On simplicial semantics of modal predicate logics

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Introduction

Kripke semantics works well for propositional modal and intermediate logics, because “most of them” are complete; moreover, they have the fmp.

How to extend Kripke's semantics to predicate logics?

There may be different options. Anyway we need

- the frame of possible worlds (W,R) for interepreting $\Box$
- the system of non-empty individual domains $D=(D_u)_{u \in W}$ for interepreting quantifiers

To keep the standard laws of classical logic, this system should be *expanding* (Kripke, 1963).

Incompleteness in Kripke semantics

However, unlike the propositional case, in first-order predicate modal (and intuitionistic) logic there is a gap between syntax and semantics.

The standard Kripke frame semantics is inadequate - "most of" modal and intermediate predicate logics are Kripke-incomplete. The first such example was discovered by Hiroakira Ono (1973).

Witinin two decades many other examples were found, and a sequence of generalizations of Kripke semantics appeared:

Kripke frames << Kripke sheaves << Kripke bundles <<

Ghilardi's frames << Metaframes << Simplicial frames

Why simplicial semantics?

The goal was to recover completeness preserving the main idea of possible worlds. *So the concept of an individual had to be changed*.

Simplicial semantics seems a satisfactory solution: we have a rather general completeness result with respect to rather natural structures.

Main references (for old results)

S. Ghilardi. Presheaf semantics and independence results for some non-classical first-order logics. Archive for Mathematical Logic, 29: 125-136, 1989.

S. Ghilardi. Quantified extensions of canonical propositional intermediate logics. Studia Logica, 51:195-214, 1992.

D. Skvortsov, V. Shehtman. Maximal Kripke-type semantics for modal and superintuitionistic predicate logics. Annals of Pure and Applied Logic, 63:69-101, 1993.

D.Gabbay, V. Shehtman, D. Skvortsov. Quantification in Nonclassical Logic, Volume 1. Elsevier, 2009.

Formulas

Modal predicate formulas (the set MF) are built from:

- the countable set of individual variables $Var=\{v_1,v_2,\ldots\}$
- countable sets of n-ary predicate letters (for every $n \geq 0$)
- $\to, \bot, \vee, \wedge, \Box$.
- $\exists, \forall$

The connectives $\neg, \Diamond$ are derived.

No equality, constants or function symbols

Intuitionistic predicate formulas (the set IF)*:* modal formulas without $\Box$.

Variable and formula substitutions

$[y_1,\dots, y_n /x_1,\dots, x_n]$ simultaneously replaces all free occurrences of $x_1,\dots, x_n$ with $y_1,\dots, y_n$ (with renaming bound variables if necessary)

To obtain $[C(x_1,\dots, x_n,y_1,\dots, y_m)/P(x_1,\dots, x_n)]A$:

(1) rename all bound variables of A that coincide with the "new" parameters $y_1,\dots, y_m$ of C,

(2) replace every occurrence of every atom $P(z_1,\dots, z_n)$ with $[z_1,\dots, z_n /x_1,\dots, x_n]C$

Strictly speaking, all substitutions are defined up to congruence (α-equivalence): formulas are congruent if they can be obtained by "legal" renaming of bound variables

Modal and superintuitionistic logics

A modal predicate logic (mpl) is a set of modal formulas containing

- the classical propositional tautologies
- the axiom of **K:** $\Box(p \to q) \to (\Box p \to \Box q)$
- the classical predicate axioms

and closed under the rules

- Modus Ponens: $A, A \to B \,/\, B$
- Necessitation: $A \,/\, \Box A$
- Generalization: $A \,/\, \forall x A$
- Substitution: A/SA (for any formula substitution S)

A superintuitionistic predicate logic (spl) is a set of intuitionistic formulas containing the Heyting axioms and closed under (MP), (Gen), (Sub).

Modal and superintuitionistic logics -2

Propositional logics can be regarded as fragments of predicate logics (with only 0-ary predicate letters, without quantifiers).

