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A SHORT NOTE ON STAR-COMPACT SPACE

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Abstract. In the paper, we intend to show that in a regular space having cellular open base, a star compact subspace and a closed subset which are disjoint from each other, are strongly separable. We also find an alternative representation of star-compactness in terms of family of closed sets by bringing some modifications to finite intersection property.

1. Introduction

In a topological space (X, τ), the star of a subset A, with respect to a collection of subsets $\mathcal{U}$ is denoted and defined as

$$St(A, \mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}.$$

The star operator was used to generalize the concept of compactness and Lindelöfness by Douwen, Reed, Roscoe and Tree [11] in the year 1991. Since then several generalization of star-compactness has been made and are studied by several authors. Some recent advances in this direction can be found in the investigations of Xuan and Song [16], Kočinac et al. [13, 14], Bonanzinga and Maesano [10] and Bal et al. [2, 3, 4, 5, 6, 8, 9]. A topological space (X, τ) is called a star compact space if for every open cover $\mathcal{U}$, we can find a finite subset F such that $St(F, \mathcal{U}) = X$ [11]. By the definition itself, it is concluded that every compact space is star compact space. Imposition of finite intersection property on the family of closed sets develops an alternative interpretation of compact sets [12]. Following the same path we established an alternative definition for star-compact space. On the other hand if a regular space has two disjoint subsets, one of which is compact and another is closed then they are strongly separable [12]. In this article, we will discuss what will happen if the first one of the disjoint subsets is star compact.

2. Preliminaries

Definition 1. [12] A family $\mathcal{F} = \{F_\alpha : \alpha \in \Lambda\}$ (where Λ is an index set) is said to have the finite intersection property if for every finite subset $\{\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n\}$ of Λ, $\bigcap_{i=1}^{n} F_{\alpha_i} \neq \emptyset$.

Definition 2. [1] A family $\mathcal{F}$ is said to be cellular if the elements of $\mathcal{F}$ are pairwise disjoint.

An open base $\mathcal{B}$ of a topological space (X, τ) will be called a cellular open base if elements of $\mathcal{B}$ are pairwise disjoint.

3. Main Results

Theorem 3.1. *Let (X, τ) be a regular space which has a cellular open base and (A, τ_A) be a star-compact subspace of (X, τ). Then for every closed set B disjoint from A there exist open sets $U, V \subseteq X$ such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$.*

Proof. Since (X, τ) is a regular space, for every $x \in A$ there exists open sets $U_x, V_x \subseteq X$ such that $x \in U_x$; $B \subseteq V_x$ and $U_x \cap V_x = \emptyset$.

Let $\mathcal{B}$ be a cellular open base for (X, τ) then there exists a $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq U_x$. Clearly, $x \in B_x$ and $B \subseteq V_x$ are such that $B_x \cap V_x = \emptyset$. Then $A \subseteq \cup_{x \in A} B_x$. Let $\mathcal{U} = \{B_x : x \in A\}$. So

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$\mathcal{U}_A = \{A \cap B_x : x \in A\}$ is a cover for the subspace (A, τ_A). But (A, τ_A) is a star-compact space. Therefore there exists a finite set $F \subseteq A$ such that

$$St(F, \mathcal{U}_A) = A.$$

But every element of $\mathcal{U}_A$ is a subset of the corresponding element of $\mathcal{U}$, therefore

$$A \subseteq St(F, \mathcal{U}).$$

We take $U = St(F, \mathcal{U})$ and $V = \cap_{x \in F} V_x = V$. Obviously U and V are open sets with $A \subseteq U$ and $B \subseteq V$. Now we have to show that $U \cap V = \emptyset$.

If possible, let $U \cap V \neq \emptyset$ and $p \in U \cap V$.

$$\implies p \in St(F, \mathcal{U}) \text{ and } p \in \bigcap_{x \in F} V_x.$$

$$\implies \{\exists B_q \in \mathcal{U} \text{ such that } p \in B_q \text{ and } F \cap B_q \neq \emptyset\} \text{ and } \{p \in V_x \forall x \in F\}.$$

Let $r \in B_q \cap F \implies r \in B_q$ and $r \in F$.

$\therefore B_q \in \mathcal{U}_A$ such that $p, r \in B_q$ and $p \in V_r$. p may or may not belong to A but $q, r \in A$. Since $B_r, B_q \in \mathcal{U}$ and $\mathcal{U}$ being a cellular open base, either $B_q \cap B_r = \emptyset$ or $B_r = B_q$.

Since $r \in B_r$ and $r \in B_q$, therefore $B_r \cap B_q \neq \emptyset$, so $B_r = B_q$.

Thus $p \in B_q = B_r$ also $p \in V_r$, $\therefore B_r \cap V_r \neq \emptyset$, which is a contradiction.

Therefore $U \cap V = \emptyset$. Hence the theorem. □

Corollary. *Let (X, τ) be a Hausdörff space which has a cellular open base and (A, τ_A) be a star-compact subspace of (X, τ). Then for every $x \in X$ such that $x \notin A$ there exist open sets $U, V \subseteq X$ such that $A \subseteq U$, $x \in V$ and $U \cap V = \emptyset$.*

Proof. In a Hausdörff space, every singleton set is closed. So the result follows directly from the aforementioned Theorem 3.1. □

Definition 3. A family $\mathcal{F}$ of subsets of a space X is said to have modified non-finite intersection property (MNI Property) if there exists a subfamily $\mathcal{E} \subseteq \mathcal{F}$ with the property $\cap\mathcal{E} = \emptyset$ for which we can find a finite subset $P \subseteq X$ such that $P \cap (X \setminus F) \neq \emptyset$ for all $F \in \mathcal{E}$.

