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GENERALIZATIONS OF REVERSE MINKOWSKI'S TYPE INEQUALITIES VIA UNIFIED INTEGRAL OPERATOR

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Abstract. In this study, our main aim is to generalize the inverse Minkowski type inequality with the help of a unified integral operator containing the Mittag–Leffler function in its kernel. Some related inequalities are also discussed. The established results are valid for various types of fractional integral operators derived from unified integral operators.

1. Introduction

The interest in the theory of fractional calculus has increased in recent years due to the study of its numerous applications. The use of fractional calculus in applied sciences develops mathematical models related to the practical problems. On the other hand, inequalities involving integrals of functions and their derivatives are of great importance in mathematics, especially in mathematical analysis and its applications. This encourages researchers to explore extensions and generalizations of classical inequalities using different fractional integrals and derivative operators.

Our study of Mikowski type integral inequalities is motivated by recent studies of these inequalities using different types of integral operators (see [5, 6, 8, 13, 17–19]). By studying inverse Minkowski type inequalities for the unified integral operator from [7]), we continue our work given in [1–4].

This unified integral operator contains in its kernel the Mittag–Leffler function, a well-known function that is a natural extension of the exponential, trigonometric and hyperbolic functions. This function is named after the great Swedish mathematician Gösta Magnus Mittag–Leffler (1846–1927), who defined it by a power series.

Definition 1.1 ([10])**.** For every $\alpha \in \mathbb{C}$ such that $\Re(\alpha) > 0$, the function E_α is given by

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)} \quad (z \in \mathbb{C}).$$

In 1905 Wiman studied the generalization of Mittag–Leffler function and defined the function $E_{\alpha,\beta}$ as follows.

Definition 1.2 ([20])**.** For every $\alpha, \beta \in \mathbb{C}$ such that $\Re(\alpha) > 0$, the function $E_{\alpha,\beta}$ is given by

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)} \quad (z \in \mathbb{C}).$$

Prabhakar defined its function in 1971, i.e., the Mittag–Leffler function of three parameters, as follows.

Definition 1.3 ([11])**.** For every $\alpha, \beta, \rho \in \mathbb{C}$ such that $\Re(\alpha) > 0$, the function $E^{\rho}_{\alpha,\beta}$ is given by

$$E^{\rho}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\rho)_n z^n}{\Gamma(\alpha n + \beta) n!} \quad (z \in \mathbb{C}).$$

After that, numerous generalizations of the Mittag–Leffler function appeared, such as those defined in [12, 14, 16]. Recently, we presented the generalized Mittag–Leffler function $E^{\delta,c,v,r}_{\rho,\sigma,\tau}(z;u)$ as follows.

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Definition 1.4 ([1])**.** Let $\rho,\sigma,\tau,\delta,c\in\mathbb{C}$, $\Re(\rho),\Re(\sigma),\Re(\tau)>0$, $\Re(c)>\Re(\delta)>0$ with $u\geq 0$, $r>0$ and $0<v\leq r+\Re(\rho)$. Then $E^{\delta,c,v,r}_{\rho,\sigma,\tau}(z;u)$ is defined by

$$E^{\delta,c,v,r}_{\rho,\sigma,\tau}(z;u) \;=\; \sum_{n=0}^{\infty}\frac{\mathrm{B}_u(\delta+nv,c-\delta)}{\mathrm{B}(\delta,c-\delta)}\frac{(c)_{nv}}{\Gamma(\rho n+\sigma)}\frac{z^n}{(\tau)_{nr}}, \tag{1.1}$$

where $(c)_{nv}$ denotes the generalized Pochhammer symbol $(c)_{nv}=\frac{\Gamma(c+nv)}{\Gamma(c)}$ and B_u is an extension of the beta function

$$\mathrm{B}_u(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}e^{-\frac{u}{t(1-t)}}\,dt \quad (\Re(x),\Re(y),\Re(u)>0)\,.$$

As proved in [1], the series (1.1) converges absolutely for all values of z provided that $v<r+\Re(\rho)$. Moreover, if $v=r+\Re(\rho)$, then $E^{\delta,c,v,r}_{\rho,\sigma,\tau}(z;u)$ converges for $|z|<\frac{r^r\,\Re(\rho)^{\Re(\rho)}}{v^v}$.

Next, we defined the unified integral operator containing $E^{\delta,c,v,r}_{\rho,\sigma,\tau}(z;u)$ in its kernel.

Definition 1.5 ([7])**.** Let $\omega,\sigma,\tau,\delta,c\in\mathbb{C}$, $\Re(\sigma),\Re(\tau)>0$, $\Re(c)>\Re(\delta)>0$ with $u\geq 0$, $\rho,r>0$ and $0<v\leq r+\rho$. Let $f\in L_1[a,b]$, $0<a<b<\infty$, be a positive function. Let $h:[a,b]\to\mathbb{R}$ be a differentiable function, strictly increasing. Also, let $\frac{\phi}{x}$ be an increasing function on $[a,\infty)$ and $x\in[a,b]$. Then the unifed integral operator is defined by

$$({}^{\phi}_{h}F^{\omega,\delta,c,v,r}_{a^+,\rho,\sigma,\tau}f)(x;u)=\int_a^x\frac{\phi(h(x)-h(t))}{h(x)-h(t)}E^{\delta,c,v,r}_{\rho,\sigma,\tau}(\omega(h(x)-h(t))^{\rho};u)h'(t)f(t)dt. \tag{1.2}$$

The following remark provides connection of Definition 1.5 with some of the already known fractional integral operators.