L+Γ := the smallest logic containing (L and Γ)

K := the minimal modal propositional logic

H := the intuitionistic propositional logic

QL := the minimal predicate logic containing the propositional logic L

Kripke frame semantics for predicate logics

A propositional Kripke frame $F=(W, R)$ ($W \neq \emptyset$, $R \subseteq W^2$)
(and R is a preorder for the intuitionistic case)

A predicate Kripke frame: $\Phi = (F,D)$, where

$D=(D_u)_{u \in W}$ is an expanding family of non-empty sets:

if $u R v$, then $D_u \subseteq D_v$

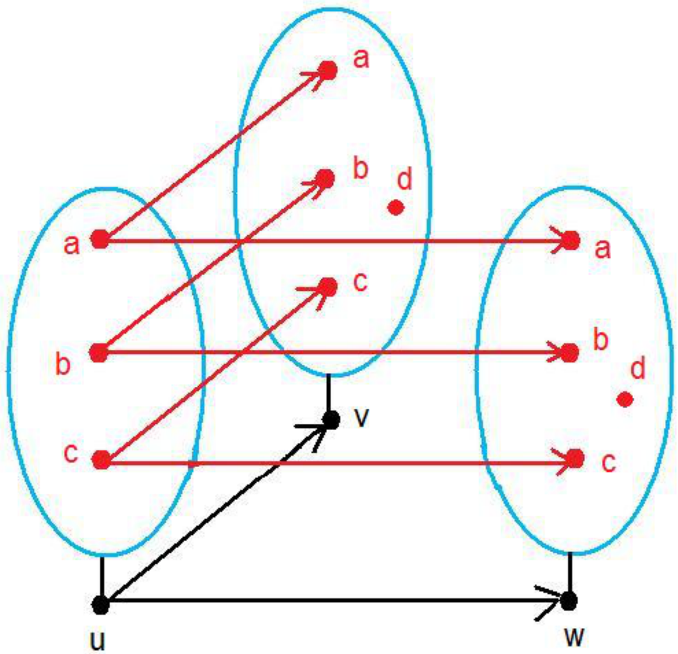
D_u is the domain at the world u

A Kripke model over Φ is a collection of classical models:

$M=(\Phi,\theta)$, where $\theta=(\theta_u)_{u \in W}$ is a valuation

$\theta_u(P)$ is an n-ary relation on D_u for each n-ary predicate letter P

In the intuitionistic case:
if $u R v$, then $\theta_u(P) \subseteq \theta_v(P)$


a
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w

Kripke frame semantics for predicate logics-2

A *variable assignment* at a world u is a function $\mathbf{a}/\mathbf{x}$ sending a finite list of different variables $\mathbf{x}$ (of length n) to a tuple $\mathbf{a} \in (D_u)^n$

For a function $\sigma : \{1,\ldots,m\} \to \{1,\ldots n\}$ put $\mathbf{x} \cdot \sigma := (x_{\sigma(1)},\ldots,x_{\sigma(m)})$.

<u>Def</u> Forcing (truth) $M, u, \mathbf{a}/\mathbf{x} \vDash B$

at a world u under an assignment $\mathbf{a}/\mathbf{x}$ for a modal formula B with parameters in $\mathbf{x}$ is defined by induction. The nontrivial cases are:

- $M,u, \mathbf{a}/\mathbf{x} \vDash P(\mathbf{x} \cdot \sigma)$ iff $(\mathbf{a} \cdot \sigma) \in \theta_u(P)$ (for m-ary P)
- $M,u, \mathbf{a}/\mathbf{x} \vDash \Box B$ iff for any v, uRv implies $M,v, \mathbf{a}/\mathbf{x} \vDash B$
- $M,u, \mathbf{a}/\mathbf{x} \vDash \forall y B$ iff for any $d \in D_u$ $M,u, \mathbf{a}d/\mathbf{x}y \vDash B$ (if $y \notin \mathbf{x}$)
- $M,u, \mathbf{a}/\mathbf{x} \vDash \forall x_i B$ iff $M,u, (\mathbf{a}-a_i)/(\mathbf{x}-x_i) \vDash \forall x_i B$

<u>Def</u> $M, u, \mathbf{a}/\mathbf{x} \Vdash B$ (for an intuitionistic B) iff

$M, u, \mathbf{a}/\mathbf{x} \vDash B^T$ (Gödel - Tarski translation)

Kripke frame semantics for predicate logics-3

<u>Def</u> Truth in a Kripke model:

$M \vDash A(x_1,\dots,x_n)$ iff for any $u \in W$ $M,u,/ \vDash \forall x_1 \dots \forall x_n A(x_1,\dots,x_n)$

