open function we obtain $(f \circ g)(g^{-1}(G)) = f(G) \in \mathscr{W}$ $\square$

5. Algorithm

When building finite topologies, manual calculations are characterized by being tedious, long and complicated. This is something that a computer can help with, and we take advantage of it using the simple algorithms presented in this section.

For any given finite Δ set, one can apply Algorithm 1 and then Algorithm 2 to create a binary topology. These algorithms follow the procedure of building a topology with a given subbase.

Algorithm 1: Finite Intersections

Input: Set $\Delta \subseteq \mathscr{P}(X) \times \mathscr{P}(Y)$
Output: Set $\mathscr{B}$, the binary base

1 $\mathscr{B} \leftarrow$ empty list;

2 foreach *possible combination of elements of Δ* **do**
3 | Intersect that finite collection of sets;
4 | **if** *the result is not in $\mathscr{B}$* **then**
5 | | Append the result to $\mathscr{B}$;
6 | **end**
7 end

8 return $\mathscr{B}$;

Algorithm 2: Finite unions

Input: Set $\mathscr{B}$, the binary base
Output: Set $\mathscr{M}$, a binary topology

1 $\mathscr{M} \leftarrow$ empty list;

2 foreach *possible combination of elements of $\mathscr{B}$* **do**
3 | Join that finite collection of sets;
4 | **if** *the result is not in $\mathscr{M}$* **then**
5 | | Append the result to $\mathscr{M}$;
6 | **end**
7 end

8 return $\mathscr{M}$;

One can also compute the binary interior and binary closure of a given set using Algorithms 3 and 4 respectively. These procedures follow the usual definitions of binary interior and binary closure, applied in the finite space.

Algorithm 3: Get binary Interior

Input : Set $(A, B) \subset \mathscr{P}(X) \times \mathscr{P}(Y)$ and binary topology $\mathscr{M}$
Output: Interior of (A, B)

1 l $\leftarrow$ empty list;

2 foreach *binary open set in $\mathscr{M}$* **do**

3 | **if** *binary open set is a subset of (A, B)* **then**
4 | | Append binary open set to list l;
5 | **end**
6 end

7 interior $\leftarrow$ union of list l;

8 return interior;

Algorithm 4: Get binary Closure

Input : Set $(A, B) \subset \mathscr{P}(X) \times \mathscr{P}(Y)$ and binary topology $\mathscr{M}$
Output: Closure of (A, B)

1 l $\leftarrow$ empty list;

2 foreach *binary closed set in $\mathscr{M}$* **do**

3   **if** (A, B) *is a subset of the binary closed set* **then**

4   Append binary closed set to list l;

5   **end**

6 end

7 closure $\leftarrow$ intersection of list l;

8 return closure;

Following the established in section 4, we also synthesize the calculation of the preimage of a function in the Algorithm 5.

Algorithm 5: Get Preimage

Input : Set $(A, B) \subset \mathscr{P}(X) \times \mathscr{P}(Y)$ and a function $f : Z \to \mathscr{P}(X) \times \mathscr{P}(Y)$
Output: Preimage of (A, B)

1 preimage $\leftarrow$ empty list;

2 foreach $z \in Z$ **do**

3   **if** $f(z)$ *is a subset of (A, B)* **then**

4   Append z to preimage;

5   **end**

6 end

7 return preimage;

By utilizing the algorithms defined earlier, we can demonstrate multiple examples, including the one presented in the following sections.

6. Applications to binary soft set

We will begin this section by recalling the concept of binary soft sets over finite universe sets X and Y, defined in [1]. Let E be a set of parameters, and consider a subset A of E. We define a function $F : A \to \mathscr{P}(X) \times \mathscr{P}(Y)$, where each element $e \in A$ is mapped to a pair (U, V), with U being a subset of X and V a subset of Y. We refer to the pair (F, A) as a binary soft set over X and Y. This notion allows us to encompass and explore the concepts presented in the previous section within the framework of binary soft sets. By doing so, we not only extend the scope of binary set theory but also open up new possibilities for its application in various fields. The flexibility and generality of binary soft sets enable us to establish connections with other mathematical theories and utilize them in innovative ways. Through the study of binary soft sets, we can establish novel concepts, techniques, and applications that go beyond traditional set theory. The interplay between binary soft sets and other mathematical frameworks gives rise to exciting opportunities for research and practical implementations in fields such as decision-making and information processing. It is evident that the concepts and topics discussed in the previous section can be encompassed within the framework of binary soft sets, leading to the emergence of novel concepts and applications within this theory. In the following discussion, we will demonstrate these new concepts and explore their practical implications. Consider $X = \{c_1, c_2, c_3, c_4\}$, $Y = \{p_1, p_2, p_3, p_4\}$, $E = \{e_1, e_2, e_3, e_4, e_5\}$ and the soft set (F, E) given by $F(e_1) = (\{c_4\}, \{p_4\})$; $F(e_2) = (\{c_1, c_2, c_3\}, \{p_1, p_2, p_3\})$; $F(e_3) = (\{c_1, c_2\}, \{p_1, p_3, p_4\})$; $F(e_4) = (\{c_3, c_4\}, \{p_1\})$ and $F(e_5) = (\{c_1, c_2\}, \{p_2, p_3, p_4\})$. Considering the attributes assigned to

