The numerical solution of linear integro-differential equations by using generalized Bernstein operators

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Abstract:

This paper presents a numerical method for solving linear integro-differential equations with a weakly singular kernel. The proposed method is based on approximating unknown functions with a generalized Bernstein operator. We give the efficient method based on matrix forms of the linear integro-differential equations with the weakly singular kernel. Furthermore, we get an estimation of the error bound of this method.

***Keywords*:** Integro-differential equations; Generalized Bernstein operators; Numerical method; Convergence order.

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1 Introduction

Many scientific phenomena in science and engineering can be modeled by linear and nonlinear integro-differential equations. Nonlinear integro-differential equations arising in chemistry, biology, physics, and engineering applications are modeled with initial value problems for a finite closed interval. The subject of integro-differential equations is one of the most useful mathematical tools in both pure and applied mathematics.

However, nonlinear integro-differential equations are usually difficult to solve analytically. So, the approximate solutions of the nonlinear integro-differential equations can be computed by numerical methods. In recent years, several numerical methods for solving linear and nonlinear integro-differential equations have been presented by many authors. For example, some linear and nonlinear integro-differential equations have been solved using the Taylor and polynomial meth-

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ods [1–10], the wavelet method [11–13], the collocation method [14–18], the homotopy perturbation method [19–23], the direct method [24,25], and related methods [26–33].

In this study, we consider a class of linear integro-differential equations with a weakly singular kernel of the form

$$\sum_{i=0}^{J} P_i(x)y^{(i)}(x) = g(x) + \lambda_1 \int_0^x \frac{y^{(k)}(t)}{\sqrt{x-t}}dt + \lambda_2 \int_0^x K(x,t)y^{(l)}(t)dt, \tag{1.1}$$

under the initial conditions

$$y^{(i)}(c) = y_i, \quad 0 \leqslant c \leqslant R, \quad i = 0, 1, \ldots, \max(J-1, l-1, k-1) = m, \tag{1.2}$$

where$P_i(x)$, $K(x,t)$ and $g(x)$ are continuous functions on [0, 1] and $y^{(i)}(x)$ stands for the ith-order derivative of $y(x)$; c, λ_i and y_i are real constants.

For $\lambda_1 = 0$, Eq.(1.1) becomes Volterra integro-differential equation (VIDE). For $\lambda_2 = 0$, Eq. (1.1) becomes VIDE with weakly singular kernel.

In the paper [34], the authors modified and developed to obtain the solution of Eq. (1.1) by means of the matrix relations between the Bernstein polynomials and their derivatives. This paper uses a generalized Bernstein operator (α-Bernstein operator).

The rest of the paper is organized as follows: In Section 2, we will introduce the Bernstein operators and their elementary properties. In Section 3, we will obtain matrix relations between the generalized Bernstein polynomials and their derivatives. In Section 4, we find an error bound with the generalized Bernstein operators approximation. Section 5 offers three examples of integro-differential equations to illustrate the properties of our method, and finally, Section 6 concludes the paper.

2 The new generalized Bernstein operators

The Bernstein's approximation, $B_n(f)$ to a function $f : [0,1] \to R$ is the polynomial

$$B_n(f(x)) = \sum_{i=0}^{n} f\left(\frac{i}{n}\right) p_{n,i}(x), \tag{2.1}$$

where $p_{n,i}$ is the polynomial of degree n,

$$p_{n,i}(x) = \begin{pmatrix} n \\ i \end{pmatrix} x^i(1-x)^{n-i}, \quad i = 0, \ldots, n. \tag{2.2}$$

Bernstein in [35] used this approximation to give the first constructive proof of the Weierstrass theorem.

Theorem 2.1. For all functions f in $C[0,1]$, the sequence $\{B_n(f); n = 1, 2, 3, \ldots\}$ converges uniformly to f, where B_n is defined by Eq. (2.1).

Chen [36] definition of a new family of the generalized Bernstein operators and their certain elementary properties, which play an important role in the theory of uniform approximation of functions. We introduce a new family of operators as follows.

Definition 2.1. Given a function $f(x)$ on [0,1], for each positive integer n and any fixed real α, we define α-Bernstein operator for $f(x)$ as

$$T_{n,\alpha}(f;x) = \sum_{i=0}^{n} f_i p_{n,i}^{(\alpha)}(x), \tag{2.3}$$

where $f_i = f\left(\frac{i}{n}\right)$. For $i = 0, 1, \cdots, n$, the α -Bernstein polynomial $p_{n,i}^{(\alpha)}(x)$ of degree n is defined by $p_{1.0}^{(\alpha)}(x) = 1 - x, p_{1,1}^{(\alpha)}(x) = x$ and

$$p_{n,i}^{(\alpha)}(x) = \left[\begin{pmatrix} n-2 \\ i \end{pmatrix}(1-\alpha)x + \begin{pmatrix} n-2 \\ i-2 \end{pmatrix}(1-\alpha)(1-x) + \begin{pmatrix} n \\ i \end{pmatrix}\alpha x(1-x)\right] x^{i-1}(1-x)^{n-i-1} \tag{2.4}$$

where $n \geq 2, x \in [0,1]$ and the binomial coefficients $\binom{k}{l}$ are given by

$$\begin{pmatrix} k \\ l \end{pmatrix} = \begin{cases} \frac{k!}{l!(k-l)!}, & \text{if } 0 \leq l \leq k, \\ 0, & \text{else.} \end{cases} \tag{2.5}$$