(/ is an empty assignment)

Validity in a frame:

$\Phi \vDash A$ iff for any M over Φ, $M \vDash A$

<u>Soundness theorem</u>

ML$(\Phi):=\{A\in MF \mid \Phi \vDash A\}$ is an mpl

Logics of this form are called *Kripke-complete*

In the intuitionistic case we obtain an spl

IL$(\Phi):=\{A\in IF \mid \Phi \Vdash A\}$

Kripke completeness

For logics of the form **QL** not so many examples are known:

- for standard logics L (classical results by Kripke, Gabbay, Cresswell et al.) **K, T, D, B, K4, S4, S5**
(with the axioms for reflexivity, transitivity, symmetry, seriality)

- for other cases, with more sophisticated proofs

S4.2 = S4 + $\Diamond\Box A \rightarrow \Box\Diamond A$ confluent frames (Ghilardi&Corsi,1989)

S4.3 = S4 + $\Box(\Box A \wedge A \rightarrow B) \vee \Box(\Box B \wedge B \rightarrow A)$ linearly ordered frames (Corsi,1989)

and some others, see our book (2009), Ch.6.

"In any case, such logics should be *very rare*" (Ghilardi, 1991).

Kripke incompleteness

In fact, in many (continuum) cases **QL** are Kripke-incomplete

E.g. for L= **GL** (Montagna, 1984)

for all nontivial extensions of **S4.1 = S4**+ $\Box\Diamond A \rightarrow \Diamond\Box A$ (Ghilardi, 1991)

Ghilardi's functor semantics

Ghilardi's frame: $\Phi = (F, D, \mathscr{F})$, where

$F=(W,R)$ is a propositional transitive Kripke frame,
$D=(D_u)_{u\in W}$ is a disjoint family of non-empty sets,
$\mathscr{F} = (\mathscr{F}(u,v))_{uRv}$ is a family of non-empty sets of functions
$f: D_u \rightarrow D_v$ for every $f \in \mathscr{F}(u,v)$
(f is a "transition function" for individuals from u to v),
such that

- $uRvRw$ & $f\in \mathscr{F}(u,v)$ & $g\in \mathscr{F}(v,w) \Rightarrow g\cdot f \in \mathscr{F}(u,w)$
- $uRu \Rightarrow id(D_u)\in \mathscr{F}(u,u)$

A model over Φ is $M=(\Phi,\theta)$, where $\theta=(\theta_u)_{u\in W}$
$\theta_u(P)$ is an n-ary relation on D_u for n-ary P

Ghilardi's semantics-2

$M, u, \mathbf{a} / \mathbf{x} \vDash B$

is defined as in Kripke semantics, with the only difference

for $\Box$:

- $M, u, \mathbf{a} / \mathbf{x} \vDash \Box B$ iff for any v with uRv, for any $f \in \mathscr{F}(u,v)$

  $M, v, (f\cdot \mathbf{a}) / \mathbf{x} \vDash B$

(where $f\cdot (a_1,\ldots, a_n) := (f(a_1),\ldots, f(a_n))$).

Similarly for the intuitionistic case and intuitionistic models:

where

$\mathbf{a} \in \theta_u(P)\ \&\ f \in \mathscr{F}(u,v) \Rightarrow f\cdot \mathbf{a} \in \theta_v(P)$

Ghilardi's semantics-3

Truth in a model:

$M \vDash A(x_1,\dots,x_n)$ iff for any $u \in W$ $M,u,/ \vDash \forall x_1 \dots \forall x_n A(x_1,\dots,x_n)$

Validity in a frame: $\Phi \vDash A$ iff for any M over Φ, $M \vDash A$

Def (*shifts*) A^n is obtained from A by substituting $P(\mathbf{x},\mathbf{z})$ for $P(\mathbf{x})$ for every predicate letter P (where $\mathbf{z}$ is a fixed list of new n variables).

Strong validity in a frame:

$\Phi \vDash^+ A$ iff for any n $\Phi \vDash A^n$.