TABLE 1. Binary topology $\mathscr{M}$

$(\{c_2\}, \{p_1\})$	$(\{c_3, c_4\}, \{p_1\})$	$(\{c_1, c_2\}, \{p_2, p_3, p_4\})$
$(\{c_4\}, \{p_1\})$	$(\{c_2\}, \{p_2, p_4\})$	$(\{c_1, c_2, c_3\}, \{p_1, p_3, p_4\})$
$(\{c_3\}, \{p_1\})$	$(\{c_4\}, \{p_1, p_4\})$	$(\{c_2, c_3, c_4\}, \{p_1, p_2, p_4\})$
$(\{c_2\}, \{p_4\})$	$(\{c_3\}, \{p_1, p_4\})$	$(\{c_1, c_2, c_4\}, \{p_2, p_3, p_4\})$
$(\{c_2\}, \{p_2\})$	$(\{c_2, c_4\}, \{p_2, p_4\})$	$(\{c_2, c_3, c_4\}, \{p_1, p_2\})$
$(\{c_4\}, \{p_4\})$	$(\{c_2, c_3\}, \{p_1, p_2\})$	$(\{c_1, c_2\}, \{p_1, p_3, p_4\})$
$(\{c_2\}, \emptyset)$	$(\{c_1, c_2\}, \{p_3, p_4\})$	$(\{c_1, c_2, c_4\}, \{p_3, p_4\})$
$(\{c_4\}, \emptyset)$	$(\{c_3, c_4\}, \{p_1, p_4\})$	$(\{c_1, c_2, c_3, c_4\}, \{p_1, p_3, p_4\})$
$(\emptyset, \{p_1\})$	$(\{c_2, c_3\}, \{p_1, p_4\})$	$(\{c_2, c_3, c_4\}, \{p_1\})$
$(\emptyset, \{p_4\})$	$(\{c_2, c_4\}, \{p_1, p_4\})$	$(\{c_2\}, \{p_1, p_2, p_4\})$
$(\emptyset, \{p_1, p_4\})$	$(\{c_2, c_4\}, \{p_2\})$	$(\{c_2, c_3, c_4\}, \{p_1, p_4\})$
$(\{c_2, c_4\}, \emptyset)$	$(\{c_1, c_2\}, \{p_1, p_2, p_3, p_4\})$	$(\{c_2\}, \{p_1, p_4\})$
$(\{c_2, c_4\}, \{p_1\})$	$(\{c_2, c_3\}, \{p_1, p_2, p_3, p_4\})$	$(\{c_1, c_2, c_4\}, \{p_1, p_3, p_4\})$
$(\{c_2, c_3\}, \{p_1\})$	$(\{c_2, c_4\}, \{p_1, p_2, p_4\})$	$(\{c_1, c_2, c_4\}, \{p_1, p_2, p_3, p_4\})$
$(\{c_2\}, \{p_1, p_2\})$	$(\{c_2, c_3\}, \{p_1, p_2, p_4\})$	$(X, Y), (\emptyset, \emptyset)$

each parameter, we can create the following list:

$$\Delta = \{(\{c_4\}, \{p_4\}), (\{c_2, c_3\}, \{p_1, p_2\}), (\{c_1, c_2\}, \{p_1, p_3, p_4\}),$$
$$(\{c_3, c_4\}, \{p_1\}), (\{c_1, c_2\}, \{p_2, p_3, p_4\})\}$$

Using the algorithm described in the previous section, we obtain the binary topology $\mathscr{M}$, as presented in Table 1.

