

ON λf -STATISTICAL CONVERGENCE

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Abstract. The aim of this study is to define λf -density, λf -statistical convergence and λf -statistical boundedness in metric spaces by using modulus functions under different conditions. In addition, we obtain some relations between the sets of λf -statistically convergent and λf -statistically bounded sequences in metric spaces.

1. INTRODUCTION AND PRELIMINARIES

The notion of statistical convergence was discovered by Zygmund [23] and first published in his monograph in Warsaw. Moreover, the concept of statistical convergence was fundamentally proposed by Steinhaus [18] and Fast [7]. Later, Schoenberg [19] autonomously rediscovered it. Numerous mathematicians have used statistical convergence as a technique to resolve a number of open issues in the areas of sequence spaces, summability theory and some other applications. Statistical convergence has been studied in many areas and under several titles during the past few decades, including number theory, ergodic theory Banach spaces, measure theory, Fourier analysis, cone metric space, trigonometric series, topological space and time scale.

The statistical convergence depends on the natural density of subsets of \mathbb{N} . The number $\delta(A)$ of a subset A of \mathbb{N} is called a natural density of A and is defined by

$$\delta(A) = \lim_{n \rightarrow \infty} \frac{1}{n} |\{a \leq n : a \in A\}|,$$

in the case the limit exists, where $|\{a \leq n : a \in A\}|$ is the number of elements of A which are less than or equal to n (see [20]).

A sequence (x_k) of numbers is said to be statistically convergent (or S -convergent) to the number x_0 if

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : |x_k - x_0| \geq \varepsilon\}| = 0$$

for each $\varepsilon > 0$ (see [20]). In this case, we write $S - \lim x_k = x_0$ or $x_k \rightarrow x_0 (S)$ and S denotes the set of all S -convergent sequences.

To generalize this idea, Mursaleen [12] proposed the notion of λ -statistical convergence by using the sequence $\lambda = (\lambda_n)$, where $\lambda = (\lambda_n)$ denotes a non-decreasing sequence of positive numbers tending to ∞ such that $\lambda_1 = 1$ and $\lambda_{n+1} \leq \lambda_n + 1$. We write I_n to denote the closed and bounded interval $[n - \lambda_n + 1, n]$. Also, we write Λ to denote the set of all such sequences $\lambda = (\lambda_n)$.

A sequence $x = (x_k)$ of numbers is called λ -statistically convergent (or S_λ -convergent) to the number x_0 if for every $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda_n} |\{k \in I_n : |x_k - x_0| \geq \varepsilon\}| = 0.$$

In this case, we write $S_\lambda - \lim x_k = x_0$ or $x_k \rightarrow x_0 (S_\lambda)$, and the set of all sequences which are λ -statistically convergent will be denoted by S_λ (see [12]). In the case $\lambda_n = n$ for each $n \in \mathbb{N}$, S_λ -convergence reduces to S -convergence.

A sequence (x_k) of numbers is said to be statistically bounded (or S -bounded) if there is $M > 0$ such that

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : |x_k| \geq M\}| = 0.$$

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We write $S(b)$ to symbolize the class of all S -bounded sequences (see [21]).

In 1953, Nakano [14] developed the concept of a modulus function. A function $f : [0, \infty) \rightarrow [0, \infty)$ is called a modulus function (or modulus) if

- (1) $f(x) = 0 \Leftrightarrow x = 0$,
- (2) $f(x + y) \leq f(x) + f(y)$ for every $x, y \in [0, \infty)$,
- (3) f is increasing,
- (4) f is continuous from the right at 0.

From these properties, it is clear that a modulus function must be continuous everywhere on $[0, \infty)$. A modulus function may be bounded or unbounded. For instance, $f(x) = x^p$ where $p \in (0, 1]$ is an unbounded modulus, but $f(x) = \frac{x}{x+1}$ is a bounded modulus.

By using modulus functions some authors have introduced and established several sequence spaces such as Ruckle [16], Gosh and Srivastava [8], Altin and Et [2], Savas and Patterson [17], Candan [4], Prakash et al. [15], and some others.