For example,

$$p_{2,0}^{(\alpha)}(x) = (1-\alpha x)(1-x), \quad p_{2,1}^{(\alpha)}(x) = 2\alpha x(1-x), \quad p_{2,2}^{(\alpha)}(x) = (1-\alpha+\alpha x)x,$$

$$p_{3,0}^{(\alpha)}(x) = (1-\alpha x)(1-x)^2, \quad p_{3,1}^{(\alpha)}(x) = (1+2\alpha-3\alpha x)x(1-x),$$

$$p_{3,2}^{(\alpha)}(x) = (1-\alpha+3\alpha x)x(1-x), \quad p_{3,3}^{(\alpha)}(x) = (1-\alpha+\alpha x)x^2,$$

$$p_{4,0}^{(\alpha)}(x) = (1-\alpha x)(1-x)^3, \quad p_{4,1}^{(\alpha)}(x) = (2+2\alpha-4\alpha x)x(1-x)^2,$$

$$p_{4,2}^{(\alpha)}(x) = (1-\alpha+6\alpha x-6\alpha x^2)x(1-x), \quad p_{4,3}^{(\alpha)}(x) = (2-2\alpha+4\alpha x)x^2(1-x),$$

$$p_{4,4}^{(\alpha)}(x) = (1-\alpha+\alpha x)x^3.$$

When $\alpha = 1$, the α -Bernstein polynomial reduces to the classical Bernstein polynomial, i.e.

$$p_{n,i}^{(1)}(x) = \begin{pmatrix} n \\ i \end{pmatrix} x^i(1-x)^{n-i}.$$

So the α-Bernstein operator has the following identity

$$T_{n,1}(f;x) = \textstyle\sum_{i=0}^{n} f_i \begin{pmatrix} n \\ i \end{pmatrix} x^i(1-x)^{n-i} = B_n(f;x),$$

which means that the class of α-Bernstein operators contains the classical Bernstein ones. For α-Bernstein operators, we give here some of their properties and results.

Theorem 2.2. If function $f(x)$ is continuous on [0,1], for any $\alpha \in [0,1]$, then the α -Bernstein operators converge uniformly to $f(x)$ on the interval [0,1].

Proof. See [36].

Theorem 2.3. Let $f(x)$ be bounded on $[0,1]$. Then, for any $x \in [0,1]$ at which $f''(x)$ exists,

$$\lim_{n\to\infty} n\left[T_{n,\alpha}(f;x) - f(x)\right] = \frac{1}{2}x(1-x)f''(x), \tag{2.6}$$

where $0<\alpha<1$.

Proof. See [36].

Before giving the following theorem, We give two definitions:

Definition 2.1. [37] Let $f(x)$ be defined on $[a,b]$. The modulus of continuity of $f(x)$ on $[a,b]$, $\omega(\delta)$, is defined for $\delta>0$ by

$$\omega(\delta)=\sup_{\substack{x_1,x_2\in[a,b]\\|x_1-x_2|\leq\delta}}\left|f\left(x_1\right)-f\left(x_2\right)\right|.$$

Definition 2.2. $\|\cdot\|$ is the uniform norm over the interval [0,1].

$$\|f(x)-T_{n,\alpha}(f;x)\|=\max_{0\leq x\leq 1}\left|f(x)-T_{n,\alpha}(f;x)\right|.$$

Theorem 2.4. If $f(x)$ is bounded on $[0,1]$, then, for $0\leq\alpha\leq 1$,

$$\|f(x)-T_{n,\alpha}(f;x)\|\leq\frac{3}{2}\omega\left(\frac{\sqrt{n+2(1-\alpha)}}{n}\right), \tag{2.7}$$

where $\omega(\delta)$ is the modulus of continuity of $f(x)$.

The α -Bernstein operator $T_{n,\alpha}(f;x)$ has the shape-preserving property as follows:

Theorem 2.5. [36] Let $f\in C[0,1]$. If $f(x)$ is monotonically increasing (or decreasing) on [0,1] for $0\leq\alpha\leq 1$, so are all its α -Bernstein operators.

Theorem 2.6. [36] Let $f\in C[0,1]$. If $f(x)$ is convex on [0,1] for $0\leq\alpha\leq 1$, so are all its α -Bernstein operators.

3 Fundamental relations

Let us consider the integro-differential equation Eq. (1.1) and use α -Bernstein operators to approximate the exact solution. Here, we will find the matrix forms of each term in the equation.

First we can convert the α-Bernstein series solution $y(x)=T_{n,\alpha}(x)$ defined by (2.3) and its derivatives $y^{(k)}(x)$ to matrix forms

$$y(x)=\mathbf{T}_{n,\alpha}(x)\mathbf{A}\quad\text{ and }\quad y^{(k)}(x)=\mathbf{T}_{n,\alpha}^{(k)}(x)\mathbf{A}, \tag{3.1}$$

where

$$\mathbf{T}_{n,\alpha}(x)=\left[P_{n,0}^{(\alpha)}(x)\quad P_{n,1}^{(\alpha)}(x)\quad\cdots\quad P_{n,n}^{(\alpha)}(x)\right],\quad\mathbf{A}=\left[\begin{array}{llll}a_0 & a_1 & \ldots & a_n\end{array}\right]^T.$$