Remark 1.1. (i) If we set $\phi(x)=x^{\sigma}$, $u=0$ and $h(x)=x$ in equation (1.2), then it reduces to the fractional integral operator defined by Salim and Faraj in [14].

(ii) If we set $\phi(x)=x^{\sigma}$, $\delta=r=1$ and $h(x)=x$, then (1.2) reduces to the fractional integral operator defined by Rahman et al. in [12].

(iii) If we set $\phi(x)=x^{\sigma}$, $u=0$, $\delta=r=1$ and $h(x)=x$, then (1.2) reduces to the fractional integral operator introduced by Srivastava and Tomovski in [16].

(iv) If we set $\phi(x)=x^{\sigma}$, $u=0$, $\delta=r=k=1$ and $h(x)=x$, then (1.2) reduces to the fractional integral operator defined by Prabhaker in [11].

(v) For $\phi(x)=x^{\sigma}$, $u=\omega=0$ and $h(x)=x$, (1.2) reduces to the left-sided Riemann–Liouville fractional integral operator.

Motivated by papers [5,15], where the authors have proved certain reverse Minkowski type integral inequalities, we present the corresponding generalized results using a unified integral operator in Section 2. We start with a condition where a quotient function is bounded by positive functions and continue with the bounds that are positive real numbers. In Section 3 we give some related fractional Minkowski type integral inequalities.

For the convenience of readers, we use simplified notations

$$\boldsymbol{E}(z;u):=E^{\delta,c,v,r}_{\rho,\sigma,\tau}(z;u),$$

$${}^{\boldsymbol{\phi}}_{\boldsymbol{h}}\boldsymbol{F}(x;u):=({}^{\phi}_{h}F^{\omega,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f)(x;u).$$

2. Inverse Fractional Minkowski Type Inequalities with a Unified Integral Operator

In this section we present inverse fractional Minkowski type integral inequalities using (1.1) (*the extended generalized Mittag–Leffler function*) along with the corresponding fractional integral operator (1.2) (*the unified integral operator*) in a real domain. In the first two theorems, the quotient function is bounded by positive functions, after which we use positive real numbers as bounds.

Theorem 2.1. *Let $w \in \mathbb{R}$, $\rho, \sigma, \tau, r > 0$, $c > \delta > 0$ with $u \geq 0$ and $0 < v \leq r+\rho$. Let $h : [a,b] \to \mathbb{R}$ be a differentiable, strictly increasing function, and let $f, g, \Psi_1, \Psi_2 \in L_p[a,b]$ be positive functions satisfying*

$$0 < \Psi_1(x) \leq \frac{f(x)}{g(x)} \leq \Psi_2(x), \quad x \in [a,b]. \tag{2.1}$$

Also, let $\frac{\phi}{x}$ be an increasing function on $[a,\infty)$. Then for $p \geq 1$, the following inequality:

$$\begin{aligned}
&\left[({}_{\boldsymbol{h}}^{\boldsymbol{\phi}}\boldsymbol{F} f^p)(x;u)\right]^{\frac{1}{p}} + \left[({}_{\boldsymbol{h}}^{\boldsymbol{\phi}}\boldsymbol{F} g^p)(x;u)\right]^{\frac{1}{p}} \\
&\leq \left[\left({}_{\boldsymbol{h}}^{\boldsymbol{\phi}}\boldsymbol{F}\left(\frac{\Psi_2}{1+\Psi_2}\right)^p (f+g)^p\right)(x;u)\right]^{\frac{1}{p}} \\
&+ \left[\left({}_{\boldsymbol{h}}^{\boldsymbol{\phi}}\boldsymbol{F}\left(\frac{1}{1+\Psi_1}\right)^p (f+g)^p\right)(x;u)\right]^{\frac{1}{p}}
\end{aligned} \tag{2.2}$$

holds.

Proof. From the hypothesis that $\frac{f(t)}{g(t)} \leq \Psi_2(t)$, we have

$$f(t) \leq \Psi_2(t)[f(t)+g(t)] - \Psi_2(t)f(t), \quad t \in [a,b],$$

from which follows the inequality

$$f(t)^p \leq \left(\frac{\Psi_2(t)}{1+\Psi_2(t)}\right)^p [f(t)+g(t)]^p, \quad p \geq 1,\ t \in [a,b]. \tag{2.3}$$

Multiplying both sides of the above inequality by

$$\frac{\phi(h(x)-h(t))}{h(x)-h(t)}\boldsymbol{E}(w(h(x)-h(t))^\rho; u)h'(t),$$

and integrating on the interval $[a,x]$, we get

$$\begin{aligned}
&\int_a^x \frac{\phi(h(x)-h(t))}{h(x)-h(t)}\boldsymbol{E}(w(h(x)-h(t))^\rho; u) f(t)^p dt \\
&\leq \int_a^x \frac{\phi(h(x)-h(t))}{h(x)-h(t)}\boldsymbol{E}(w(h(x)-h(t))^\rho; u)\left(\frac{\Psi_2(t)}{1+\Psi_2(t)}\right)^p [f(t)+g(t)]^p dt,
\end{aligned}$$

which implies

$$({}_{\boldsymbol{h}}^{\boldsymbol{\phi}}\boldsymbol{F} f^p)(x;u) \leq \left({}_{\boldsymbol{h}}^{\boldsymbol{\phi}}\boldsymbol{F}\left(\frac{\Psi_2}{1+\Psi_2}\right)^p (f+g)^p\right)(x;u). \tag{2.4}$$

Further, for the lower bound $\frac{f(t)}{g(t)} \geq \Psi_1(t)$, we have

$$g(t) \leq \frac{1}{\Psi_1(t)}[f(t)+g(t)] - \frac{1}{\Psi_1(t)}g(t), \quad t \in [a,b],$$

and

$$g(t)^p \leq \left(\frac{1}{1+\Psi_1(t)}\right)^p [f(t)+g(t)]^p, \quad p \geq 1,\ t \in [a,b]. \tag{2.5}$$