In 2014, by using unbounded modulus functions Aizpuru et al. [1] presented the definition of f -statistical convergence as follows. Let f be an unbounded modulus function. Then, it is said that the sequence (x_k) of numbers is f -statistically convergent (or S^f -convergent) to the number x_0 if for every $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{f(n)} f(|\{k \leq n : |x_k - x_0| \geq \varepsilon\}|) = 0.$$

Some other applications and generalizations on the notion of statistical convergence are available in [3, 5, 6, 9–11, 13, 22].

2. MAIN RESULTS

Definition 1. Let f be an unbounded modulus, $\lambda = (\lambda_n) \in \Lambda$ and $A \subset \mathbb{N}$. Then, the number $\delta_\lambda^f(A)$ of the set A is named the λf -density of A and is defined by

$$\delta_\lambda^f(A) = \lim_{n \rightarrow \infty} \frac{1}{f(\lambda_n)} f(|\{a \in I_n : a \in A\}|),$$

in the case this limit exists.

Remark 1. It is not necessary for the equality $\delta_\lambda^f(A) + \delta_\lambda^f(\mathbb{N} \setminus A) = 1$ to remain true, in general. Indeed, if take $f(x) = \log(x + 1)$, $(\lambda_n) = (n)$ and $A = \{1, 3, 5, \dots\}$, then $\delta_\lambda^f(A) = \delta_\lambda^f(\mathbb{N} \setminus A) = 1$. That is, $\delta_\lambda^f(A) + \delta_\lambda^f(\mathbb{N} \setminus A) \neq 1$.

Definition 2. Let (x_k) be a sequence in a metric space (X, d) , f be an unbounded modulus and $\lambda = (\lambda_n) \in \Lambda$. Then, (x_k) is called λf -statistically convergent in a metric space (X, d) (or simply $S_\lambda^f(X, d)$ -convergent) if there exists some $x_0 \in X$ such that

$$\lim_{n \rightarrow \infty} \frac{1}{f(\lambda_n)} f(|\{k \in I_n : d(x_k, x_0) \geq \varepsilon\}|) = 0$$

for each $\varepsilon > 0$, where $f(\lambda_n)$ denotes the n th term of the sequence $(f(\lambda_n))$. In this case, we write $x_k \rightarrow x_0 \left(S_\lambda^f(X, d) \right)$ or $x_k \xrightarrow{S_\lambda^f} x_0(X, d)$. Throughout the study, the class of all λf -statistically convergent sequences in a metric space (X, d) will be denoted by $S_\lambda^f(X, d)$, that is,

$$S_\lambda^f(X, d) = \left\{ (x_k) : \lim_{n \rightarrow \infty} \frac{1}{f(\lambda_n)} f(|\{k \in I_n : d(x_k, x_0) \geq \varepsilon\}|) = 0 \text{ for some } x_0 \in X \right\}.$$

$S_\lambda^f(X, d)$ -convergence reduces to $S_\lambda(X, d)$ -convergence in the case $f(x) = x$. And, $S_\lambda^f(X, d)$ -convergence reduces to $S^f(X, d)$ -convergence in the case $(\lambda_n) = (n)$. Also, in the special case $f(x) = x$ and $(\lambda_n) = (n)$, $S_\lambda^f(X, d)$ -convergence reduces to $S(X, d)$ -convergence.

We write $S_{\lambda,0}^f(X, d)$ to denote the class of all $S_{\lambda,0}^f(X, d)$ -null sequences. It is clear that $S_{\lambda,0}^f(X, d) \subset S_\lambda^f(X, d)$.

Theorem 2.1. Let (x_k) and (y_k) be sequences in a metric space (X, d) , f be an unbounded modulus and $\lambda \in \Lambda$.

- (1) If $x_k \xrightarrow{S_\lambda^f} x_0(X, d)$, then $cx_k \xrightarrow{S_\lambda^f} cx_0(X, d)$ for each $c \in \mathbb{C}$.
- (2) If $x_k \xrightarrow{S_\lambda^f} x_0(X, d)$ and $y_k \xrightarrow{S_\lambda^f} y_0(X, d)$, then $(x_k + y_k) \xrightarrow{S_\lambda^f} (x_0 + y_0)(X, d)$.