We write $y(x)$ in the form of a matrix product as follows:

$$\left[\mathbf{T}_{n,\alpha}(x)\right]^T=\left[\begin{array}{c}P_{n,0}^{(\alpha)}(x)\\P_{n,1}^{(\alpha)}(x)\\\vdots\\P_{n,n}^{(\alpha)}(x)\end{array}\right]=\mathbf{P}(\mathbf{X}(x))^T \tag{3.2}$$

where

$$\mathbf{P} = \begin{bmatrix} p_{00} & p_{01} & \cdots & p_{0n} \\ p_{10} & p_{11} & \cdots & p_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n0} & p_{n1} & \cdots & p_{nn} \end{bmatrix}, \quad \mathbf{X}(x) = \begin{bmatrix} 1 & x & \cdots & x^n \end{bmatrix}. \tag{3.3}$$

From (2.4), we get

$$\begin{aligned} P_{n,i}^{(\alpha)}(x) &= \left[\begin{pmatrix} n-2 \\ i \end{pmatrix}(1-\alpha)x + \begin{pmatrix} n-2 \\ i-2 \end{pmatrix}(1-\alpha)(1-x) + \begin{pmatrix} n \\ i \end{pmatrix}\alpha x(1-x)\right] x^{i-1}(1-x)^{n-i-1} \\ &= \begin{pmatrix} n-2 \\ i \end{pmatrix}(1-\alpha)x^i(1-x)^{n-i-1} + \begin{pmatrix} n-2 \\ i-2 \end{pmatrix}(1-\alpha)x^{i-1}(1-x)^{n-i} + \begin{pmatrix} n \\ i \end{pmatrix}\alpha x^i(1-x)^{n-i} \\ &= \begin{pmatrix} n-2 \\ i \end{pmatrix}(1-\alpha)x^i\left(\sum_{j_1=0}^{n-i-1}\begin{pmatrix} n-i-1 \\ j_1 \end{pmatrix}(-1)^{n-i-j_1-1}x^{n-i-j_1-1}\right) \\ &\quad + \begin{pmatrix} n-2 \\ i-2 \end{pmatrix}(1-\alpha)x^{i-1}\left(\sum_{j_2=0}^{n-i}\begin{pmatrix} n-i \\ j_2 \end{pmatrix}(-1)^{n-i-j_2}x^{n-i-j_2}\right) \\ &\quad + \begin{pmatrix} n \\ i \end{pmatrix}\alpha x^i\left(\sum_{j_3=0}^{n-i}\begin{pmatrix} n-i \\ j_3 \end{pmatrix}(-1)^{n-i-j_3}x^{n-i-j_3}\right) \\ &= \sum_{j_1=0}^{n-i-1}(-1)^{n-i-j_1-1}(1-\alpha)\begin{pmatrix} n-2 \\ i \end{pmatrix}\begin{pmatrix} n-i-1 \\ j_1 \end{pmatrix}x^{n-j_1-1} \\ &\quad + \sum_{j_2=0}^{n-i}(-1)^{n-i-j_2}(1-\alpha)\begin{pmatrix} n-2 \\ i-2 \end{pmatrix}\begin{pmatrix} n-i \\ j_2 \end{pmatrix}x^{n-j_2-1} \\ &\quad + \sum_{j_3=0}^{n-i}(-1)^{n-i-j_3}\alpha\begin{pmatrix} n \\ i \end{pmatrix}\begin{pmatrix} n-i \\ j_3 \end{pmatrix}x^{n-j_3} \end{aligned} \tag{3.4}$$

so, we get the coefficient of x^j, the first of (3.4): $j_1 := n-j-1$, $j_2 := n-j-1$, and $j_3 := n-j$

$$\begin{aligned} p_{ij} &= (-1)^{j-i}(1-\alpha)\begin{pmatrix} n-2 \\ i \end{pmatrix}\begin{pmatrix} n-i-1 \\ n-j-1 \end{pmatrix} + (-1)^{j-i+1}(1-\alpha)\begin{pmatrix} n-2 \\ i-2 \end{pmatrix}\begin{pmatrix} n-i \\ n-j-1 \end{pmatrix} \\ &\quad + (-1)^{j-i}\alpha\begin{pmatrix} n \\ i \end{pmatrix}\begin{pmatrix} n-i \\ n-j \end{pmatrix} \end{aligned}$$

. (3.5)

Similarly, we give the equation of the matrix $\mathbf{X}(x)$ and its derivative $\mathbf{X}^{(1)}(x)$,

$$\mathbf{X}^{(k)}(x) = \mathbf{X}^{(k-1)}(x)\mathbf{B} = \cdots = \mathbf{X}(x)\mathbf{B}^k \tag{3.6}$$

where

$$\mathbf{B}=\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & n \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Consequently, we have the matrix relation

$$y^{(k)}(x)=\mathbf{X}(x)\mathbf{B}^k\mathbf{P}^T\mathbf{A} \tag{3.7}$$

$$\int_0^x \frac{t^n}{\sqrt{x-t}}dt=\frac{\sqrt{\pi}x^{\left(\frac{1}{2}+n\right)}\Gamma(n+1)}{\Gamma\left(n+\frac{3}{2}\right)} \tag{3.8}$$