Similarly, if we multiply the above inequality by

$$\frac{\phi(h(x)-h(t))}{h(x)-h(t)}\boldsymbol{E}(w(h(x)-h(t))^\rho; u)h'(t)$$

and integrate on $[a,x]$, we get

$$\int_a^x \frac{\phi(h(x)-h(t))}{h(x)-h(t)} \boldsymbol{E}(w(h(x)-h(t))^\rho;u)h'(t)g(t)^p dt$$
$$\leq \int_a^x \frac{\phi(h(x)-h(t))}{h(x)-h(t)} \boldsymbol{E}(w(h(x)-h(t))^\rho;u)h'(t)\left(\frac{1}{1+\Psi_1(t)}\right)^p [f(t)+g(t)]^p\, dt,$$

from which we obtain

$$\left[\left({}_{\boldsymbol{h}}^{\boldsymbol{\phi}}\boldsymbol{F} g^p\right)(x;u)\right]^{\frac{1}{p}} \leq \left[\left({}_{\boldsymbol{h}}^{\boldsymbol{\phi}}\boldsymbol{F}\left(\frac{1}{1+\Psi_1}\right)^p (f+g)^p\right)(x;u)\right]^{\frac{1}{p}}. \tag{2.6}$$

The resulting inequality (2.2) now follows by adding (2.4) and (2.6). □

Theorem 2.2. *Suppose the assumptions of Theorem* 2.1 *hold. Then*

$$\left[\left({}_{\boldsymbol{h}}^{\boldsymbol{\phi}}\boldsymbol{F}\left(\frac{1+\Psi_2}{\Psi_2}\right)^p f^p\right)(x;u)\right]^{\frac{1}{p}} \left[\left({}_{\boldsymbol{h}}^{\boldsymbol{\phi}}\boldsymbol{F}\,(1+\Psi_1)^p g^p\right)(x;u)\right]^{\frac{1}{p}}$$
$$-2\left[\left({}_{\boldsymbol{h}}^{\boldsymbol{\phi}}\boldsymbol{F} f^p\right)(x;u)\right]^{\frac{1}{p}}\left[\left({}_hF^{\phi}_{\rho,\alpha,l,\omega,a+}, r,\delta,k, cg^p\right)(x;u)\right]^{\frac{1}{p}}$$
$$\leq \left[\left({}_{\boldsymbol{h}}^{\boldsymbol{\phi}}\boldsymbol{F} f^p\right)(x;u)\right]^{\frac{2}{p}} + \left[\left({}_{\boldsymbol{h}}^{\boldsymbol{\phi}}\boldsymbol{F} g^p\right)(x;u)\right]^{\frac{2}{p}}. \tag{2.7}$$

Proof. From (2.3), we have

$$\left(\frac{1+\Psi_2(t)}{\Psi_2(t)}\right)^p f(t)^p \leq [f(t)+g(t)]^p, \quad p\geq 1,\ t\in[a,b]. \tag{2.8}$$

Multiplying both sides of the above inequality by

$$\frac{\phi(h(x)-h(t))}{h(x)-h(t)} \boldsymbol{E}(w(h(x)-h(t))^\rho;u)h'(t)$$

and integrating on the interval $[a,x]$, we get

$$\int_a^x \frac{\phi(h(x)-h(t))}{h(x)-h(t)} \boldsymbol{E}(w(h(x)-h(t))^\rho;u)h'(t)\,(1+\Psi_2(t))^p f(t)^p dt$$
$$\leq \int_a^x \frac{\phi(h(x)-h(t))}{h(x)-h(t)} \boldsymbol{E}(w(h(x)-h(t))^\rho;u)h'(t)\,[f(t)+g(t)]^p\, dt,$$

from which we obtain

$$\left[\left({}_{\boldsymbol{h}}^{\boldsymbol{\phi}}\boldsymbol{F}\left(\frac{1+\Psi_2}{\Psi_2}\right)^p f^p\right)(x;u)\right]^{\frac{1}{p}} \leq \left[\left({}_{\boldsymbol{h}}^{\boldsymbol{\phi}}\boldsymbol{F}(f+g)^p\right)(x;u)\right]^{\frac{1}{p}}. \tag{2.9}$$

Similarly, from (2.5), we have

$$(1+\Psi_1(t))^p g(t)^p \leq [f(t)+g(t)]^p, \quad p\geq 1,\ t\in[a,b].$$

Multiplying both sides of the above inequality by

$$\frac{\phi(h(x)-h(t))}{h(x)-h(t)} \boldsymbol{E}(w(h(x)-h(t))^\rho;u)h'(t)$$

and integrating on the interval $[a,x]$, we get

$$\begin{aligned}&\int_a^x \frac{\phi(h(x)-h(t))}{h(x)-h(t)}\boldsymbol{E}(w(h(x)-h(t))^{\rho};u)h'(t)\left(1+\Psi_1(t)\right)^p g(t)^p dt\\ &\leq \int_a^x \frac{\phi(h(x)-h(t))}{h(x)-h(t)}\boldsymbol{E}(w(h(x)-h(t))^{\rho};u)h'(t)\left[f(t)+g(t)\right]^p dt,\end{aligned}$$

which implies

$$\left[\left({}_{\boldsymbol{h}}^{\boldsymbol{\phi}}\boldsymbol{F}\left(1+\Psi_1\right)^p g^p\right)(x;u)\right]^{\frac{1}{p}} \leq \left[\left({}_{\boldsymbol{h}}^{\boldsymbol{\phi}}\boldsymbol{F}(f+g)^p\right)(x;u)\right]^{\frac{1}{p}}. \tag{2.10}$$

