# Kakutani duality for groups

*Based on a joint work V. Marra*

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# Embedding spaces

It is well known that

> *every compact Hausdorff space $X$ can be embedded in some hypercube $[0,1]^J$ for some index set $J$.*

Suppose that $X$ is now endowed with a function $\delta\colon X \to \mathbb{N}$.

**Problem**

Given a pair $\langle X, \delta \rangle$, is there a continuous embedding $\iota\colon X \to [0,1]^J$ in such a way that the denominators of the points in $\iota[X]$ agree with $\delta$?

Let us assume that "agree" means that $\delta(x) = \mathtt{den}(\iota(x))$.

# Denominators



Recall that $\mathbb{N}$ forms a complete lattice under the divisibility order: the top being 0 and the bottom being 1.

Let $J$ be a set and $\overline{p} \in [0,1]^J$. If $\overline{p} \in \mathbb{Q}^J$ we define its denominator to be the natural number

$$\mathtt{den}(\overline{p}) = \mathtt{lcd}\{p_i \mid i \in J\}$$

where lcd stands for the least common denominator. If $\overline{p} \notin \mathbb{Q}^J$ we set $\mathtt{den}(\overline{p}) = 0$.

1. A function $f\colon [0,1]^J \to [0,1]$ preserves denominators if for any $\overline{x} \in [0,1]^J$, $\mathtt{den}(f(x)) = \mathtt{den}(x)$.
2. A function $f\colon [0,1]^J \to [0,1]$ respects denominators if for any $\overline{x} \in [0,1]^J$, $\mathtt{den}(f(x)) \mid \mathtt{den}(x)$.

# An easy counter-example

Consider $X = [0, 1]$ with its Euclidean topology and endow it with a constant $\delta$:

$$\forall x \in X \quad \delta(x) = 1.$$

The only points with denominator equal 1 in $[0, 1]^J$ are the so-called lattice points i.e., points whose coordinates are either 0 or 1.

The only way $\iota$ could agree with $\delta$ is to send all points in one lattice point —failing injectivity— or by sending the points in different lattice points —failing continuity.

# MV-algebras

The above mentioned problem is crucial in the duality theory of MV-algebras —the equivalent algebraic semantics of Łukasiewicz logic.
An MV-algebra is a structure $\langle A, \oplus, \neg, 0\rangle$ such that

1. $\langle A, \oplus, 0\rangle$ is a commutative monoid,
2. $\neg\neg x = x$,
3. $\neg 0 \oplus x = \neg 0$
4. $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

**Example**

The interval $[0, 1]$ in the real numbers has a natural MV-structure given by the truncated sum $x \oplus y = \min\{x + y, 1\}$ and $\neg x = 1 - x$. The importance of this structure comes from the fact that it generates the whole variety of MV-algebras.

# MV-algebras and compact spaces



## Theorem (Marra, S. 2012)

*Semisimple MV-algebras with their homomorphisms form a category that is dually equivalent to the category of compact Hausdorff spaces embedded in some hypercube, with $\mathbb{Z}$-maps among them.*

## Definition

For $I, J$ arbitrary sets, a map from $\mathbb{R}^I$ into $\mathbb{R}^J$ is called $\mathbb{Z}$-map if it is continuous and piecewise (affine) linear map, where each (affine) linear piece has integer coefficients.

## Remark

Since every $\mathbb{Z}$-map $f$ acts on each point as an linear function with integer coefficients, it respect denominators i.e.,

$$\mathrm{den}(f(x)) \mid \mathrm{den}(x).$$

# Mundici's functor

An abelian $\ell$-group with order unit (u$\ell$-group, for short), is a partially ordered Abelian group $G$ whose order is a lattice, and that possesses an element $u$ such that

$$\text{for all } g \in G, \text{ there exists } n \in \mathbb{N} \text{ such that } (n)u \geq g.$$

The functor $\Gamma$ that takes an u$\ell$-group $\langle G, +, -, 0, u\rangle$ to its unital interval $[0, u]$ with operation $\oplus$ and $\neg$ defined as follows:

$$x \oplus y = \min\{u, x + y\} \qquad \text{and} \qquad \neg x = u - x,$$

is full, faithful, and dense hence it has a quasi-inverse $\Xi$ and

**Theorem (Mundici 1986)**

*The pair $\Gamma, \Xi$ gives an equivalence of categories between the category of MV-algebras with their morphisms, and the category of u$\ell$-groups with ordered group morphisms preserving the order unit.*