Proof. (1) If $c = 0$, then the proof is clear. If $c \neq 0$, then for every $\varepsilon > 0$, the proof follows the fact

$$f(|\{k \in I_n : d(cx_k, cx_0) \geq \varepsilon\}|) = f\left(\left|\left\{k \in I_n : d(x_k, x_0) \geq \frac{\varepsilon}{|c|}\right\}\right|\right).$$

(2) The proof follows the fact

$$\begin{aligned} f(|\{k \in I_n : d(x_k + y_k, x_0 + y_0) \geq \varepsilon\}|) &\leq f\left(\left|\left\{k \in I_n : d(x_k, x_0) \geq \frac{\varepsilon}{2}\right\}\right|\right) \\ &\quad + f\left(\left|\left\{k \in I_n : d(y_k, y_0) \geq \frac{\varepsilon}{2}\right\}\right|\right). \end{aligned}$$

□

Theorem 2.2. Let (x_k) be a sequence in a metric space (X, d) , f be an unbounded modulus, $\lambda \in \Lambda$ and $x_0 \in X$. Then, $x_k \xrightarrow{S_\lambda^f} x_0(X, d)$ implies $x_k \xrightarrow{S_\lambda} x_0(X, d)$.

Proof. Suppose $x_k \xrightarrow{S_\lambda^f} x_0(X, d)$. Then, for every $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{f(\lambda_n)} f(|\{k \in I_n : d(x_k, x_0) \geq \varepsilon\}|) = 0.$$

So, for any $m \in \mathbb{N}$, there is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$f(|\{k \in I_n : d(x_k, x_0) \geq \varepsilon\}|) \leq \frac{1}{m} f(\lambda_n) \leq \frac{1}{m} m f\left(\frac{1}{m} \lambda_n\right) = f\left(\frac{1}{m} \lambda_n\right).$$

Since f is a modulus function, we have

$$|\{k \in I_n : d(x_k, x_0) \geq \varepsilon\}| \leq \frac{1}{m} \lambda_n.$$

Since the above inequality holds for each $m \in \mathbb{N}$, we get that

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda_n} |\{k \in I_n : d(x_k, x_0) \geq \varepsilon\}| = 0.$$

Therefore, $x_k \xrightarrow{S_\lambda} x_0(X, d)$. □

Remark 2. The converse of Theorem 2.2 is not true, in general. That is, $x_k \xrightarrow{S_\lambda} x_0(X, d)$ does not imply $x_k \xrightarrow{S_\lambda^f} x_0(X, d)$ for every unbounded modulus f and for each $\lambda \in \Lambda$. The following example provides this situation.

Example. Consider the sequence (x_k) in a usual metric space (\mathbb{R}, d) as

$$x_k = \begin{cases} k & \text{if } k = m^2 \\ \frac{k}{4k+2} & \text{if } k \neq m^2 \end{cases} \quad m \in \mathbb{N}.$$

If we take $f(x) = \log(x + 1)$ and $(\lambda_n) = (n)$, then for every $\varepsilon > 0$, we have

$$\left|\left\{k \in I_n : d\left(x_k, \frac{1}{4}\right) \geq \varepsilon\right\}\right| \leq \sqrt{\lambda_n}.$$

That is,

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda_n} \left|\left\{k \in I_n : d\left(x_k, \frac{1}{4}\right) \geq \varepsilon\right\}\right| \leq \lim_{n \rightarrow \infty} \frac{1}{\lambda_n} \sqrt{\lambda_n} = 0.$$

This means that $x_k \xrightarrow{S_\lambda} \frac{1}{4}(\mathbb{R}, d)$. However, for every $\varepsilon > 0$, we have

$$\lim_{n \rightarrow \infty} \frac{1}{f(\lambda_n)} f \left(\left| \left\{ k \in I_n : d \left(x_k, \frac{1}{4} \right) \geq \varepsilon \right\} \right| \right) \geq \lim_{n \rightarrow \infty} \frac{1}{f(\lambda_n)} f \left(\sqrt{\lambda_n} - 1 \right) = \frac{1}{2}.$$

So that $x_k \not\xrightarrow{S_\lambda} \frac{1}{4}(\mathbb{R}, d)$.

We get the following result by taking $(\lambda_n) = (n)$ from Theorem 2.2.