Soundness theorem (Skvortsov)

$\mathbf{ML}(\Phi) := \{A \in MF \mid \Phi \vDash^+ A\}$ is an mpl

Logics of this form are called *complete in Ghilardi's semantics.*

Ghilardi's semantics-4

Similarly we obtain a superintuitionistic logic for an S4-frame Φ

$$\mathbf{IL}(\Phi) := \{A \in IF \mid \Phi \Vdash^{+} A\}$$

Completeness theorem (Ghilardi, 1992)

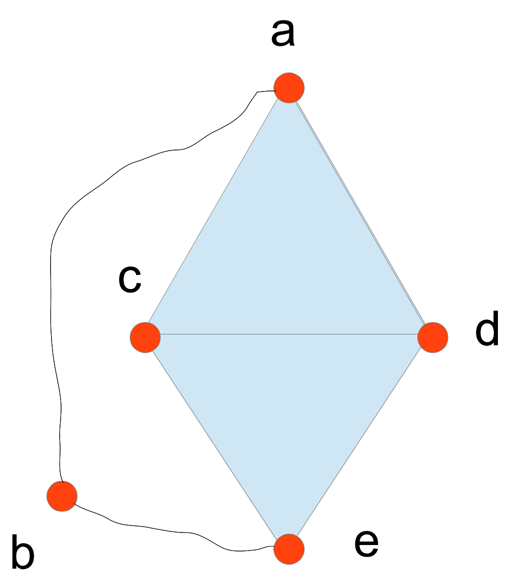
*If L is a canonical superintuitionistic propositional logic, then **QL** is complete in Ghilardi's semantics.*

Def L is *canonical* if it is valid in every canonical frame with arbitrarily many propositional letters ("d-persistence").

As we shall see later, this theorem does not extend to modal logics

Simplicial complexes

Geometric simplicial complex



Abstract simplicial complex

{acd, cde, ac, ad, cd, de, ce, ab, be, a,b,c,d,e}

$$X \in S \ \& \ Y \subset X \Rightarrow Y \in S$$

Simplicial sets

(J.P. May, 1967)

Δ is the category:

Ob **Δ** = ω,

Δ(m,n) = (non-strict) monotonic maps (m+1) → (n+1)

A *simplicial set* is a contravariant functor X: $\mathbf{\Delta}^{\circ} \rightsquigarrow$ **SET**

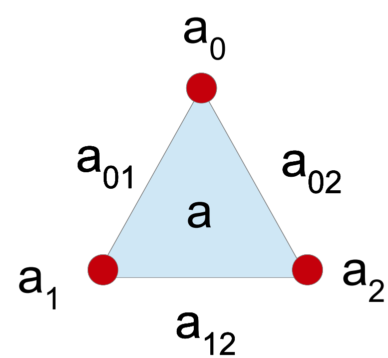
X(n) is the set of n-dimensional simplices

For every f ∈ **Δ**(m,n), X(f): X(n) → X(m) is a face map selecting an m-dimensional face of an n-dimensional simplex (it may be degenerate – if f is not injective)

Simplicial sets-2

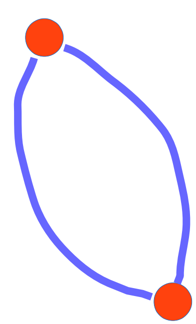
Example: If $a \in \mathbf{X}(2)$ is a triangle,

$f \in \mathbf{\Delta}(1,2)$, $f(0)=0$, $f(1)=2$, then $X(f)$ chooses the second side of a (it can be denoted by a_{02}) .



Two differences between simplicial complexes and simplicial sets:

- simplicial sets include degenerate simplices (such as a_{11}, a_{002})
- in simplicial sets two different simplices may have the same proper faces.



Simplicial frames

Introduced by Dmitry Skvortsov (1990); the first publication (abstract) in 1991; the paper in 1993.

In these publications simplicial frames we called 'Kripke metaframes'. Later the names were changed:

Kripke metaframes >> Simplicial frames

Cartesian metaframes >> Kripke metaframes

A *simplicial frame* is a modification of a simplicial set.

- $\mathbf{\Delta}$ is replaced by another category $\mathbf{\Sigma}$

Ob $\mathbf{\Sigma}$ = ω,

$\mathbf{\Sigma}_{mn}$ = *all* maps $I_m \to I_n$ (where $I_n=\{1,\ldots,n\}$, $I_0=\varnothing$).

