Topological Completeness of Modal Logics for Spaces Constructed from Trees

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Setting the Scene

Signature

- Propositional letters: $p, q, r, \ldots, p_0, p_1, p_2, \ldots$
- Classical connectives: $\neg$ and $\rightarrow$
- Modal connective: $\Box$
- Typical abbreviations: $\Diamond\varphi := \neg\Box\neg\varphi$, $\varphi \vee \psi := \neg\varphi \rightarrow \psi$, and $\varphi \wedge \psi := \neg(\varphi \rightarrow \neg\psi)$

Topological Interpretation

Given a space X:

- Letters $\Rightarrow$ subsets of X
- Classical connectives $\Rightarrow$ Boolean operations in $\wp(X)$
- Modal box $\Rightarrow$ interior operator $\mathbf{i}$ of X;
  hence, diamond $\Rightarrow$ closure operator $\mathbf{c}$ of X

Topological Semantics and S4

Valid Modal formulas

Call a formula φ *valid* in X provided it evaluates to X for any interpretation of the letters; in symbols $X \Vdash \varphi$.

Valid Formulas	Corresponding Property
$\Box\top \leftrightarrow \top$	$\mathbf{i}X = X$
$\Box p \to p$	$\mathbf{i}A \subseteq A$
$\Box p \to \Box\Box p$	$\mathbf{i}A \subseteq \mathbf{ii}A$
$\Box(p \wedge q) \leftrightarrow (\Box p \wedge \Box q)$	$\mathbf{i}(A \cap B) = \mathbf{i}A \cap \mathbf{i}B$

Put $\mathrm{Log}(X) = \{\varphi \mid X \Vdash \varphi\}$

Theorem

For any space X, $\mathrm{Log}(X)$ is a normal extension of S4

Relating Topological and Kripke Semantics

Generalizing Kripke Semantics for S4

- An S4-*frame* is $\mathfrak{F} = (W, R)$ where R is a reflexive and transitive relation on W
- An R-*upset* in $\mathfrak{F}$ is $U \subseteq W$ such that $w \in U$ and wRv imply $v \in U$
- The set of R-upsets forms the *Alexandroff* topology τ_R on W

A Result, a Consequence, and an Observation

- For an S4-frame $\mathfrak{F} = (W, R)$, $\mathfrak{F} \Vdash \varphi$ iff $(W, \tau_R) \Vdash \varphi$
- A logic extending S4 that is Kripke complete is topologically complete
- Such topological completeness is typically not for spaces satisfying "nice" separation axioms; e.g. Tychonoff spaces

Some Motivating Completeness Results I.1

McKinsey and Tarski 1944

For a separable crowded metrizable space X, $\text{Log}(X) = \text{S4}$

S4

Some Motivating Completeness Results I.2

Rasiowa and Sikorski 1963

For a crowded metrizable space X, $\mathrm{Log}(X) = \mathsf{S4}$

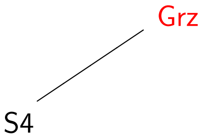
S4

Some Motivating Completeness Results II

Abashidze 1987 and Blass 1990 (independently)

For any ordinal space $\alpha \geq \omega^\omega$, $\mathsf{Log}(\alpha) = \mathsf{Grz}$

$\mathsf{Grz} := \mathsf{S4} + \Box(\Box(p \to \Box p) \to p) \to p$

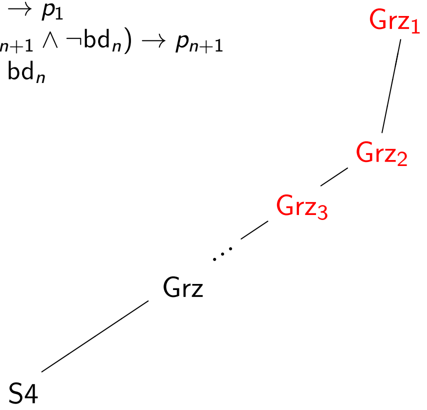


Some Motivating Completeness Results III

Abashidze 1987 (Bezhanishvili and Morandi 2010)

For an ordinal α such that $\omega^{n-1} + 1 \leq \alpha \leq \omega^n$, $\mathsf{Log}(\alpha) = \mathsf{Grz}_n$

$$\begin{aligned} \mathsf{bd}_1 &:= \Diamond\Box p_1 \to p_1 \\ \mathsf{bd}_{n+1} &:= \Diamond(\Box p_{n+1} \wedge \neg \mathsf{bd}_n) \to p_{n+1} \\ \mathsf{Grz}_n &:= \mathsf{Grz} + \mathsf{bd}_n \end{aligned}$$