Samely, we get

$$\lambda_1\int_0^x \frac{y^{(k)}(t)}{\sqrt{x-t}}dt=\lambda_1\left(\int_0^x \frac{\mathbf{X}(t)}{\sqrt{x-t}}dt\right)\mathbf{B}^k\mathbf{P}^T\mathbf{A}=\lambda_1\mathbf{Q}_x\mathbf{B}^k\mathbf{P}^T\mathbf{A}, \tag{3.9}$$

and

$$\lambda_2\int_0^x K(x,t)y^{(l)}(t)dt=\lambda_2\left(\int_0^x K(x,t)\mathbf{X}(t)dt\right)\mathbf{B}^l\mathbf{P}^T\mathbf{A}=\lambda_2\mathbf{V}_x\mathbf{B}^l\mathbf{P}^T\mathbf{A}, \tag{3.10}$$

where

$$\mathbf{Q}_x=\begin{bmatrix} \frac{\sqrt{\pi}x^{\left(\frac{1}{2}\right)}\Gamma(1)}{\Gamma\left(\frac{3}{2}\right)} & \frac{\sqrt{\pi}x^{\left(\frac{3}{2}\right)}\Gamma(2)}{\Gamma\left(\frac{5}{2}\right)} & \cdots & \frac{\sqrt{\pi}x^{\left(\frac{1}{2}+n\right)}\Gamma(n+1)}{\Gamma\left(n+\frac{3}{2}\right)} \end{bmatrix},$$

$$\mathbf{V}_x=\left[\int_0^x K(x,t)dt \quad \int_0^x K(x,t)tdt \quad \cdots \quad \int_0^x K(x,t)t^n dt\right].$$

So, (1.1) can be written as

$$\sum_{i=0}^{J}P_i(x)\mathbf{X}(x)\mathbf{B}^i\mathbf{P}^T\mathbf{A}=g(x)+\lambda_1\mathbf{Q}_x\mathbf{B}^k\mathbf{P}^T\mathbf{A}+\lambda_2\mathbf{V}_x\mathbf{B}^l\mathbf{P}^T\mathbf{A} \tag{3.11}$$

By using the nodes $\{x_i \mid i=0,1,\ldots,n; 0=x_0<x_1<\ldots<x_n=1\}$ in (3.11), we get the system of matrix equations

$$\sum_{i=0}^{J}P_i\left(x_j\right)\mathbf{X}\left(x_j\right)\mathbf{B}^i\mathbf{P}^T\mathbf{A}=g\left(x_j\right)+\lambda_1\mathbf{Q}_{x_i}\mathbf{B}^k\mathbf{P}^T\mathbf{A}+\lambda_2\mathbf{V}_{x_i}\mathbf{B}^l\mathbf{P}^T\mathbf{A},\quad i=0,1,\ldots,n \tag{3.12}$$

or briefly the fundamental matrix equation

$$\left[\sum_{i=0}^{J}\mathbf{P}_i\mathbf{X}\mathbf{B}^i\mathbf{P}^T-\lambda_1\mathbf{Q}\mathbf{B}^k\mathbf{P}^T-\lambda_2\mathbf{V}\mathbf{B}^l\mathbf{P}^T\right]\mathbf{A}=\mathbf{G}, \tag{3.13}$$

where

$$\mathbf{P}_i=\begin{bmatrix} P_i\left(x_0\right) & 0 & 0 & \cdots & 0 \\ 0 & P_i\left(x_1\right) & 0 & \cdots & 0 \\ 0 & 0 & P_i\left(x_2\right) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & \cdots & P_i\left(x_n\right) \end{bmatrix},\quad \mathbf{G}=\begin{bmatrix} g\left(x_0\right) \\ g\left(x_1\right) \\ g\left(x_2\right) \\ \vdots \\ g\left(x_n\right) \end{bmatrix},\quad \mathbf{Q}=\begin{bmatrix} \mathbf{Q}_{x_0} \\ \mathbf{Q}_{x_1} \\ \vdots \\ \mathbf{Q}_{x_n} \end{bmatrix},$$

$$\mathbf{V}=\begin{bmatrix}\mathbf{V}_{x_0}\\ \mathbf{V}_{x_1}\\ \vdots\\ \mathbf{V}_{x_n}\end{bmatrix},\quad \mathbf{X}=\begin{bmatrix}\mathbf{X}(x_0)\\ \mathbf{X}(x_1)\\ \mathbf{X}(x_2)\\ \vdots\\ \mathbf{X}(x_n)\end{bmatrix}=\begin{bmatrix}1 & x_0 & x_0^2 & \cdots & x_0^n\\ 1 & x_1 & x_1^2 & \cdots & x_1^n\\ 1 & x_2 & x_2^2 & \cdots & x_2^n\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & x_n & x_n^2 & \cdots & x_n^n\end{bmatrix}$$

$$\mathbf{WA}=\mathbf{G}, \tag{3.14}$$

where

$$\mathbf{W}=\sum_{i=0}^{J}\mathbf{P}_i\mathbf{X}\mathbf{B}^i\mathbf{P}^T-\lambda_1\mathbf{Q}\mathbf{B}^k\mathbf{P}^T-\lambda_2\mathbf{V}\mathbf{B}^l\mathbf{P}^T. \tag{3.15}$$

Here, (3.14) corresponds to a system of $(n+1)$ linear algebraic equations with unknown coefficients $a_0, a_1, \ldots, a_n$.