Let $\Sigma = \cup\{ \mathbf{\Sigma}_{mn} \mid m,n\geq 0\}$

- Accessibility relations are also involved

Simplicial frames-2

Roughly, a simplicial frame is a layered Kripke frame. The worlds are at level 0, individuals at level 1 (0–simplices), abstract n–tuples of individuals at level n ((n–1)–simplices).

Def A *simplicial frame* over a propositional Kripke frame F=(W,R) is $\Phi = (F, D, \mathbf{R}, \pi)$, where

- $D=(D^n)_{n\geq 0}$, $\mathbf{R}=(R^n)_{n\geq 0}$, (D^n,R^n) is a propositional frame, $(D^0,R^0) = F$,
- $\pi = (\pi_\sigma)_{\sigma\in\Sigma}$, $\pi_\sigma : D^n \to D^m$ for $\sigma \in \Sigma_{mn}$

$\Sigma_{0n} = \{\varnothing_n\}$ (the empty map).

$\pi_{\varnothing_n}$ sends every absract n–tuple to “its possible world”.

D^n_u denotes $(\pi_{\varnothing_n})^{-1}(u)$, the set of “n–tuples living in the world u”,

Simplicial frames-3

A *Kripke metaframe* is a simplicial frame, in which the abstract tuples are real:

$D^n_u = (D^1_u)^n$, the n–th Cartesian power of D^1_u

and $\pi_\sigma(\mathbf{a}) = \mathbf{a} \cdot \sigma$.

Ghilardi's frame $(F, D, \mathcal{F})$ corresponds to a metaframe $(F, D, \mathbf{R}, \pi)$, where

- $(D^0, R^0) = F$,
- $\mathbf{a} R^n \mathbf{b}$ iff

$$\exists\, u,v\ \exists\, f\ (uRv\ \&\ f \in \mathcal{F}(u,v)\ \&\ \mathbf{a} \in D^n_u\ \&\ \mathbf{b} \in D^n_v\ \&\ \mathbf{b} = f \cdot \mathbf{a}).$$

Simplicial frames-4

<u>Definition</u> A *valuation* in **F** is a function ξ such that $\xi_u(P) \subseteq D^n{}_u$ for every n-ary predicate letter P.

M=(**F**, ξ) is a *simplicial model* over **F**.

An *assignment* of length n at u is a pair (**x, a**), where **x** is a list of different variables of length n, $\mathbf{a} \in D^n{}_u$. (We still denote it by **a**/**x**.)

Simplicial frames-5

Definition (truth of a formula A in a simplicial model M at u under an assignment (**x, a**) involving the formula parameters)

This makes sense if **a** lives in u

Notation: M, **a**/**x**, u ⊨ A.

M, **a**/**x**, u ⊨ P(**x**· σ) iff $\pi_\sigma(\mathbf{a}) \in \xi_u(P)$,

M, **a** /**x**, u ⊨ □B (for **a** $\in D^n{}_u$) iff

∀v,**b** (uRv & **b** $\in D^n{}_v$ & **a**R^n**b** ⇒ M, **b**/**x**, v ⊨ B)

M, **a** /**x**, u ⊨ ∃y B (for y ∉ **x**, **a** $\in D^n{}_u$) iff

$\exists \mathbf{c} \in D^{n+1}{}_u\ (\pi_{\delta_{n+1}}(\mathbf{c}) = \mathbf{a}$ & M,**c**/**x**y ⊨ B),

M, **a** /**x**, u ⊨ $\exists x_i$ B iff M, $\pi_{\delta_i}(\mathbf{a})/(\mathbf{x}\cdot \delta_i)$, u ⊨ B, where δ_i is the monotonic inclusion map $I_n \to I_{n+1}$ skipping i.

Simplicial frames-6

Truth in a model:

$M \vDash A(x_1,\dots,x_n)$ iff for any $u \in W$ $M,u,/ \vDash \forall x_1 \dots \forall x_n A(x_1,\dots,x_n)$

Validity in a frame: $\Phi \vDash A$ iff for any M over Φ, $M \vDash A$

Strong validity in a frame: $\Phi \vDash^+ A$ iff for any n $\Phi \vDash A^n$.