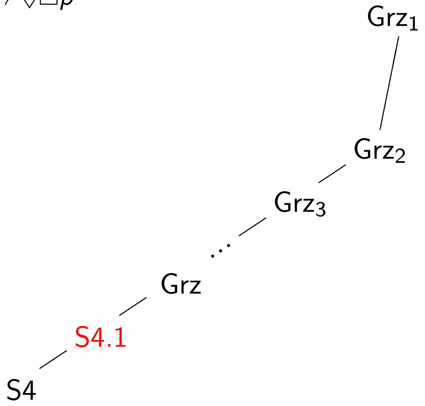


Some Motivating Completeness Results IV

Bezhanishvili and Harding 2012

The logic of the Pelczynski compactification of ω is S4.1

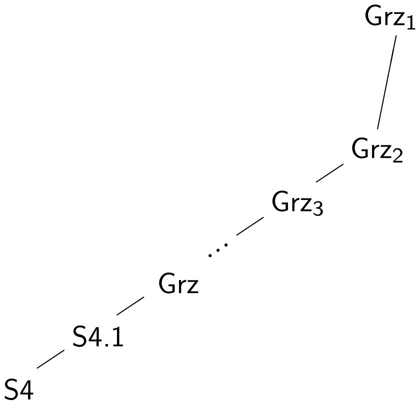
$\mathsf{S4.1} := \mathsf{S4} + \Box\Diamond p \to \Diamond\Box p$



Some Motivating Completeness Results V

Bezhanishvili, Gabelaia, and Lucero-Bryan 2015

Metrizable spaces yield exactly these logics: S4, S4.1, Grz, or Grz_n



A Short Interlude

Question

What sort of Tychonoff space X has $\mathrm{Log}(X)$ not listed above?

An Obvious Answer

Non-metrizable spaces... (as a typical example)
Our focus: Extremally disconnected Tychonoff spaces

Definition

A space X is *extremally disconnected* (ED) provided the closure of each open set is open

Lemma

X is ED iff $X \Vdash \Diamond\Box p \to \Box\Diamond p$

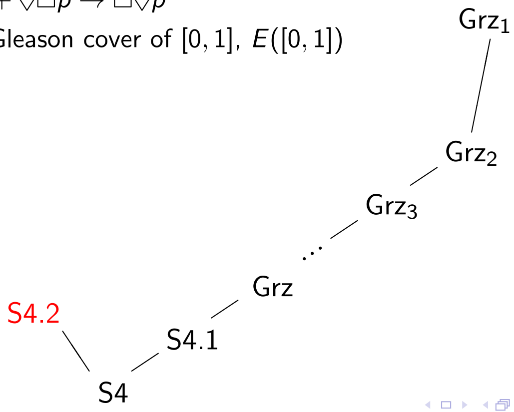
Some Motivating Completeness Results VI.1

Bezhanishvili and Harding 2012

Let X be an infinite ED Stone space.
If X is not weakly scattered, then $\text{Log}(X) = \text{S4.2}$

$\text{S4.2} := \text{S4} + \Diamond\Box p \to \Box\Diamond p$

Example: Gleason cover of $[0,1]$, $E([0,1])$



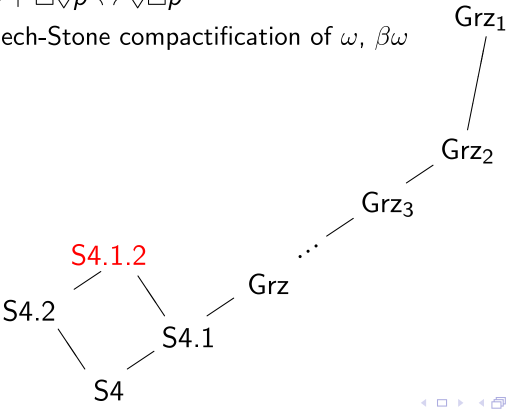
Some Motivating Completeness Results VI.2

Bezhanishvili and Harding 2012

Let X be an infinite ED Stone space.
If X is weakly scattered, then $\mathsf{Log}(X) = \mathsf{S4.1.2}$