Now, taking the product of inequalities (2.9) and (2.10), we obtain

$$\begin{aligned}&\left[\left({}_{\boldsymbol{h}}^{\boldsymbol{\phi}}\boldsymbol{F}\left(\frac{1+\Psi_2}{\Psi_2}\right)^p f^p\right)(x;u)\right]^{\frac{1}{p}}\left[\left({}_{\boldsymbol{h}}^{\boldsymbol{\phi}}\boldsymbol{F}\left(1+\Psi_1\right)^p g^p\right)(x;u)\right]^{\frac{1}{p}}\\ &\leq \left[\left({}_{\boldsymbol{h}}^{\boldsymbol{\phi}}\boldsymbol{F}(f+g)^p\right)(x;u)\right]^{\frac{2}{p}}.\end{aligned}$$

On the right-hand side, by applying Minkowski's inequality, we obtain

$$\begin{aligned}&\left[\left({}_{\boldsymbol{h}}^{\boldsymbol{\phi}}\boldsymbol{F}\left(\frac{1+\Psi_2}{\Psi_2}\right)^p f^p\right)(x;u)\right]^{\frac{1}{p}}\left[\left({}_{\boldsymbol{h}}^{\boldsymbol{\phi}}\boldsymbol{F}\left(1+\Psi_1\right)^p g^p\right)(x;u)\right]^{\frac{1}{p}}\\ &\leq \left[\left(\left({}_{\boldsymbol{h}}^{\boldsymbol{\phi}}\boldsymbol{F}f^p\right)(x;u)\right)^{\frac{1}{p}}+\left(\left({}_{\boldsymbol{h}}^{\boldsymbol{\phi}}\boldsymbol{F}g^p\right)(x;u)\right)^{\frac{1}{p}}\right]^2,\end{aligned}$$

from which we can easily obtain inequality (2.7). □

Remark 2.1. If we set $\phi(x)=x^{\sigma}$ in (1.2), then we obtain a generalized fractional operator ${}_h\Upsilon^{w,\delta,c,v,r}_{a^+,\rho,\sigma,\tau}f$ from [9]:

$$\left({}_h\Upsilon^{w,\delta,c,v,r}_{a^+,\rho,\sigma,\tau}f\right)(x;u)=\int_a^x (h(x)-h(t))^{\sigma-1}E^{\delta,c,v,r}_{\rho,\sigma,\tau}(w(h(x)-h(t))^{\rho};u)h'(t)f(t)dt. \tag{2.11}$$

Applying this in Theorem 2.1, we get the inequality given in [3, Theorem 3]. Analogously, Theorem 2.2 is a generalization of [3, Theorem 4].

If we take $\Psi_1(x)$ and $\Psi_2(x)$ as constant functions in (2.1), i.e., $\Psi_1(x)=m$ and $\Psi_2(x)=M$ for all $x\in[a,b]$, then we obtain the following result.

Theorem 2.3. *Let $w\in\mathbb{R}$, $\rho,\sigma,\tau,r>0$, $c>\delta>0$ with $u\geq 0$ and $0<v\leq r+\rho$. Let $h:[a,b]\to\mathbb{R}$ be a differentiable, strictly increasing function, and let $f,g\in L_p[a,b]$ be positive functions satisfying*

$$0<m\leq\frac{f(x)}{g(x)}\leq M,\quad x\in[a,b]. \tag{2.12}$$

Also, let $\frac{\phi}{x}$ be an increasing function on $[a,\infty)$. Then for $p\geq 1$, the following inequality:

$$\left[\left({}_{\boldsymbol{h}}^{\boldsymbol{\phi}}\boldsymbol{F}f^p\right)(x;u)\right]^{\frac{1}{p}}+\left[\left({}_{\boldsymbol{h}}^{\boldsymbol{\phi}}\boldsymbol{F}g^p\right)(x;u)\right]^{\frac{1}{p}}\leq c_1\left[\left({}_{\boldsymbol{h}}^{\boldsymbol{\phi}}\boldsymbol{F}(f+g)^p\right)(x;u)\right]^{\frac{1}{p}} \tag{2.13}$$

holds, where

$$c_1=\frac{M(m+2)+1}{(m+1)(M+1)}. \tag{2.14}$$

Theorem 2.4. *Under the assumptions of Theorem* 2.3*, we have*

$$\left[\left({}_{\boldsymbol{h}}^{\boldsymbol{\phi}}\boldsymbol{F}f^p\right)(x;u)\right]^{\frac{2}{p}}+\left[\left({}_{\boldsymbol{h}}^{\boldsymbol{\phi}}\boldsymbol{F}g^p\right)(x;u)\right]^{\frac{2}{p}}$$
$$\geq c_2\left[\left({}_{\boldsymbol{h}}^{\boldsymbol{\phi}}\boldsymbol{F}f^p\right)(x;u)\right]^{\frac{1}{p}}\left[\left({}_{\boldsymbol{h}}^{\boldsymbol{\phi}}\boldsymbol{F}g^p\right)(x;u)\right]^{\frac{1}{p}}, \tag{2.15}$$

where

$$c_2=\frac{m(M+1)-(M-1)}{M}. \tag{2.16}$$

Remark 2.2. If we set $h(x)=x$ and $\phi(x)=x^\sigma$ in (1.2), then we obtain a generalized fractional operator $\varepsilon_{a^+,\rho,\sigma,\tau}^{w,\delta,c,v,r}f$ from [1]:

$$\left(\varepsilon_{a^+,\rho,\sigma,\tau}^{w,\delta,c,v,r}f\right)(x;u)=\int_a^x(x-t)^{\sigma-1}E_{\rho,\sigma,\tau}^{\delta,c,v,r}(w(x-t)^\rho;u)f(t)dt. \tag{2.17}$$

Using this in Theorem 2.3 and Theorem 2.4, we generalize [2, Theorem 2.1] and [2, Theorem 2.2], respectively.