Corollary. *Let (x_k) be a sequence in a metric space (X, d) , f be any unbounded modulus and $x_0 \in X$. Then, $x_k \xrightarrow{S^f} x_0(X, d)$ implies $x_k \xrightarrow{S} x_0(X, d)$.*

Theorem 2.3. *Let (x_k) be a sequence in a metric space (X, d) , f be an unbounded modulus, $\lambda \in \Lambda$ and $x_0 \in X$. If $\lim_{n \rightarrow \infty} \inf \frac{f(\lambda_n)}{\lambda_n} > 0$, then $x_k \xrightarrow{S_\lambda} x_0(X, d)$ implies $x_k \xrightarrow{S_\lambda^f} x_0(X, d)$.*

Proof. Suppose $x_k \xrightarrow{S_\lambda} x_0(X, d)$. Since f is a modulus, we have

$$|\{k \in I_n : d(x_k, x_0) \geq \varepsilon\}| \geq \frac{1}{f(1)} f(|\{k \in I_n : d(x_k, x_0) \geq \varepsilon\}|)$$

for every $\varepsilon > 0$. That is,

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda_n} |\{k \in I_n : d(x_k, x_0) \geq \varepsilon\}| \geq \lim_{n \rightarrow \infty} \frac{f(\lambda_n)}{\lambda_n} \frac{1}{f(1)} \frac{f(|\{k \in I_n : d(x_k, x_0) \geq \varepsilon\}|)}{f(\lambda_n)}.$$

Since $x_k \xrightarrow{S_\lambda} x_0(X, d)$ and $\lim_{n \rightarrow \infty} \inf \frac{f(\lambda_n)}{\lambda_n} > 0$, we get that

$$\lim_{n \rightarrow \infty} \frac{1}{f(\lambda_n)} f(|\{k \in I_n : d(x_k, x_0) \geq \varepsilon\}|) = 0.$$

Therefore, $x_k \xrightarrow{S_\lambda} x_0(X, d)$ implies $x_k \xrightarrow{S_\lambda^f} x_0(X, d)$ if $\lim_{n \rightarrow \infty} \inf \frac{f(\lambda_n)}{\lambda_n} > 0$. \square

We get the following result by taking $(\lambda_n) = (n)$ in Theorem 2.3.

Corollary. *Let (x_k) be a sequence in a metric space (X, d) , f be an unbounded modulus and $x_0 \in X$. If $\lim_{n \rightarrow \infty} \inf \frac{f(n)}{n} > 0$, then $x_k \xrightarrow{S} x_0(X, d)$ implies $x_k \xrightarrow{S^f} x_0(X, d)$.*

From Theorem 2.2 and Theorem 2.3, we get the following result.

Corollary. *Let (x_k) be a sequence in a metric space (X, d) , $\lambda \in \Lambda$ and $x_0 \in X$. Then, for any unbounded modulus f providing $\lim_{n \rightarrow \infty} \inf \frac{f(\lambda_n)}{\lambda_n} > 0$ we have $x_k \xrightarrow{S_\lambda} x_0(X, d)$ if and only if $x_k \xrightarrow{S_\lambda^f} x_0(X, d)$.*

Theorem 2.4. *Let (x_k) be a sequence in a metric space (X, d) , f be an unbounded modulus, $\lambda \in \Lambda$ and $x_0 \in X$. If $\lim_{n \rightarrow \infty} \inf \frac{f(\lambda_n)}{n} > 0$, then $x_k \xrightarrow{S} x_0(X, d)$ implies $x_k \xrightarrow{S_\lambda^f} x_0(X, d)$.*

Proof. Suppose $x_k \xrightarrow{S} x_0(X, d)$. Obviously, for every $\varepsilon > 0$, we have

$$|\{k \leq n : d(x_k, x_0) \geq \varepsilon\}| \geq |\{k \in I_n : d(x_k, x_0) \geq \varepsilon\}|.$$

So,

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : d(x_k, x_0) \geq \varepsilon\}| &\geq \lim_{n \rightarrow \infty} \frac{1}{n} |\{k \in I_n : d(x_k, x_0) \geq \varepsilon\}| \\ &\geq \lim_{n \rightarrow \infty} \frac{1}{n} \frac{1}{f(1)} f(|\{k \in I_n : d(x_k, x_0) \geq \varepsilon\}|) \\ &= \lim_{n \rightarrow \infty} \frac{f(\lambda_n)}{n} \frac{1}{f(1)} \frac{f(|\{k \in I_n : d(x_k, x_0) \geq \varepsilon\}|)}{f(\lambda_n)}. \end{aligned}$$