$\mathsf{S4.1.2} := \mathsf{S4} + \Box\Diamond p \leftrightarrow \Diamond\Box p$

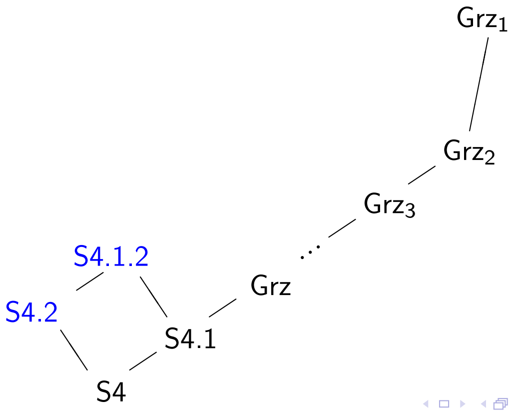
Example: Čech-Stone compactification of ω, $\beta\omega$



Some Motivating Completeness Results VI.3

Bezhanishvili and Harding 2012

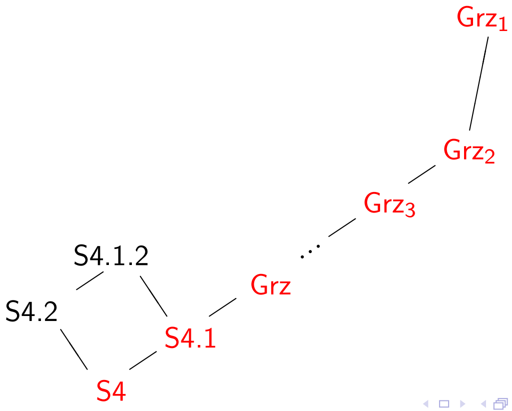
Note regarding the two previous results: require set theoretic axioms beyond ZFC (control cardinality of MAD families)



To Come...

Via a Unified Approach to Topologizing Trees

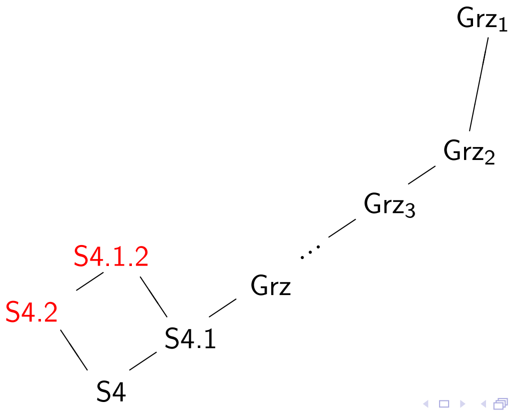
We realize many previous results regarding both metrizable and non-metrizable spaces for the highlighted logics



To Come...

Via a Unified Approach to Topologizing Trees

Within ZFC, extend to ED Tychonoff spaces yielding analogous results for the highlighted logics (but not for $E([0,1])$ or $\beta\omega$)



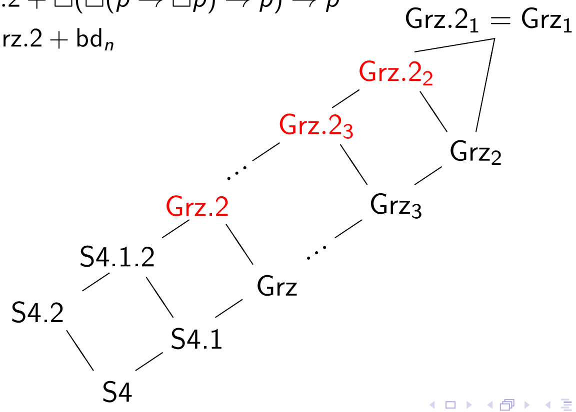
To Come...