3. Some Related Unified Integral Operator Inequalities

Continuing with generalizations, we present several unified integral operator inequalities satisfying conditions of Theorem 2.3.

Theorem 3.1. *Suppose that the assumptions of Theorem* 2.3 *hold. Let* $p,q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$*. Then*

$$\left[\left({}_{\boldsymbol{h}}^{\boldsymbol{\phi}}\boldsymbol{F}f\right)(x;u)\right]^{\frac{1}{p}}\left[\left({}_{\boldsymbol{h}}^{\boldsymbol{\phi}}\boldsymbol{F}g\right)(x;u)\right]^{\frac{1}{q}}$$
$$\leq\left(\frac{M}{m}\right)^{\frac{1}{pq}}\left({}_{\boldsymbol{h}}^{\boldsymbol{\phi}}\boldsymbol{F}\left(f^{\frac{1}{p}}g^{\frac{1}{q}}\right)\right)(x;u). \tag{3.1}$$

Proof. Under the condition $\frac{f(t)}{g(t)}\leq M$, we obtain

$$f(t)^{\frac{1}{q}}\leq M^{\frac{1}{q}}g(t)^{\frac{1}{q}},$$

and after multiplication by $f^{\frac{1}{p}}(t)$, we get

$$f(t)\leq M^{\frac{1}{q}}f(t)^{\frac{1}{p}}g(t)^{\frac{1}{q}},\quad t\in[a,b].$$

Multiplying both sides of the above inequality by

$$\frac{\phi(h(x)-h(t))}{h(x)-h(t)}\boldsymbol{E}(w(h(x)-h(t))^\rho;u)h'(t)$$

and integrating on the interval $[a,x]$, we get

$$\int_a^x\frac{\phi(h(x)-h(t))}{h(x)-h(t)}\boldsymbol{E}(w(h(x)-h(t))^\rho;u)h'(t)f(t)^{\frac{1}{q}}dt$$
$$\leq M^{\frac{1}{q}}\int_a^x\frac{\phi(h(x)-h(t))}{h(x)-h(t)}\boldsymbol{E}(w(h(x)-h(t))^\rho;u)h'(t)g(t)^{\frac{1}{q}}dt,$$

from which we get

$$\left({}_{\boldsymbol{h}}^{\boldsymbol{\phi}}\boldsymbol{F}f\right)(x;u)\leq M^{\frac{1}{q}}\left({}_hF_{\rho,\alpha,l,\omega,a+}^{\phi.r,\delta,k,c}\left(f^{\frac{1}{p}}g^{\frac{1}{q}}\right)\right)(x;u),$$

and taking power $\frac{1}{p}$ on both sides, we get

$$\left[\left({}_{\boldsymbol{h}}^{\boldsymbol{\phi}}\boldsymbol{F}f\right)(x;u)\right]^{\frac{1}{p}}\leq M^{\frac{1}{pq}}\left[\left({}_{\boldsymbol{h}}^{\boldsymbol{\phi}}\boldsymbol{F}\left(f^{\frac{1}{p}}g^{\frac{1}{q}}\right)\right)(x;u)\right]^{\frac{1}{p}}. \tag{3.2}$$

Next, from the lower bound $m \leq \frac{f(t)}{g(t)}$, we have

$$g(t) \leq \frac{1}{m^{\frac{1}{p}}} f^{\frac{1}{p}}(t) g^{\frac{1}{q}}(t).$$

Multiplying the above inequality by

$$\frac{\phi(h(x)-h(t))}{h(x)-h(t)} \boldsymbol{E}(w(h(x)-h(t))^{\rho}; u) h'(t)$$

and integrating on the interval $[a, x]$, we get

$$\begin{aligned} &\int_a^x \frac{\phi(h(x)-h(t))}{h(x)-h(t)} \boldsymbol{E}(w(h(x)-h(t))^{\rho}; u) h'(t) g(t) dt \\ &\leq \frac{1}{m^{\frac{1}{p}}} \int_a^x \frac{\phi(h(x)-h(t))}{h(x)-h(t)} \boldsymbol{E}(w(h(x)-h(t))^{\rho}; u) h'(t) f^{\frac{1}{p}}(t) g^{\frac{1}{q}}(t) dt, \end{aligned}$$

which implies

$$\left({}^{\boldsymbol{\phi}}_{\boldsymbol{h}}\boldsymbol{F} g\right)(x; u) \leq \frac{1}{m^{\frac{1}{p}}} \left({}^{\boldsymbol{\phi}}_{\boldsymbol{h}}\boldsymbol{F} \left(f^{\frac{1}{p}} g^{\frac{1}{q}}\right)\right)(x; u),$$

and taking power $\frac{1}{q}$ on both sides, we get

$$\left[\left({}^{\boldsymbol{\phi}}_{\boldsymbol{h}}\boldsymbol{F} g\right)(x; u)\right]^{\frac{1}{q}} \leq \frac{1}{m^{\frac{1}{pq}}} \left[\left({}^{\boldsymbol{\phi}}_{\boldsymbol{h}}\boldsymbol{F} \left(f^{\frac{1}{p}} g^{\frac{1}{q}}\right)\right)(x; u)\right]^{\frac{1}{q}}. \tag{3.3}$$