We use the same method [27] to obtain the corresponding matrix forms for the conditions (1.2), by means of the relation (3.7), as follows

$$\mathbf{X}(c)\mathbf{B}^i\mathbf{P}^T\mathbf{A}=[y_i], \quad 0\leqslant c\leqslant R, \quad i=0,1,\ldots,m. \tag{3.16}$$

On the other hand, the augmented matrix form for conditions can be written as

$$\mathbf{U}_i\mathbf{A}=[y_i] \text{ or } [\mathbf{U}_i; y_i], \quad i=0,1,\ldots,m \tag{3.17}$$

where

$$\mathbf{U}_i=\mathbf{X}(c)\mathbf{B}^i\mathbf{P}^T=\begin{bmatrix}u_{i0} & u_{i1} & \cdots & u_{in}\end{bmatrix}, \quad i=0,1,\ldots,m. \tag{3.18}$$

Replacing the row matrices (3.17) by any $m+1$ rows of the matrix (3.14), we get the new augmented matrix $[\widetilde{\mathbf{W}};\widetilde{\mathbf{G}}]$ as

$$[\widetilde{\mathbf{W}};\widetilde{\mathbf{G}}]=\begin{bmatrix}w_{00} & w_{01} & w_{02} & \cdots & w_{0n}; & g(x_0)\\ w_{10} & w_{11} & w_{12} & \cdots & w_{1n}; & g(x_1)\\ w_{20} & w_{21} & w_{22} & \cdots & w_{2n}; & g(x_2)\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ w_{(n-m-1)0} & w_{(n-m-1)1} & w_{(n-m-1)2} & \cdots & w_{(n-m-1)n}; & g(x_{n-m-1})\\ u_{00} & u_{01} & u_{02} & \cdots & u_{0n}; & y_0\\ u_{10} & u_{11} & u_{12} & \cdots & u_{1n}; & y_1\\ u_{20} & u_{21} & u_{22} & \cdots & u_{2n}; & y_2\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ u_{m0} & u_{m1} & u_{m2} & \cdots & u_{mn}; & y_m\end{bmatrix} \tag{3.19}$$

Note that $\operatorname{rank}\widetilde{\mathbf{W}}=\operatorname{rank}[\widetilde{\mathbf{W}};\widetilde{\mathbf{G}}]=n+1$. Thus, we can write

$$\mathbf{A}=(\widetilde{\mathbf{W}})^{-1}\widetilde{\mathbf{G}} \tag{3.20}$$

and hence the elements $a_0, a_1, \ldots, a_n$ of $\mathbf{A}$ are uniquely determined.

4 Estimation of error bound by using generalized Bernstein operators

We give an error bound for this solution in the following theorem.

Theorem 4.1. Consider the integral equation of (1.1). Assume $P_i(x)$, $K(x,t)$ and $g(x)$ are continuous functions on [0,1] and $y^{(i)}(x)$ stands for the ith-order derivative of $y(x)$; c, λ_i and y_i are real constants. We obtained matrix $\mathbf{W}$ from relation (3.15). Then

$$\sup_{x_i\in[0,1]} y\left(x_i\right) - T_{n,\alpha}\left(y_n\left(x_i\right)\right) | \leqslant \frac{1}{8n}\left[\|y''\| + (M + C(1+J))\|\mathbf{W^{-1}}\|\|y^{(m)}\|\right] \tag{4.1}$$

Where $x_i = \frac{i}{n}, i = 0,\ldots,n, y(x)$ is exact solution, $M = max\left(|\lambda_1 c_1|, |\lambda_2 c_2|\right)$, $c_1 = \frac{\sqrt{\pi}\Gamma(1)}{\Gamma\left(\frac{3}{2}\right)}$, $c_2 = \sup_{x,t\in[0,1]}|k(x,t)|$, $C = max_{x\in[0,1]}(|P_i(x)|)$, $\|y^{(m)}\| = max\left(\|y^{(i+2)}\|\right)$ and $T_{n,\alpha}\left(y_n(x)\right)$ is the proposed method solution.

Proof. It is clear that:

$$\begin{aligned}\sup_{x_i\in[0,1]}\left|y\left(x_i\right) - T_{n,\alpha}\left(y_n\left(x_i\right)\right)\right| &\leq \sup_{x_i\in[0,1]}\left|y\left(x_i\right) - T_{n,\alpha}\left(y\left(x_i\right)\right)\right| + \sup_{x_i\in[0,1]}\left|T_{n,\alpha}\left(y\left(x_i\right)\right) - T_{n,\alpha}\left(y_n\left(x_i\right)\right)\right| \\ &= T_1 + T_2\end{aligned} \tag{4.2}$$

From Theorem 2.3, we have the following bound:

$$\sup_{x_i\in[0,1]}|y(x) - T_{n,\alpha}(y;x)| \leq \frac{1}{2n}x(1-x)\left\|y''\right\| \leq \frac{1}{8n}\left\|y''\right\|. \tag{4.3}$$