Via a Unified Approach to Topologizing Trees

Further extend to yield new completeness results with respect to Tychonoff spaces regarding the highlighted logics

$\mathsf{Grz.2} := \mathsf{S4.2} + \Box(\Box(p \to \Box p) \to p) \to p$

$\mathsf{Grz.2}_n := \mathsf{Grz.2} + \mathsf{bd}_n$



Formalization

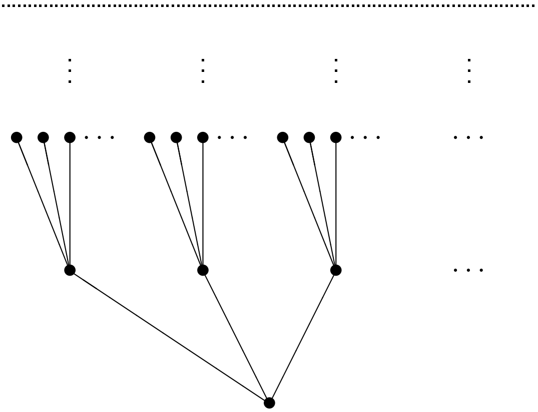
Definition

- κ nonzero cardinal (when infinite viewed as initial ordinal)
- A *sequence in* κ: $\sigma : \alpha \to \kappa$ for $\alpha \leq \omega$
  - *finite* of length α when $\alpha < \omega$
  - *infinite* when $\alpha = \omega$
- The *initial segment order* on $L_\kappa := \{\sigma \mid \sigma \text{ a sequence in } \kappa\}$:

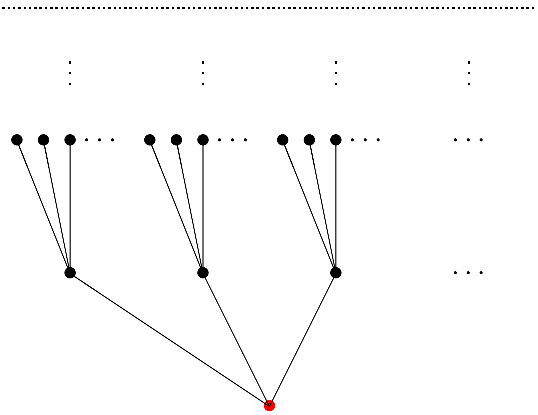
$$\sigma \leq \varsigma \text{ iff } \sigma(n) = \varsigma(n) \text{ for all } n \in \mathrm{Dom}(\sigma) \leq \mathrm{Dom}(\varsigma)$$

- The *κ-ary tree with limits* is $(L_\kappa, \leq)$
  Note: the infinite sequences are the "limits" (also the leafs)

A Picture of $(L_\omega, \leq)$ and Key Players

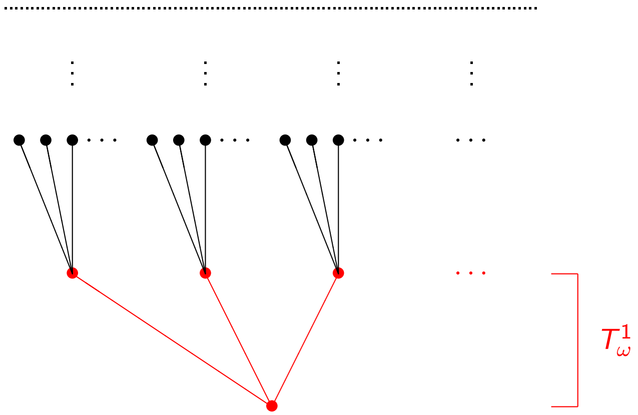


A Picture of $(L_\omega, \leq)$ and Key Players: T_ω^n Sequences of Length $\leq n$



$\sqsupset$ T_ω^0

A Picture of $(L_\omega, \leq)$ and Key Players: T^n_ω Sequences of Length $\leq n$



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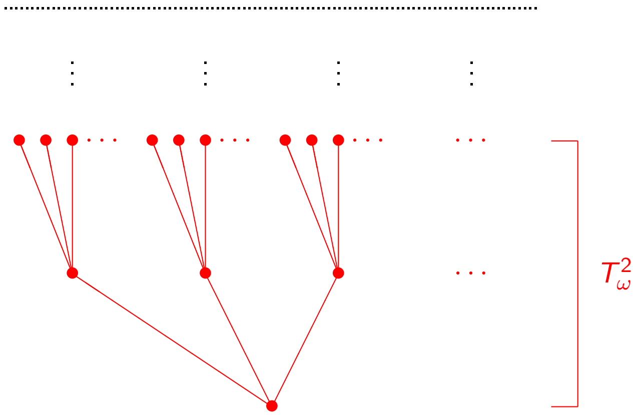
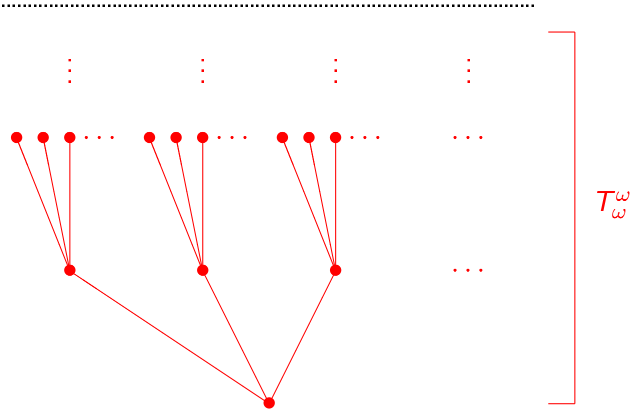