One can get (3.1) by multiplying inequality (3.2) by (3.3). □

Next, we need two following inequalities for $x, y \geq 0$. The first is

$$(x+y)^p \leq 2^{p-1}(x^p + y^p), \quad p > 1 \tag{3.4}$$

and the second is the Young inequality for the products

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q}, \quad p, q > 1,\ p^{-1} + q^{-1} = 1. \tag{3.5}$$

Theorem 3.2. *Suppose that the assumptions of Theorem* 2.3 *hold. Let* $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$\begin{aligned} \left({}^{\boldsymbol{\phi}}_{\boldsymbol{h}}\boldsymbol{F}(fg)\right)(x; u) \leq &\frac{2^{p-1}}{p} \left(\frac{M}{M+1}\right)^p \left({}^{\boldsymbol{\phi}}_{\boldsymbol{h}}\boldsymbol{F}\left(f^p + g^p\right)\right)(x; u) \\ &+ \frac{2^{q-1}}{q} \left(\frac{1}{m+1}\right)^q \left({}^{\boldsymbol{\phi}}_{\boldsymbol{h}}\boldsymbol{F}\left(f^q + g^q\right)\right)(x; u). \end{aligned} \tag{3.6}$$

Proof. From inequalities (2.4) and (2.6), with $\Psi_1(x) = m$ and $\Psi_2(x) = M$, we have

$$\frac{1}{p}\left({}^{\boldsymbol{\phi}}_{\boldsymbol{h}}\boldsymbol{F} f^p\right)(x; u) \leq \frac{M^p}{p(M+1)^p} \left({}^{\boldsymbol{\phi}}_{\boldsymbol{h}}\boldsymbol{F}(f+g)^p\right)(x; u) \tag{3.7}$$

and

$$\frac{1}{q}\left({}^{\boldsymbol{\phi}}_{\boldsymbol{h}}\boldsymbol{F} g^q\right)(x; u) \leq \frac{1}{q(m+1)^q} \left({}^{\boldsymbol{\phi}}_{\boldsymbol{h}}\boldsymbol{F}(f+g)^q\right)(x; u). \tag{3.8}$$

Using Young's inequality (3.5), we have

$$f(t)g(t) \leq \frac{f(t)^p}{p} + \frac{g(t)^q}{q}.$$

Multiplying both sides of the above inequality by

$$\frac{\phi(h(x)-h(t))}{h(x)-h(t)} \boldsymbol{E}(w(h(x)-h(t))^{\rho}; u) h'(t)$$

and integrating on the interval $[a,x]$, we get

$$\begin{aligned}&\int_a^x \frac{\phi(h(x)-h(t))}{h(x)-h(t)}\boldsymbol{E}(w(h(x)-h(t))^\rho;u)h'(t)f(t)g(t)dt\\ &\leq \frac{1}{p}\int_a^x \frac{\phi(h(x)-h(t))}{h(x)-h(t)}\boldsymbol{E}(w(h(x)-h(t))^\rho;u)h'(t)f(t)^p dt\\ &+\frac{1}{q}\int_a^x \frac{\phi(h(x)-h(t))}{h(x)-h(t)}\boldsymbol{E}(w(h(x)-h(t))^\rho;u)h'(t)g(t)^q dt,\end{aligned}$$

from which we have

$$\left({}_{\boldsymbol{h}}^{\boldsymbol{\phi}}\boldsymbol{F}(fg)\right)(x;u) \leq \frac{1}{p}\left({}_{\boldsymbol{h}}^{\boldsymbol{\phi}}\boldsymbol{F}f^p\right)(x;u)+\frac{1}{q}\left({}_{\boldsymbol{h}}^{\boldsymbol{\phi}}\boldsymbol{F}g^q\right)(x;u). \tag{3.9}$$

From (3.7), (3.8) and (3.9), we obtain

$$\left({}_{\boldsymbol{h}}^{\boldsymbol{\phi}}\boldsymbol{F}(fg)\right)(x;u) \leq \frac{M^p}{p(M+1)^p}\left({}_{\boldsymbol{h}}^{\boldsymbol{\phi}}\boldsymbol{F}(f+g)^p\right)(x;u)+\frac{1}{q(m+1)^q}\left({}_{\boldsymbol{h}}^{\boldsymbol{\phi}}\boldsymbol{F}(f+g)^q\right)(x;u). \tag{3.10}$$

Using elementary inequality (3.4), we obtain

$$\left({}_{\boldsymbol{h}}^{\boldsymbol{\phi}}\boldsymbol{F}\,(f+g)^p\right)(x;u) \leq 2^{p-1}\left({}_{\boldsymbol{h}}^{\boldsymbol{\phi}}\boldsymbol{F}\,(f^p+g^p)\right)(x;u) \tag{3.11}$$

and

$$\left({}_{\boldsymbol{h}}^{\boldsymbol{\phi}}\boldsymbol{F}(f+g)^q\right)(x;u) \leq 2^{q-1}\left({}_{\boldsymbol{h}}^{\boldsymbol{\phi}}\boldsymbol{F}\,(f^q+g^q)\right)(x;u). \tag{3.12}$$

Hence, from (3.10), (3.11) and (3.12), we get (3.6). □

Remark 3.1. If in the obtained results we use the power function $\phi(x)=x^\sigma$, then we obtain inequalities involving generalized fractional operator ${}_h\Upsilon^{w,\delta,c,v,r}_{a^+,\rho,\sigma,\tau}f$ as in (2.11), thus generalizing the corresponding results from [3].