# Norm induced by the order unit



### Definition

Let $(G, u)$ be a $u\ell$-group. The order unit $u$ induces a **seminorm** $\| \ \|_u$ defined as folows:

$$\|g\|_u := \inf\left\{\frac{p}{q} \in \mathbb{Q} \mid p, q \in \mathbb{N}, q \neq 0 \text{ and } q|g| \leq pu\right\}$$

The seminorm $\| \ \|_u \colon G \to R^+$ is in fact a norm if, and only if, $G$ is archimedean. **Any semisimple MV-algebra $A$ inherits a norm** from its enveloping (archimedean) group $\Xi(A)$.

### Definition

An **norm-complete MV-algebra** is a semisimple MV-algebra which is Cauchy-complete w.r.t. its induced norm.

# Kakutani duality



## Theorem (Kakutani-Yosida duality 1941)

*A unital real vector lattice $(V, u)$ is isomorphic to $(\mathrm{C}(X), 1)$ for some compact Hausdorff space $X$, if, and only if, $V$ is Archimedean and norm-complete (with respect to the norm $\| \|_u$ induced by the unit).*

## Question

What if we want to substitute $u\ell$-group for real vector lattice in the above statement?

## Remark

An answer was already given by Stone: compact Hausdorff spaces correspond to Archimedean, complete and divisible $u\ell$-groups.

# Denominator preserving maps

**Theorem (Goodearl-Handelman 1980)**

*Let $X$ be a compact Hausdorff space. For each $x \in X$ choose $A_x$ to be either $A_x = \mathbb{R}$ or $A_x = (\frac{1}{n})\mathbb{Z}$. Then, the algebra of functions*

$$\{f \in \mathrm{C}(X) \mid f(x) \in A_x \text{ for all } x \in X\},$$

*is a norm-complete $u\ell$-group and every such a group can be represented in this way.*

As a corollary we obtain

**Corollary**

*The norm-completion of the algebra of $\mathbb{Z}$-maps is given by all continuous maps which respect denominators.*

# A duality for norm-complete MV-algebras

## The category $\mathbb{MV}$

Let $\mathbb{MV}$ be the category whose objects are semisimple MV-algebras and arrows are MV-homomorphisms.

## The category $\mathbb{A}$

Let $\mathbb{A}$ be the category whose objects are pairs $\langle X, \delta \rangle$, where $X$ is a compact Hausdorff space and $\delta$ is a map from $X$ into $\mathbb{N}$. An arrow between two objects $\langle X, \delta \rangle$ and $\langle Y, \delta' \rangle$ is a continuous map $f\colon X \to Y$ that respects denominators, i.e.,

$$\delta'(f(x)) \mid \delta(x).$$

# A duality for norm-complete MV-algebras



### The functor $\mathscr{L}$

Let $\mathscr{L} : \mathbb{A} \to \mathbb{MV}$ be the assignment that associates to every object $\langle X, \delta \rangle$ in $\mathbb{A}$ the MV-algebra

$$\mathscr{L}(\langle X, \delta \rangle) := \{g \in \mathtt{C}(X) \mid \forall x \in X \quad \mathtt{den}(g(x)) \mid \delta(x)\},$$

and to any $\mathbb{A}$-arrow $f \colon \langle X, \delta \rangle \to \langle Y, \delta' \rangle$ the $\mathbb{MV}$-arrow that sends each $h \in \mathscr{L}(\langle Y, \delta' \rangle)$ into the map $h \circ f$.

# A duality for norm-complete MV-algebras

## The functor $\mathscr{M}$

Let $\mathscr{M}\colon \mathbb{MV} \to \mathbb{A}$ be the assignment that associates to each MV-algebra $A$, the pair $\langle \mathtt{Max}(A), \delta_A \rangle$, where $\mathtt{Max}(A)$ is maximal spectrum of $A$ and, for any $\mathfrak{m} \in \mathtt{Max}(A)$,

$$\delta_A(\mathfrak{m}) := \begin{cases} n & \text{if } A/\mathfrak{m} \text{ has } n+1 \text{ elements} \\ 0 & \text{otherwise.} \end{cases}$$

Let also $\mathscr{M}$ assign to every MV-homomorphism $h\colon A \to B$ the map that sends every $\mathfrak{m} \in \mathscr{M}(B)$ into its inverse image under $h$, in symbols $\mathscr{M}(h)(\mathfrak{m}) = h^{-1}[\mathfrak{m}] \in \mathtt{Max}(A)$.