Then it is enough to find a bound for T_2. If we substitute $T_{n,\alpha}(y^{(i)};x)$, $T_{n,\alpha}(y^{(k)};x)$, $T_{n,\alpha}(y^{(l)};x)$ instead of $y^{(i)}(x)$, $y^{(k)}(x)$, $y^{(l)}(x)$ in the integral equation

$$g(x) = \sum_{i=0}^{J} P_i(x)y^{(i)}(x) - \lambda_1\int_0^x \frac{y^{(k)}(t)}{\sqrt{x-t}}dt - \lambda_2\int_0^x K(x,t)y^{(l)}(t)dt \tag{4.4}$$

then the left-hand side of the integral equation is exchanged by a new function that we denote it by $\widehat{g(x)}$. We have

$$\widehat{g(x)} = \sum_{i=0}^{J} P_i(x)T_{n,\alpha}(y^{(i)}(x)) - \lambda_1\int_0^x \frac{T_{n,\alpha}(y^{(k)}(t))}{\sqrt{x-t}}dt - \lambda_2\int_0^x K(x,t)T_{n,\alpha}(y^{(l)}(t))dt \tag{4.5}$$

and consequently, from (4.4), (4.5) and by using Theorem 2.3 we have:

$$
\begin{aligned}
\sup_{x_i\in[0,1]}\left|g(x)-\widehat{g(x)}\right| &\leq \sum_{i=0}^{J} P_i(x)\sup_{x_i\in[0,1]}\left|T_{n,\alpha}(y^{(i)};x)-y^{(i)}(x)\right| - \lambda_1\int_0^x \frac{\sup_{x_i\in[0,1]}\left|T_{n,\alpha}(y^{(k)};x)-y^{(k)}(x)\right|}{\sqrt{x-t}}dt \\
&\quad - \lambda_2\int_0^x K(x,t)\sup_{x_i\in[0,1]}\left|T_{n,\alpha}(y^{(t)};x)-y^{(t)}(x)\right|dt \\
&\leq \sum_{i=0}^{J}\frac{1}{8n}P_i(x)\|y^{(i+2)}\| + \lambda_1 c_1\|y^{(k+2)}\| + \lambda_2 c_2\|y^{(l+2)}\|
\end{aligned}
\tag{4.6}
$$

Where we define :
$M = max\left(|\lambda_1 c_1|, |\lambda_2 c_2|\right),\ C = max_{x\in[0,1]}(|P_i(x)|),\ \|y^{(m)}\| = max\left(\|y^{(i+2)}\|, \|y^{(k+2)}\|, \|y^{(l+2)}\|\right),$
$i = 0,\ldots,J.$

$$
\sup_{x_i\in[0,1]}\left|g(x)-\widehat{g(x)}\right| \leq \frac{1}{8n}\left(M + C(J+1)\right)\|y^{(m)}\| \tag{4.7}
$$

On the other hand we have $WX = Y$ and $W\bar{X} = \bar{Y}$ where:

$$
X = \left[T_{n,\alpha}\left(y_n\left(x_j\right)\right)\right],\quad Y = \left[g\left(x_i\right)\right],\quad \bar{X} = \left[T_{n,\alpha}\left(y\left(x_j\right)\right)\right],\quad \bar{Y} = \left[\widehat{g}\left(x_i\right)\right]
$$

therefore by using (4.7), the result is:

$$
\begin{aligned}
\sup_{x_i\in[0,1]}\left|T_{n,\alpha}\left(y\left(x_i\right)\right) - T_{n,\alpha}\left(y_n\left(x_i\right)\right)\right| &\leq \|\mathbf{W^{-1}}\|\cdot\sup_{x_i\in[0,1]}\left|g(x)-\widehat{g(x)}\right| \\
&\leq \frac{(M + C(1+J))}{8n}\|\mathbf{W^{-1}}\|\|y^{(m)}\|
\end{aligned}
\tag{4.8}
$$

Finally, by applying (4.8) and (4.3) in (4.2) the proof is completed.

5 Illustrative examples

In this section, several numerical examples are given to illustrate the properties of the method. All calculations were made in Maple 2019. We note that

$$
\|e_n\| = \left(\int_0^1 e_n^2(x)dx\right)^{1/2} \approx \left(\frac{1}{n}\sum_{i=0}^{n} e_n^2\left(x_i\right)\right)^{1/2}
$$

where

$$
e_n\left(x_i\right) = y\left(x_i\right) - T_{n,\alpha}\left(y_n\left(x_i\right)\right),\quad i = 0,\ldots,n.
$$

where $x_i = i/n, i = 0,\ldots,n$ and $T_{n,\alpha}\left(y_n(x)\right), y(x)$ are the approximate solution of order n with α-Bernstein's approximation and exact solutions of the integral equations, respectively.

Example 5.1 We consider the Volterra integro-differential equation with a weakly singular kernel.

$$
\begin{cases}
y''(x) + y(x) + \frac{1}{\sqrt{\pi}}\int_0^x \frac{y''(t)}{\sqrt{x-t}}dt = f(x) \\
y(0) = y'(0) = 1
\end{cases}
$$

where the exact solution is $1 + x + x^2$ when $y(x)$ is $3 + x + x^2 + \frac{4\sqrt{x}}{\sqrt{4}}$.
For n = 7, α =0.01 the fundamental matrix equation of the problem is

$$\left\{\mathbf{X}\mathbf{B}^2\mathbf{P}^T + \mathbf{X}\mathbf{P}^T + \frac{1}{\sqrt{\pi}}\mathbf{Q}\mathbf{B}^2\mathbf{P}^T\right\}\mathbf{A} = \mathbf{G}$$

where

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 1 & -6.01 & 15.06 & -20.15 & 15.20 & -6.15 & 1.06 & -0.01 \\ 0 & 5.02 & -25.17 & 50.55 & -50.90 & 25.80 & -5.37 & 0.07 \\ 0 & 0.99 & 5.16 & -30.75 & 51.60 & -36.75 & 9.96 & -0.21 \\ 0 & 0 & 4.95 & -9.55 & -1.4 & 12 & -6.35 & 0.35 \\ 0 & 0 & 0 & 9.9 & -24.4 & 18.75 & -3.9 & -0.35 \\ 0 & 0 & 0 & 0 & 9.9 & -18.6 & 8.49 & 0.21 \\ 0 & 0 & 0 & 0 & 0 & 4.95 & -4.88 & -0.07 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.99 & 0.01 \end{bmatrix},$$