Theorem 3.3. *Suppose that the assumptions of Theorem 2.3 hold. Then*

$$\frac{1}{M}\left({}_{\boldsymbol{h}}^{\boldsymbol{\phi}}\boldsymbol{F}(fg)\right)(x;u) \leq \frac{1}{(m+1)(M+1)}\left({}_{\boldsymbol{h}}^{\boldsymbol{\phi}}\boldsymbol{F}(f+g)^2\right)(x;u) \leq \frac{1}{m}\left({}_{\boldsymbol{h}}^{\boldsymbol{\phi}}\boldsymbol{F}(fg)\right)(x;u). \tag{3.13}$$

Proof. Since the quotient function $\frac{f(t)}{g(t)}$ is bounded, that is,

$$m \leq \frac{f(t)}{g(t)} \leq M,$$

which implies

$$\frac{1}{M} \leq \frac{g(t)}{f(t)} \leq \frac{1}{m},$$

from the above inequalities, we obtain

$$(m+1)g(t) \leq f(t)+g(t) \leq (M+1)g(t) \tag{3.14}$$

and

$$\left(\frac{M+1}{M}\right)f(t) \leq f(t)+g(t) \leq \left(\frac{m+1}{m}\right)f(t). \tag{3.15}$$

Multiplying inequalities (3.14) and (3.15) we get

$$\frac{1}{M}f(t)g(t) \leq \frac{(f(t)+g(t))^2}{(M+1)(m+1)} \leq \frac{1}{m}f(t)g(t).$$

Further, multiplying the above inequalities by

$$\frac{\phi(h(x)-h(t))}{h(x)-h(t)}\boldsymbol{E}(w(h(x)-h(t))^{\rho};u)h'(t)$$

and integrating on the interval $[a,x]$, we get

$$\begin{aligned}&\frac{1}{M}\int\limits_a^x \frac{\phi(h(x)-h(t))}{h(x)-h(t)}\boldsymbol{E}(w(h(x)-h(t))^{\rho};u)h'(t)f(t)g(t)dt\\ &\leq \frac{1}{(M+1)(m+1)}\int\limits_a^x \frac{\phi(h(x)-h(t))}{h(x)-h(t)}\boldsymbol{E}(w(h(x)-h(t))^{\rho};u)h'(t)\left(f(t)+g(t)\right)^2 dt\\ &\leq \frac{1}{M}\int\limits_a^x \frac{\phi(h(x)-h(t))}{h(x)-h(t)}\boldsymbol{E}(w(h(x)-h(t))^{\rho};u)h'(t)f(t)g(t)dt,\end{aligned}$$

which implies inequalities (3.13). □

In the next theorem, we add a positive parameter ϑ such that $\vartheta < m$, i.e.,

$$0<\vartheta<m\leq \frac{f(x)}{g(x)}\leq M, \quad x\in[a,b].$$

Theorem 3.4. *Suppose that the assumptions of Theorem* 2.3 *hold. Let* $\vartheta > 0$ *be such that* $\vartheta < m$ *in* (2.12)*. Then*

$$\begin{aligned}&\frac{M+1}{M-\vartheta}\left({}^{\boldsymbol{\phi}}_{\boldsymbol{h}}\boldsymbol{F}(f-\vartheta g)^{\frac{1}{p}}\right)(x;u)\\ &\leq \left[\left({}^{\boldsymbol{\phi}}_{\boldsymbol{h}}\boldsymbol{F}f^p\right)(x;u)\right]^{\frac{1}{p}}+\left[\left({}^{\boldsymbol{\phi}}_{\boldsymbol{h}}\boldsymbol{F}g^p\right)(x;u)\right]^{\frac{1}{p}}\\ &\leq \frac{m+1}{m-\vartheta}\left({}^{\boldsymbol{\phi}}_{\boldsymbol{h}}\boldsymbol{F}(f-\vartheta g)^{\frac{1}{p}}\right)(x;u).\end{aligned} \tag{3.16}$$

Proof. From the given condition $0<\vartheta<m\leq M$, we have

$$m\vartheta\leq M\vartheta \Rightarrow m\vartheta+m\leq M\vartheta+M,$$

that is,

$$m-M\vartheta\leq M-m\vartheta,$$

from which we have

$$\frac{M+1}{M-\vartheta}\leq\frac{m+1}{m-\vartheta}.$$

Also, from the given condition, we have

$$m-\vartheta\leq\frac{f(t)-\vartheta g(t)}{g(t)}\leq M-\vartheta$$

which implies

$$\frac{[f(t)-\vartheta g(t)]^p}{(M-\vartheta)^p}\leq g^p(t)\leq\frac{[f(t)-\vartheta g(t)]^p}{(m-\vartheta)^p}.$$

Multiplying the above inequalities by

$$\frac{\phi(h(x)-h(t))}{h(x)-h(t)}\boldsymbol{E}(w(h(x)-h(t))^{\rho};u)h'(t)$$

and integrating on the interval $[a,x]$, we get

$$\begin{aligned}&\frac{1}{(M-\vartheta)^p}\int_a^x \frac{\phi(h(x)-h(t))}{h(x)-h(t)}\boldsymbol{E}(w(h(x)-h(t))^\rho;u)h'(t)\left[f(t)-\vartheta g(t)\right]^p dt\\&\leq \int_a^x \frac{\phi(h(x)-h(t))}{h(x)-h(t)}\boldsymbol{E}(w(h(x)-h(t))^\rho;u)h'(t)g^p(t)dt\\&\leq \frac{1}{(m-\vartheta)^p}\int_a^x \frac{\phi(h(x)-h(t))}{h(x)-h(t)}\boldsymbol{E}(w(h(x)-h(t))^\rho;u)h'(t)\left[f(t)-\vartheta g(t)\right]^p dt,\end{aligned}$$

which implies

$$\begin{aligned}\frac{1}{(M-\vartheta)^p}\left({}_{\boldsymbol{h}}^{\boldsymbol{\phi}}\boldsymbol{F}(f-\vartheta g)^p\right)(x;u) &\leq \left({}_{\boldsymbol{h}}^{\boldsymbol{\phi}}\boldsymbol{F}g^p\right)(x;u)\\&\leq \frac{1}{(m-\vartheta)^p}\left({}_{\boldsymbol{h}}^{\boldsymbol{\phi}}\boldsymbol{F}(f-\vartheta g)^p\right)(x;u).\end{aligned}$$