# A duality for norm-complete MV-algebras

**Theorem**

*The functors $\mathscr{L}$ and $\mathscr{M}$ form a contravariant adjunction.*

So, what is left to do in order to find a duality is to characterise the fixed points on each side.

It is quite easy to see the the fixed points on the algebraic side are exactly the norm-complete MV-algebras.
What are the fixed points on the topological side?

# A-normal spaces

## Definition

An object $\langle X, \delta \rangle$ in A is said a-normal (for arithmetically normal) if for any pair of points $x, y \in X$ such that $x \neq y$,

1. if $\delta(y) \neq 0$, then, letting $d := \frac{1}{\delta(y)}$, there exists a family of open sets $\{O_q \mid q \in (0, d) \cap \mathbb{Q}\}$
2. if $\delta(y) = 0$, then there are infinitely many $d \in [0, 1]$ such that for each of those there exists a family of open sets $\{O_q \mid q \in (0, d) \cap \mathbb{Q}\}$

the families $\{O_p\}$ are such for any $p, q \in (0, d) \cap \mathbb{Q}$ and $n \in \mathbb{N}$

1. $p < q$ implies $\{x\} \subseteq O_p \subseteq \overline{O_p} \subseteq O_q \subseteq \overline{O_q} \subseteq \{y\}^c$.
2. $\delta^{-1}[\{n\}] \subseteq \bigcup \{O_p \mid \mathrm{den}(p) \mid n\}$.

# A-normal spaces



**Theorem**

*For any set $I$, the a-space $\langle [0,1]^I, \mathrm{den} \rangle$ is a-normal.*

**Lemma (A-normality is weakly hereditary)**

*If an a-space $\langle X, \delta \rangle$ is a-normal, then so are all its closed a-subspaces.*

**Theorem**

*An a-space $\langle X, \delta \rangle$ is a-normal if, and only if, there exist a set $I$ and an a-iso from $X$ into an a-subspace of $\langle [0,1]^I, \mathrm{den} \rangle$.*

# Sketch of the proof



The key step in the proof is to show that there are enough **good** functions to separate points:

### Theorem

*Let $\langle X, \delta \rangle$ be an a-normal space. For any pair of distinct points $x, y \in X$,*

1. *if $\delta(y) \neq 0$, then there exists a denominator respecting, continuous function $f\colon X \to [0,1]$ such that $f(x) = 0$ and $f(y) = \frac{1}{\delta(y)}$*
2. *if $\delta(y) = 0$, then there are infinitely many $d \in [0,1]$ such that for each of them there is a denominator respecting, continuous function $f\colon X \to [0,1]$ such that $f(x) = 0$ and $f(y) = d$*

# Sketch of the proof



Then we can use

**Theorem (Kelley's Embedding lemma)**

*Let $X$ and $Y$ be topological spaces and $\mathcal{F}$ be a family of functions from $X$ to $Y$. Suppose that all functions in $\mathcal{F}$ are* continuous *and that they* separate points. *Then the evaluation map* $\mathrm{ev}\colon X \to Y^{\mathcal{F}}$ *given by*

$$\mathrm{ev}(x) = (f(x))_{f\in\mathcal{F}}$$

*is continuous and injective.*

# Sketch of the proof



It is immediate to see that if all functions in $\mathcal{F}$ respect denominators, then so does $\mathtt{ev}$. Additionally, if $x \in X$, then

1. if $\delta(x) \neq 0$, then the value $\frac{1}{\delta(x)}$ is attained by some $f$ on $x$, so the function $\mathtt{ev}$ actually preserves $\delta(x)$;
2. if $\delta(x) = 0$, then there infinitely a-maps $f$ that on $x$ attain infinitely different values, hence $\mathtt{den}(\mathtt{ev}(x)) = 0$.

# Kakutani duality, for groups

**Corollary**

*The category of norm-complete archimedean $u\ell$-groups is dually equivalent to the full subcategory of $\mathbb{A}$ given by all a-normal spaces.*

**Corollary**

*The category of norm-complete MV-algebras is dually equivalent to the full subcategory of $\mathbb{A}$ given by all a-normal spaces.*

Thanks for your attention!