Since $x_k \xrightarrow{S} x_0(X, d)$ and $\liminf_{n \rightarrow \infty} \frac{f(\lambda_n)}{n} > 0$, we get that

$$\lim_{n \rightarrow \infty} \frac{1}{f(\lambda_n)} f(|\{k \in I_n : d(x_k, x_0) \geq \varepsilon\}|) = 0.$$

Therefore, $x_k \xrightarrow{S} x_0(X, d)$ implies $x_k \xrightarrow{S_\lambda^f} x_0(X, d)$ if $\liminf_{n \rightarrow \infty} \frac{f(\lambda_n)}{n} > 0$. \square

From Theorem 2.4, we get the following result by taking $f(x) = x$.

Corollary. *Let (x_k) be a sequence in a metric space (X, d) , $\lambda \in \Lambda$ and $x_0 \in X$. If $\liminf_{n \rightarrow \infty} \frac{\lambda_n}{n} > 0$, then $x_k \xrightarrow{S} x_0(X, d)$ implies $x_k \xrightarrow{S_\lambda} x_0(X, d)$.*

From Theorem 2.4, we get the following result by taking $(\lambda_n) = (n)$.

Corollary. *Let (x_k) be a sequence in a metric space (X, d) , f be an unbounded modulus and $x_0 \in X$. If $\liminf_{n \rightarrow \infty} \frac{f(n)}{n} > 0$, then $x_k \xrightarrow{S} x_0(X, d)$ implies $x_k \xrightarrow{S^f} x_0(X, d)$.*

Definition 3. Let (x_k) be a sequence in a metric space (X, d) , f be an unbounded modulus and $\lambda = (\lambda_n) \in \Lambda$. Then, (x_k) is called λf -statistically bounded in a metric space (X, d) (or simply $S_\lambda^f(X, d)$ -bounded) if there exists $q \in X$ such that

$$\lim_{n \rightarrow \infty} \frac{1}{f(\lambda_n)} f(|\{k \in I_n : d(x_k, q) \geq M\}|) = 0$$

for some $M \in \mathbb{R}^+$. Throughout the study, the set of all $S_\lambda^f(X, d)$ -bounded sequences will be denoted by $BS_\lambda^f(X, d)$.

In the case $f(x) = x$, we write $BS_\lambda(X, d)$ instead of $BS_\lambda^f(X, d)$. In the case $(\lambda_n) = (n)$, we write $BS^f(X, d)$ instead of $BS_\lambda^f(X, d)$. Also, in the special case $f(x) = x$ and $(\lambda_n) = (n)$, we write $BS(X, d)$ instead of $BS_\lambda^f(X, d)$.

Theorem 2.5. *Let (x_k) be a sequence in a metric space (X, d) , f be an unbounded modulus and $\lambda \in \Lambda$. If (x_k) is $S_\lambda^f(X, d)$ -convergent, then it is $S_\lambda^f(X, d)$ -bounded.*

Proof. Suppose (x_k) is $S_\lambda^f(X, d)$ -convergent and $x_k \xrightarrow{S_\lambda^f} x_0(X, d)$. Obviously, for every $\varepsilon > 0$, we have

$$|\{k \in I_n : d(x_k, x_0) \geq \varepsilon\}| \geq |\{k \in I_n : d(x_k, x_0) > M\}|$$

for some $M \in \mathbb{R}^+$ such that $M > \varepsilon$. Since f is a modulus, the above inequality implies

$$\frac{1}{f(\lambda_n)} f(|\{k \in I_n : d(x_k, x_0) \geq \varepsilon\}|) \geq \frac{1}{f(\lambda_n)} f(|\{k \in I_n : d(x_k, x_0) > M\}|).$$

Since $x_k \xrightarrow{S_\lambda^f} x_0(X, d)$, we get that

$$\lim_{n \rightarrow \infty} \frac{1}{f(\lambda_n)} f(|\{k \in I_n : d(x_k, x_0) > M\}|) = 0.$$

Therefore, (x_k) is $S_\lambda^f(X, d)$ -bounded. \square

Remark 3. The converse of the above theorem does not have to be correct, in general. That is, a $S_\lambda^f(X, d)$ -bounded sequence does not have to be $S_\lambda^f(X, d)$ -convergent for every unbounded modulus f and for each $\lambda \in \Lambda$. The following example provides this situation.