Taking power $\frac{1}{p}$ on both sides, we get

$$\begin{aligned}\frac{1}{M-\vartheta}\left[\left({}_{\boldsymbol{h}}^{\boldsymbol{\phi}}\boldsymbol{F}(f-\vartheta g)^p\right)(x;u)\right]^{\frac{1}{p}} &\leq \left[\left({}_{\boldsymbol{h}}^{\boldsymbol{\phi}}\boldsymbol{F}g^p\right)(x;u)\right]^{\frac{1}{p}}\\&\leq \frac{1}{m-\vartheta}\left[\left({}_{\boldsymbol{h}}^{\boldsymbol{\phi}}\boldsymbol{F}(f-\vartheta g)^p\right)(x;u)\right]^{\frac{1}{p}}. \qquad (3.17)\end{aligned}$$

Further, from $\frac{1}{M}\leq\frac{g(t)}{f(t)}\leq\frac{1}{m}$, we obtain

$$\frac{m-\vartheta}{m}\leq\frac{f(t)-\vartheta g(t)}{f(t)}\leq\frac{M-\vartheta}{M},$$

from which we have

$$\frac{M^p\left[f(t)-\vartheta g(t)\right]^p}{(M-\vartheta)^p}\leq f^p(t)\leq\frac{m^p\left[f(t)-\vartheta g(t)\right]^p}{(m-\vartheta)^p}.$$

Again, multiplying the above inequalities by

$$\frac{\phi(h(x)-h(t))}{h(x)-h(t)}\boldsymbol{E}(w(h(x)-h(t))^\rho;u)h'(t)$$

and integrating over $[a,x]$, we obtain

$$\begin{aligned}&\frac{M^p}{(M-\vartheta)^p}\int_a^x \frac{\phi(h(x)-h(t))}{h(x)-h(t)}\boldsymbol{E}(w(h(x)-h(t))^\rho;u)h'(t)\left[f(t)-\vartheta g(t)\right]^p dt\\&\leq \int_a^x \frac{\phi(h(x)-h(t))}{h(x)-h(t)}\boldsymbol{E}(w(h(x)-h(t))^\rho;u)h'(t)f^p(t)dt\\&\leq \frac{m^p}{(m-\vartheta)^p}\int_a^x \frac{\phi(h(x)-h(t))}{h(x)-h(t)}\boldsymbol{E}(w(h(x)-h(t))^\rho;u)h'(t)\left[f(t)-\vartheta g(t)\right]^p dt,\end{aligned}$$

which implies

$$\begin{aligned}\frac{M^p}{(M-\vartheta)^p}\left({}_{\boldsymbol{h}}^{\boldsymbol{\phi}}\boldsymbol{F}(f-\vartheta g)^p\right)(x;u) &\leq \left({}_{\boldsymbol{h}}^{\boldsymbol{\phi}}\boldsymbol{F}f^p\right)(x;u)\\&\leq \frac{m^p}{(m-\vartheta)^p}\left({}_{\boldsymbol{h}}^{\boldsymbol{\phi}}\boldsymbol{F}(f-\vartheta g)^p\right)(x;u).\end{aligned}$$

Taking power $\frac{1}{p}$ on both sides, we obtain

$$\frac{M}{M-\vartheta}\left[\left({}^{\boldsymbol{\phi}}_{\boldsymbol{h}}\boldsymbol{F}(f-\vartheta g)^p\right)(x;u)\right]^{\frac{1}{p}} \leq \left[\left({}^{\boldsymbol{\phi}}_{\boldsymbol{h}}\boldsymbol{F}f^p\right)(x;u)\right]^{\frac{1}{p}}$$
$$\leq \frac{m}{m-\vartheta}\left[\left({}^{\boldsymbol{\phi}}_{\boldsymbol{h}}\boldsymbol{F}(f-\vartheta g)^p\right)(x;u)\right]^{\frac{1}{p}}. \tag{3.18}$$

Adding inequalities (3.17) and (3.18), we get (3.16). □

Remark 3.2. Using the identity function for the function h and the power function $\phi(x) = x^\sigma$ in the above two theorems, we obtain the inequalities from [2] which involve generalized fractional operator $\varepsilon^{w,\delta,c,v,r}_{a^+,\rho,\sigma,\tau} f$ as in (2.17).

Concluding Remarks

This research provides certain reverse fractional Minkowski type inequalities for the unified integral operator thereby generalizing the known inequalities. Special cases also provide the results for fractional integral operators and their refinements.

The right-sided versions of all inequalities in this paper can be established by using

$$({}^{\phi}_{h}F^{\omega,\delta,c,v,r}_{b^-,\rho,\sigma,\tau} f)(x;u) = \int_a^x \frac{\phi(h(t)-h(x))}{h(t)-h(x)} E^{\delta,c,v,r}_{\rho,\sigma,\tau}(\omega(h(t)-h(x))^\rho;u)h'(t)f(t)dt$$

and can be proved analogously.

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