Then, we calculate the preimage of each element in $\mathscr{M}$ to obtain a basis β and consequently a topology on E.

$$\beta = \{\emptyset, \{e_1\}, \{e_3\}, \{e_4\}, \{e_5\}, \{e_1, e_4\}, \{e_1, e_3\}, \{e_1, e_5\}, \{e_2, e_3, e_5\},$$
$$\{e_1, e_3, e_4\}, \{e_3, e_5\}, \{e_1, e_2, e_3, e_4, e_5\}, \{e_1, e_3, e_5\}\}$$

This procedure turns F into an binary continuous function. The idea is to consider the closure of the parameters (elements of the domain) and calculate the image of this closure. This allows us to determine which other parameters are "similar" to the originally assigned parameters and thus expand the range of assignment for decision making. For example, for the parameter e_3, we obtain that $\mathrm{Cl}(\{e_3\}) = \{e_2, e_3\}$, and therefore $F(\mathrm{Cl}(\{e_3\})) = \{(\{c_1, c_2, c_3\}, \{p_1, p_2, p_3\}), (\{c_1, c_2\}, \{p_1, p_3, p_4\})\}$. This allows for the interpretation that it is possible to expand the parameter assignment and consider the set $(\{c_1, c_2, c_3\}, \{p_1, p_2, p_3, p_4\})$ as parameters associated with e_3, thereby expanding the

possibilities for decision-making. Observe that we obtain new attributes associated with the selected parameter when the parameter in question is not a closed set. In the case where the parameter is a closed set, the assignment remains unchanged. This is the reason why the parameter e_3 was chosen to illustrate in this example.

6.1. **Adittional examples.** Now, in order to complement the theoretical concepts presented in Section 3, we will provide some examples obtained through the algorithm proposed in Section 6. Consider $X = \{c_1, c_2, c_3, c_4\}$, $Y = \{p_1, p_2, p_3, p_4\}$ and $\mathscr{M}$ defined as in the Table 1, then:

(1) $(A, B) = (\{c_3, c_4\}, \{p_4\})$ and $(C, D) = (\{c_1\}, \{p_4\})$ are binary t-sets but $(A \cup C, B \cup D) = (\{c_1, c_3, c_4\}, \{p_4\})$ isn't a binary t-set.

(2) $(A, B) = (\{c_1, c_2, c_4\}, \{p_2\})$ and $(C, D) = (\{c_3\}, \{p_1, p_4\})$ are binary α^*-sets but $(A \cup C, B \cup D) = (\{c_1, c_2, c_3, c_4\}, \{p_1, p_2, p_4\})$ isn't a binary α^*-set.

(3) $(A, B) = (\{c_1\}, \{p_4\})$ and $(C, D) = (\{c_1, c_2\}, \{p_3\})$ are binary semi open sets but $(A \cap C, B \cap D) = (\{c_1\}, \emptyset)$ isn't a binary semi open set.

6.2. **Benchmarking with other works.** It is important to note that the functions with which we have worked throughout this article, and their properties, differ subtly from the functions introduced in [10]. For example, in [10] functions $f : Z \to X \times Y$ are considered, whereas we consider functions $f : Z \to \mathscr{P}(X) \times \mathscr{P}(Y)$. In the work done by [10], they define $f^{-1}(A, B) = \{z \in Z : f(z) = (x, y) \in (A, B)\}$, and under these conditions, they prove that

$$Z \setminus f^{-1}(A, B) = f^{-1}(X \setminus A, B) \cup f^{-1}(A, Y \setminus B) \cup f^{-1}(X \setminus A, Y \setminus B)$$

However, in our context, this is generally not true, as shown in the following example.

Example. Consider $X = \{c_1, c_2, c_3, c_4\}$, $Y = \{p_1, p_2, p_3, p_4\}$, $E = \{e_1, e_2, e_3, e_4, e_5\}$ and the function $f : Z \to \mathscr{P}(X) \times \mathscr{P}(Y)$ as follow: $f(e_1) = (\{c_1\}, \{p_1\})$; $f(e_2) = (\{c_2, c_3, c_4\}, \{p_1\})$; $f(e_3) = (\{c_1\}, \{p_2, p_3, p_4\})$; $f(e_4) = (\emptyset, \{p_2, p_3, p_4\})$ and $f(e_5) = (\{c_2, c_3, c_4\}, \emptyset)$. Then, for $(A, B) = (\emptyset, \{p_2, p_3, p_4\})$ we have: $f^{-1}(A, B) = \{e_4\}$; $f^{-1}(X \setminus A, B) = \{e_3, e_4, e_5\}$; $f^{-1}(A, Y \setminus B) = \emptyset$ and $f^{-1}(X \setminus A, Y \setminus B) = \{e_1, e_2, e_5\}$.

So, $f^{-1}(X \setminus A, B) \cup f^{-1}(A, Y \setminus B) \cup f^{-1}(X \setminus A, Y \setminus B) = Z \neq Z \setminus f^{-1}(A, B) = Z \setminus \{e_4\}$

This small yet subtle and significant change in the range of the function is what allows us to establish a relationship between binary sets and soft sets.

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