Soundness theorem (Skvortsov)

$\mathbf{ML}(\Phi) := \{A \in MF \mid \Phi \vDash^+ A\}$ is an mpl if Φ satisfies the conditions

- $\pi_{\varnothing 1}$ is surjective,
- $\pi_{\sigma \cdot \tau} = \pi_\tau \cdot \pi_\sigma$; $\pi_{id(I_n)} = id(D^n)$. [id(X) is the identity map on X]
- for $\sigma \in \Sigma_{mn}$ $\pi_\sigma : (D^n, R^n) \to (D^m, R^m)$ is a p-morphism, i.e.,

$$\pi_\sigma(R^n(\mathbf{a})) = R^m(\pi_\sigma(\mathbf{a})) \text{ for any } \mathbf{a} \in D^n.$$

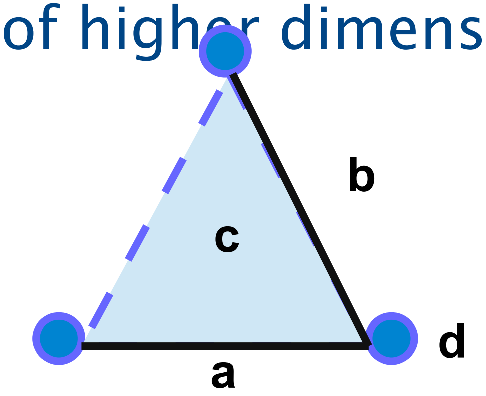
Simplicial frames-7

- if $\pi_{\delta_{m+1}}(\mathbf{b}) = \pi_\sigma(\mathbf{a})=\mathbf{d}$, $\sigma \in \Sigma_{mn}$, then

  for some $\mathbf{c} \in D^{n+1}$ $\pi_{\sigma_+}(\mathbf{c})=\mathbf{b}$ & $\pi_{\delta_{n+1}}(\mathbf{c}) = \mathbf{a}$.

($\sigma_+ \in \Sigma_{m+1,n+1}$ extends σ by $\sigma_+(m+1)=n+1$)

In particular, this means that two simplices with a common face are faces of a simplex of higher dimension:



In metaframes: $\mathbf{d}=a_{\sigma(1)}\ldots a_{\sigma(m)}$, $\mathbf{b}=\mathbf{d}b_{m+1}$; then $\mathbf{c}=\mathbf{a}b_{m+1}$

Completeness theorem

Logics of the form **ML**(Φ) are called *complete in simplicial semantics*.

Theorem (Skvortsov & Sh., 1993) If a propositional modal logic L is canonical, then **QL** is complete in simplicial semantics.

For the proof we construct the *canonical simplicial model*; its n-th level consists of n-types in **QL** (maximal consistent sets of formulas in parameters $x_1,\dots,x_n$).

a R^n **b** iff for any A, $\Box A \in$ **a** implies $A \in$ **b**

$\pi_\sigma(\mathbf{a}) := \{A(\mathbf{x}) \mid A(\mathbf{x}\cdot\sigma) \in \mathbf{a}$

$\xi_u(P) := \{\mathbf{a} \in D^n_u \mid P(x_1,\dots,x_n) \in \mathbf{a}\}$ for n-ary P

Incompleteness theorem

Theorem (Sh., 2018) If a propositional modal logic L is between **K4.1** and **SL4**, then **QL** is incomplete with respect to metaframes.

K4.1 $= \mathbf{K} + \Box p \to \Box\Box p + \Box\Diamond p \to \Diamond\Box p$

SL4 $= \mathbf{K} + \Box p \to \Box\Box p + \Box p \leftrightarrow \Diamond p$

(this is the logic of the two-world frame

with the first world irreflexive and the second one reflexive)

Corollary The logics **QK4.1**, **QSL4** are complete in simplicial semantics, but incomplete w.r.t. metaframes (and so, in Ghilardi's semantics).

Incompleteness theorem-2

Idea of the proof

Consider the formula

$A = \Box\Diamond\forall x\forall y(\Box\Diamond P(x,y) \to \exists x'\exists y'(P(x',y') \wedge \Diamond P(x,y')))$.

1. If a metaframe $\Phi \vDash^{+}$ **K4.1**, then $\Phi \vDash^{+} A$.

2. There is a simplicial frame $\Phi \vDash^{+}$ **SL4** such that

$\Phi \nvDash A$.

Thank you!