Theorem 3.2. *For a topological space (X, τ), following statements are equivalent:*

(1) *X is star compact space.*

(2) *For a given family $\mathcal{U}$ of open sets, if for every finite subset $F \subseteq X$, $St(F, \mathcal{U}) \neq X$ then $\mathcal{U}$ can not cover X.*

(3) *For a given family $\mathcal{F}$ of closed subsets of X, if $\bigcap \mathcal{F} = \emptyset$, then the family $\mathcal{F}$ will have the MNI Property.*

(4) *Every family of closed subsets of X not having the MNI property have non-empty intersection.*

Proof.

(1) $\implies$ (2)

Let X be a star compact space . If possible, suppose that $\mathcal{U}$ is a family of open sets which covers X. In this case, by condition (2), for every finite subset $F \subseteq X$, $St(F, \mathcal{U}) \neq X$, which contradicts the fact that X is star compact. So $\mathcal{U}$ can not be a cover of X.

(2) $\implies$ (1)

Let $\mathcal{U}$ be a arbitrary open cover of X and condition (2) holds. Clearly $\mathcal{U}$ is a collection of open sets. If possible, suppose that for every finite set $F \subseteq X$, $St(F, \mathcal{U}) \neq X$. So, by condition (2), $\mathcal{U}$ is not a cover of X, which is a contradiction. So there must exists a finite subset $F \subseteq X$ such that $St(F, \mathcal{U}) = X$.

$\therefore X$ is a star compact space.

(1) $\implies$ (3)

Let X be a star compact space. Suppose $\mathcal{F}$ be a family of closed sets such that $\bigcap \mathcal{F} = \emptyset$.

$$\therefore X \setminus (\bigcap \mathcal{F}) = X \setminus \emptyset.$$

$$\implies \bigcup_{F\in\mathcal{F}} (X \setminus F) = X.$$

Thus $\mathcal{G} = \{G = (X \setminus F) : F \in \mathcal{F}\}$ is an open cover of X. But X is a star compact space. So there exists a finite set $P \subseteq X$ such that $St(P, \mathcal{G}) = X$.

$$\implies \bigcup\{G \in \mathcal{G} : P \cap G \neq \emptyset\} = X.$$

$$\implies \bigcup\{(X \setminus F) \in \mathcal{G} : P \cap (X \setminus F) \neq \emptyset\} = X.$$

$$\implies \bigcup\{(X \setminus F) : F \in \mathcal{F} \text{ and } P \cap (X \setminus F) \neq \emptyset\} = X.$$

$$\implies X \setminus \bigcap\{F \in \mathcal{F} : P \cap (X \setminus F) \neq \emptyset\} = X.$$

$$\implies \bigcap\{F \in \mathcal{F} : P \cap (X \setminus F) \neq \emptyset\} = \emptyset.$$

Suppose, $\mathcal{E} = \{F \in \mathcal{F} : P \cap (X \setminus F) \neq \emptyset\}$. Therefore, $\mathcal{E} \subseteq \mathcal{F}$ is such that $\bigcap \mathcal{E} = \emptyset$ and $P \subseteq X$ is finite such that $P \cap (X \setminus F) \neq \emptyset$ for all $F \in \mathcal{E}$. Therefore, $\mathcal{F}$ has the MNI property.

(3) $\implies$ **(1)**
Let condition (3) holds and $\mathcal{U}$ be a arbitrary open cover of X. Therefore, $\bigcup \mathcal{U} = X$.

$$\implies X \setminus (\bigcup \mathcal{U}) = X \setminus X = \emptyset.$$

$$\implies \bigcap\{X \setminus U : U \in \mathcal{U}\} = \emptyset.$$

Suppose, $\mathcal{F} = \{F = X \setminus U : U \in \mathcal{U}\}$. So, $\mathcal{F}$ is a family of closed sets with $\bigcap \mathcal{F} = \emptyset$. Now, by condition (3), the family $\mathcal{F}$ has the MNI property. Thus there exists a sub-family $\mathcal{E} \subseteq \mathcal{F}$ with $\bigcap \mathcal{E} = \emptyset$ for which there exists a finite subset $P \subseteq X$ such that $P \cap (X \setminus F) \neq \emptyset$ for all $F \in \mathcal{E}$.

$$\text{Now, } \mathcal{E} \subseteq \mathcal{F} \implies \{X \setminus F : F \in \mathcal{E}\} \subseteq \{X \setminus F : F \in \mathcal{F}\}.$$

$$\implies \mathcal{U}' = \{U = X \setminus F : F \in \mathcal{E}\} \subseteq \mathcal{U}.$$

$$\text{Also, } \bigcup \mathcal{U}' = X \setminus (\bigcap\{F : F \in \mathcal{E}\}) = X \setminus (\bigcap \mathcal{E}) = X \setminus \emptyset = X.$$

But $P \cap (X \setminus F) \neq \emptyset$ for all $F \in \mathcal{E}$.

$$\text{So, } St(P, \mathcal{U}') = \bigcup\{U = X \setminus F \in \mathcal{U}' : P \cap (X \setminus F) \neq \emptyset\} = \bigcup \mathcal{U}' = X.$$

$$\implies St(P, \mathcal{U}) = X (\text{ Since, } \mathcal{U}' \subseteq \mathcal{U}).$$

Therefore, X is a star compact spaces.

(3) $\iff$ **(4)**
Statement (3) and (4) are contrapositive to each other, hence equivalent.

□

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