Example. Consider the sequence (x_k) in a usual metric space (\mathbb{R}, d) as

$$x_k = \begin{cases} 1 & \text{if } k \text{ is odd,} \\ 0 & \text{if } k \text{ is even.} \end{cases}$$

Then, (x_k) is $S_\lambda^f(\mathbb{R}, d)$ -bounded but (x_k) is not $S_\lambda^f(\mathbb{R}, d)$ -convergent if we take $(\lambda_n) = (n)$ and $f(x) = \sqrt{x}$.

From Theorem 2.5, we get the following result by taking $f(x) = x$.

Corollary. *Let (x_k) be a sequence in a metric space (X, d) and $\lambda \in \Lambda$. If (x_k) is $S_\lambda(X, d)$ -convergent, then it is $S_\lambda(X, d)$ -bounded but the converse is not true, in general.*

From Theorem 2.5, we get the following result by taking $(\lambda_n) = (n)$.

Corollary. *Let (x_k) be a sequence in a metric space (X, d) and f be an unbounded modulus. If (x_k) is $S^f(X, d)$ -convergent, then it is $S^f(X, d)$ -bounded.*

Theorem 2.6. *Let (x_k) be a sequence in a metric space (X, d) , f be an unbounded modulus and $\lambda \in \Lambda$. If (x_k) is $S_\lambda^f(X, d)$ -bounded, then it is $S_\lambda(X, d)$ -bounded.*

Proof. Suppose (x_k) is $S_\lambda^f(X, d)$ -bounded. Then, there is $q \in X$ such that

$$\lim_{n \rightarrow \infty} \frac{1}{f(\lambda_n)} f(|\{k \in I_n : d(x_k, q) \geq M\}|) = 0$$

for some $M \in \mathbb{R}^+$. So, for any $m \in \mathbb{N}$, there is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$f(|\{k \in I_n : d(x_k, q) \geq M\}|) \leq \frac{1}{m} f(\lambda_n) \leq \frac{1}{m} m f\left(\frac{1}{m} \lambda_n\right) = f\left(\frac{1}{m} \lambda_n\right).$$

Since f is a modulus function, we have

$$|\{k \in I_n : d(x_k, q) \geq M\}| \leq \frac{1}{m} \lambda_n.$$

This means that,

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda_n} |\{k \in I_n : d(x_k, q) \geq M\}| = 0.$$

Therefore, (x_k) is $S_\lambda(X, d)$ -bounded. □

Remark 4. The converse of the above theorem does not have to be correct, in general. That is, a $S_\lambda(X, d)$ -bounded sequence need not be $S_\lambda^f(X, d)$ -bounded for every unbounded modulus f and each $\lambda \in \Lambda$. The following example provides this situation.

Example. Consider the sequence $(x_k) = (1, 0, 0, 4, 0, 0, 0, 9, \dots)$ in a usual metric space (\mathbb{R}, d) . Then, for any $M > 0$ and each $q \in \mathbb{R}$, we have $\{k \in \mathbb{N} : d(x_k, q) > M\} = A \setminus H$, where $A = \{1, 4, 9, \dots\}$ and H is a finite subset of \mathbb{N} . Now, if we take $f(x) = \log(x+1)$ and $(\lambda_n) = (n)$, we get

$$\delta_\lambda^f(A) = \lim_{n \rightarrow \infty} \frac{1}{f(\lambda_n)} f(|\{k \in I_n : k \in \{1, 4, 9, \dots\}\}|) = \frac{1}{2} \neq 0.$$

This means that (x_k) is not $S_\lambda^f(X, d)$ -bounded. However,

$$\delta(A) = \lim_{n \rightarrow \infty} \frac{1}{\lambda_n} |\{k \in I_n : k \in \{1, 4, 9, \dots\}\}| = 0.$$

So that (x_k) is $S_\lambda(X, d)$ -bounded.

From Theorem 2.6, we get the following result by taking $(\lambda_n) = (n)$.