$$\mathbf{X} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & (\frac{1}{7}) & (\frac{1}{7})^2 & (\frac{1}{7})^3 & (\frac{1}{7})^4 & (\frac{1}{7})^5 & (\frac{1}{7})^6 & (\frac{1}{7})^7 \\ 1 & (\frac{2}{7}) & (\frac{2}{7})^2 & (\frac{2}{7})^3 & (\frac{2}{7})^4 & (\frac{2}{7})^5 & (\frac{2}{7})^6 & (\frac{2}{7})^7 \\ 1 & (\frac{3}{7}) & (\frac{3}{7})^2 & (\frac{3}{7})^3 & (\frac{3}{7})^4 & (\frac{3}{7})^5 & (\frac{3}{7})^6 & (\frac{3}{7})^7 \\ 1 & (\frac{4}{7}) & (\frac{4}{7})^2 & (\frac{4}{7})^3 & (\frac{4}{7})^4 & (\frac{4}{7})^5 & (\frac{4}{7})^6 & (\frac{4}{7})^7 \\ 1 & (\frac{5}{7}) & (\frac{5}{7})^2 & (\frac{5}{7})^3 & (\frac{5}{7})^4 & (\frac{5}{7})^5 & (\frac{5}{7})^6 & (\frac{5}{7})^7 \\ 1 & (\frac{6}{7}) & (\frac{6}{7})^2 & (\frac{6}{7})^3 & (\frac{6}{7})^4 & (\frac{6}{7})^5 & (\frac{6}{7})^6 & (\frac{6}{7})^7 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} 3 \\ 4.016239 \\ 4.573635 \\ 5.089641 \\ 5.603908 \\ 6.131799 \\ 6.681189 \\ 7.256758 \end{bmatrix},$$

$$\mathbf{Q} = \begin{bmatrix} 0.6325 & 0.0421 & 0.0034 & 0.2891\times10^{-3} & 0.2570\times10^{-4} & 0.2336\times10^{-5} & 0.2157\times10^{-6} & 0.2013\times10^{-7} \\ 0.7559 & 0.0720 & 0.0082 & 0.0010 & 0.1279\times10^{-3} & 0.1661\times10^{-4} & 0.2191\times10^{-5} & 0.2921\times10^{-6} \\ 1.0690 & 0.2036 & 0.0465 & 0.0114 & 0.0029 & 0.7519\times10^{-3} & 0.1983\times10^{-3} & 0.5288\times10^{-4} \\ 1.3093 & 0.3741 & 0.1283 & 0.0471 & 0.0179 & 0.0070 & 0.0028 & 0.0011 \\ 1.5119 & 0.5759 & 0.2633 & 0.1290 & 0.0655 & 0.0340 & 0.0179 & 0.0096 \\ 1.6903 & 0.8049 & 0.4599 & 0.2816 & 0.1788 & 0.1161 & 0.0765 & 0.0510 \\ 1.8516 & 1.0581 & 0.7255 & 0.5331 & 0.4061 & 0.3165 & 0.2504 & 0.2003 \\ 2 & 1.3333 & 1.0667 & 0.9143 & 0.8127 & 0.7388 & 0.6820 & 0.6365 \end{bmatrix},$$

$$[\mathbf{W};\mathbf{G}] = \begin{bmatrix} 31.12 & -50.34 & 10.32 & 9.9 & 0 & 0 & 0 & 0; & 3 \\ 25.335826 & -29.411782 & -5.857459 & 4.005563 & 4.831761 & 1.808127 & 0.34832 & 0.035586; & 4.016239 \\ 16.269295 & -10.811952 & -9.704065 & -3.480806 & 2.459566 & 3.917923 & 1,730908 & 0.247962; & 4.573635 \\ 10.209425 & -2.460924 & -5.759776 & -6.307126 & -2.606899 & 2.698031 & 3.990197 & 1.316604; & 5.089641 \\ 6.732903 & 0.616962 & -0.397316 & -4.727566 & -6.334275 & -2.375886 & 4.550125 & 4.332585; & 5.603908 \\ 4.990979 & -1.516737 & 2.297612 & 0.047102 & -5.487912 & -8.753002 & -1.543187 & 10.515137; & 6.131799 \\ -6.01 & 5.02 & 0.99 & 0 & 0 & 0 & 0 & 0; & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0; & 0 \end{bmatrix}$$

Solving this system, α-Bernstein coefficients matrix is obtained as

$$\mathbf{A} = \begin{bmatrix} 1 & 1.135789 & 1.321553 & 1.557293 & 1.843007 & 2.178693 & 2.564361 & 3 \end{bmatrix}$$

and from (3.1) the α-Bernstein series solution

$$T_{n,\alpha}(x) = \mathbf{X}_7(x)\mathbf{P}^T\mathbf{A} = 1+x+x^2+5.3\times10^{-17}x^3-3.8\times10^{-16}x^4+1.032\times10^{-15}x^5-1.2102\times10^{-15}x^6+5.1608\times10^{-16}x^7$$

Results has been shown in Table 1 for n=2, 5, 7, 20 and α=0.01, 0.1. Also, Figure 1 shows the exact and approximate solution for n=7 and α=0.01.