Corollary. *Let (x_k) be a sequence in a metric space (X, d) and f be an unbounded modulus. If (x_k) is $S^f(X, d)$ -bounded, then it is $S(X, d)$ -bounded but the converse is not true, in general.*

Theorem 2.7. *Let (x_k) be a sequence in a metric space (X, d) , f be an unbounded modulus and $\lambda \in \Lambda$. If $\liminf_{n \rightarrow \infty} \frac{f(\lambda_n)}{\lambda_n} > 0$ and the sequence (x_k) is $S_\lambda(X, d)$ -bounded, then it is $S_\lambda^f(X, d)$ -bounded.*

Proof. Suppose (x_k) is $S_\lambda(X, d)$ -bounded. Then, there is $q \in X$ such that

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda_n} |\{k \in I_n : d(x_k, q) \geq M\}| = 0$$

for some $M \in \mathbb{R}^+$. Since f is a modulus, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{\lambda_n} |\{k \in I_n : d(x_k, q) \geq M\}| &\geq \lim_{n \rightarrow \infty} \frac{1}{\lambda_n} \frac{1}{f(1)} f(|\{k \in I_n : d(x_k, q) \geq M\}|) \\ &\geq \lim_{n \rightarrow \infty} \frac{f(\lambda_n)}{\lambda_n} \frac{1}{f(1)} \frac{f(|\{k \in I_n : d(x_k, q) \geq M\}|)}{f(\lambda_n)}. \end{aligned}$$

Since (x_k) is $S_\lambda(X, d)$ -bounded and $\liminf_{n \rightarrow \infty} \frac{f(\lambda_n)}{\lambda_n} > 0$, we get that

$$\lim_{n \rightarrow \infty} \frac{1}{f(\lambda_n)} f(|\{k \in I_n : d(x_k, q) \geq M\}|) = 0.$$

Thus, (x_k) is $S_\lambda^f(X, d)$ -bounded. □

From Theorem 2.6 and Theorem 2.7, we get the following result.

Corollary. *Let (x_k) be a sequence in a metric space (X, d) , f be an unbounded modulus such that $\liminf_{n \rightarrow \infty} \frac{f(\lambda_n)}{\lambda_n} > 0$ and $\lambda \in \Lambda$. Then, (x_k) is $S_\lambda(X, d)$ -bounded if and only if it is $S_\lambda^f(X, d)$ -bounded.*

Theorem 2.8. *Let (x_k) be a sequence in a metric space (X, d) , f be an unbounded modulus and $\lambda \in \Lambda$. If $\liminf_{n \rightarrow \infty} \frac{f(\lambda_n)}{f(n)} > 0$ and the sequence (x_k) is $S^f(X, d)$ -bounded, then it is $S_\lambda^f(X, d)$ -bounded.*

Proof. Suppose (x_k) is $S^f(X, d)$ -bounded. Then, there is $q \in X$ such that

$$\lim_{n \rightarrow \infty} \frac{1}{f(n)} f(|\{k \leq n : d(x_k, q) \geq M\}|) = 0$$

for some $M \in \mathbb{R}^+$. Obviously, we have

$$|\{k \leq n : d(x_k, q) \geq M\}| \geq |\{k \in I_n : d(x_k, q) \geq M\}|.$$

Since f is a modulus, we may write

$$\begin{aligned} \frac{1}{f(n)} f(|\{k \leq n : d(x_k, q) \geq M\}|) &\geq \frac{1}{f(n)} f(|\{k \in I_n : d(x_k, q) \geq M\}|) \\ &= \frac{f(\lambda_n)}{f(n)} \frac{1}{f(\lambda_n)} f(|\{k \in I_n : d(x_k, q) \geq M\}|). \end{aligned}$$

Since (x_k) is $S^f(X, d)$ -bounded and $\liminf_{n \rightarrow \infty} \frac{f(\lambda_n)}{f(n)} > 0$, we get

$$\lim_{n \rightarrow \infty} \frac{1}{f(\lambda_n)} f(|\{k \in I_n : d(x_k, q) \geq M\}|) = 0.$$

Therefore, (x_k) is $S_\lambda^f(X, d)$ -bounded. □

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REFERENCES

1. A. Aizpuru, M.C. Listàn-García, F. Rambla-Barreno, Density by moduli and statistical convergence. *Quaest. Math.* **3**(4) (2014), 525–530. <https://doi.org/10.2989/16073606.2014.981683>
2. Y. Altin and M. Et, Generalized difference sequence spaces defined by a modulus function in a locally convex space. *Soochow J. Math.* **31**(2) (2005), 233–243.
3. M. Ayman-Mursaleen and S. Serra-Capizzano, Statistical convergence via q -calculus and a Korovkin's type approximation theorem. *Axioms*, **11**(2) (2022), 1–10. <https://doi.org/10.3390/axioms11020070>
4. M. Candan, Some new sequence spaces defined by a modulus function and an infinite matrix in a seminormed space, *J. Math. Anal.* **3**(2) (2012), 1–9.
5. K. Demirci, F. Dirik and S. Yildiz, Approximation via equi-statistical convergence in the sense of power series method. *Rev. la Real Acad. Ciencias Exactas, Fisicas y Nat. Ser. A. Matematicas*, **116**(2) (2022), 1–7. <https://doi.org/10.1007/s13398-021-01191-4>

6. E. Dundar, F. Nuray and U. Ulusu, Wijsman statistical convergence of double sequences of sets. *Iran. J. Math. Sci. Informatics*, **16**(1) (2021), 55–64. <https://doi.org/10.29252/ijmsi.16.1.55>
7. H. Fast, Sur la convergence statistique. *In Colloquium Mathematicae*, **2**(3-4) (1951), 241–244.
8. D. Ghosh and P. D. Srivastava, On some vector valued sequence spaces defined using a modulus function. *Indian J. pure appl. Math.*, **30**(8) (1999), 819–826.
9. I.S. Ibrahim and R. Colak, On strong lacunary summability of order α with respect to modulus functions. *An. Univ. Craiova Ser. Math. Inform*, **48**(1) (2021), 127–136. <https://doi.org/10.52846/ami.v48i1.1399>
10. O. Kisi, Lacunary statistical convergence for sequence of sets in intuitionistic fuzzy metric spac. *J. Appl. Math. informatics*, **40**(1-2) (2022), 69–83. DOI:<https://doi.org/10.14317/jami.2022.069>
11. S. A. Mohiuddine, A. Asiri and B. Hazarika, Weighted statistical convergence through difference operator of sequences of fuzzy numbers with application to fuzzy approximation theorems, *Int. J. Gen. Syst.*, **48**(5) (2019), 492–506. DOI:<https://doi.org/10.1080/03081079.2019.1608985>
12. M. Mursaleen, λ -statistical convergence, *Math. slovac*, **50**(1) (2000), 111–115.
13. M. Mursaleen and P. Baliarsingh, On the convergence and statistical convergence of difference sequences of fractional order, *J. Anal.*, **30**(2) (2022), 469–481. DOI:<https://doi.org/10.1007/s41478-021-00352-3>
14. H. Nakano, Concave modulars, *J. Math. Soc. Japan*, **5**(1) (1953), 29–49.
15. S.T.V.G Prakash, M. Chandramouleeswaran and N. Subramanian, The strongly generalized triple difference Γ_3 sequence spaces defined by a modulus. *Math. Moravica*, **20**(1) (2016), 115–123.
16. W.H. Ruckle, FK Spaces in which the sequence of coordinate vectors is bounded, *Can. J. Math.*, **25**(3) (1973), 973–978. DOI:<https://doi.org/10.4153/CJM-1973-102-9>
17. E. Savas and R. Patterson, Double sequence spaces defined by a modulus. *Math. Slovaca*, **61**(2) (2011), 245–256. DOI:<https://doi.org/10.2478/s12175-011-0009-2>
18. H. Steinhaus, Sur la convergence ordinaire et la convergence asymptotique, *Colloq. Math.*, **2**(2) (1951), 73–74.
19. I. J. Schoenberg, The integrability of certain functions and related summability methods, *Am. Math. Mon.*, **66**(5) (1959), 361–775.
20. T. Salat On statistically convergent sequences of real numbers, *Math. Slovaca*, **30**(2) (1980), 139–150.
21. F. Temizsu and M. Et, Some results on generalizations of statistical boundedness, *Math. Methods Appl. Sci.*, **44**(9) (2021), 7471–7478. DOI:<https://doi.org/10.1002/mma.6271>
22. E. Yilmaz, T. Gulsen, Y. Altin and H. Koyunbakan, λ -Wijsman statistical convergence on time scales. *Commun. Stat. Methods*, **30** (2021), 1–15. DOI:<https://doi.org/10.1080/03610926.2021.2006716>
23. K. Zygmund, *Trigonometric Series*, Cambridge University Press, London, 1979.

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