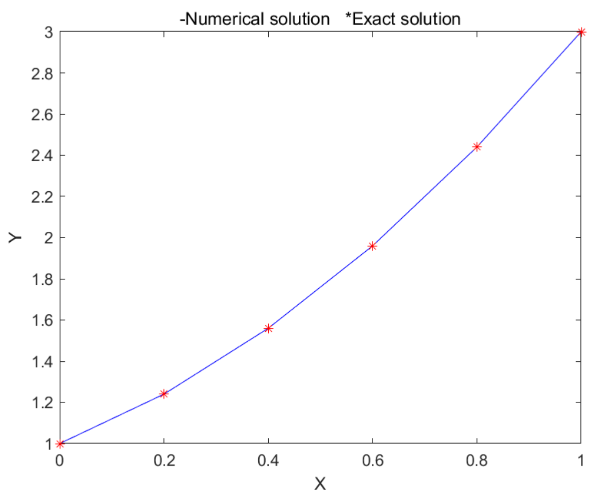

Figure 1: Results for Example 5.1 with n=7, α=0.01

Example 5.2 Let us consider the Volterra integro-differential equation.

$$\begin{cases} y''(x) + y(x) + \int_0^x xy(t)dt + \int_0^x \frac{y(t)}{\sqrt{x-t}}dt = 2 + x^2 + \frac{16}{15}x^{\frac{5}{2}} + \frac{x^4}{3} \\ y(0) = y'(0) = 0 \end{cases}$$

with the exact solution is $y(x) = x^2$.

Results have been shown in Table 1 for n=2, 5, 7, 20 and α=0.01, 0.5. Also, Figure 2 shows the exact and approximate solution for n=7 and α=0.5.

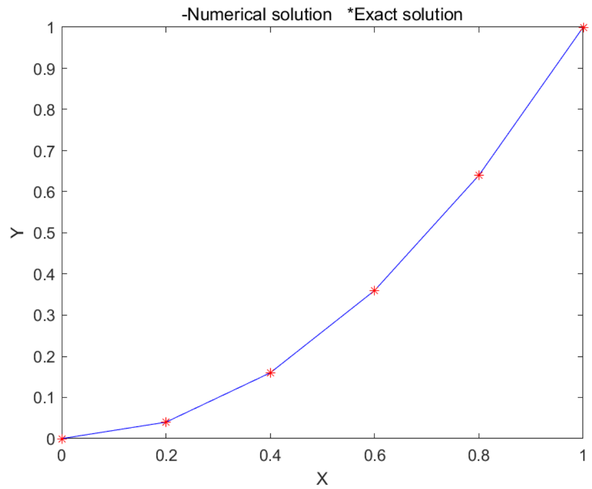

Figure 2: Results for Example 5.2 with n=7, α=0.5

Example 5.3 Consider the Volterra integral equation.

$$y(x) = cos(x) - e^x sin(x) + \int_0^x e^x y(t)dt, 0 \leq x \leq 1 \ .$$

where the exact solution is $y(x) = cos(x)$.

Table 1 shows the results for n=2, 5, 7, 20 and α=0.01, 0.8. Also, Figure 3 shows the exact and approximate solution for n=7 and α=0.01.

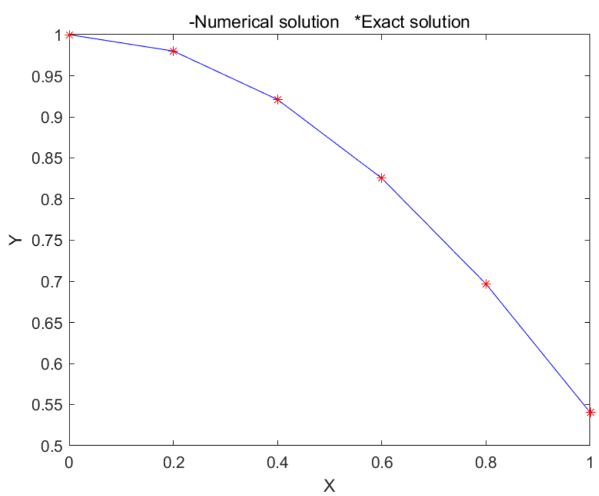

Figure 3: Results for Example 5.3 with n=7, α=0.01

Table 1: Computed errors for various for Examples 5.1-5.3

	Example 5.1		Example 5.2		Example 5.3	
n	α=0.01	α=0.1	α=0.01	α=0.5	α=0.01	α=0.8
2	0	0	0	0	0.122E-2	0.132E-2
5	0.3821E-18	0.9483E-19	0.2015E-19	0.1061E-19	0.1839E-4	0.1839E-4
7	0.3626E-17	0.1813E-16	0.4947E-19	0.3388E-19	0.526E-7	0.526E-7
20	0.9069E-10	0.4586E-10	0.9602E-12	0.5527E-11	0.6211E-10	0.4011E-9

6 Conclusions

In this paper, we present a numerical method for solving linear integro-differential equations with the weakly singular kernel by a generalized Bernstein operator. The generalized Bernstein operators have the same elementary properties as the class Bernstein operators. Especially, α-Bernstein has the monotony-preserving and convexity-preserving properties. It is observed from the examples that the proposed method for integro-differential equations is a good approximation. Furthermore, we get an estimation of the error bound of this method.

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