A Picture of $(L_\omega, \leq)$ and Key Players: T_ω^ω All Finite Sequences



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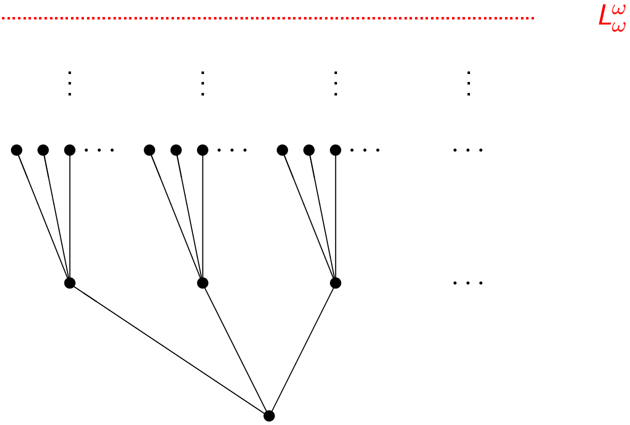


Some Topologies on L_κ

Definition

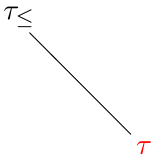
- $\uparrow\sigma := \{\varsigma \in L_\kappa \mid \sigma \leq \varsigma\}$
- $\tau_{\leq}$—the Alexandroff topology of $(L_\kappa, \leq)$
- $(L_\kappa, \tau_{\leq})$ is T_0 and compact, but not Tychonoff

$\tau_{\leq}$

Some Topologies on L_κ

Definition

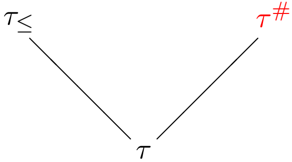
- $\mathscr{S} := \{\uparrow\sigma \mid \sigma \text{ is finite}\}$
- τ is the topology generated by $\mathscr{S}$
- (L_κ, τ) is a (non-Tychonoff) spectral space (sober & coherent)
- Compact opens are finite unions of elements in $\mathscr{S}$



Some Topologies on L_κ

Definition

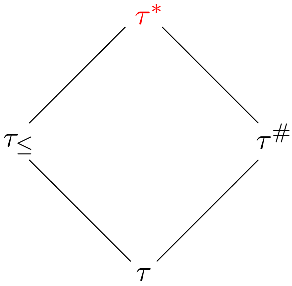
- $\mathscr{S} := \{\uparrow\sigma \mid \sigma \text{ is finite}\}$
- $\mathscr{B}$ is the Boolean algebra generated by $\mathscr{S}$
- $\tau^{\#}$ is the topology generated by $\mathscr{B}$ (patch topology of τ)
- $(L_\kappa, \tau^{\#})$ is a Stone space (compact, T_2, and zero-dimensional)



Some Topologies on L_κ

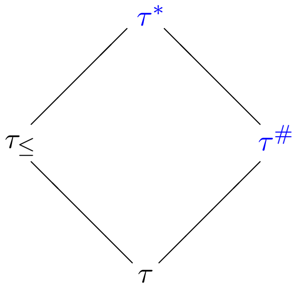
Definition

- $\mathscr{S} := \{\uparrow\sigma \mid \sigma \text{ is finite}\}$
- $\mathscr{A}$ is the Boolean σ-algebra generated by $\mathscr{S}$
- τ^* is the topology generated by $\mathscr{A}$ (σ-*patch* topology of τ)
- (L_κ, τ^*) is Tychonoff and $\bigcap_{n\in\omega} U_n \in \tau^*$ when each $U_n \in \tau^*$



Primary Topologies of Interest: $\tau^{\#}$ and τ^*

(for uncountable arity: $\kappa > \omega$)

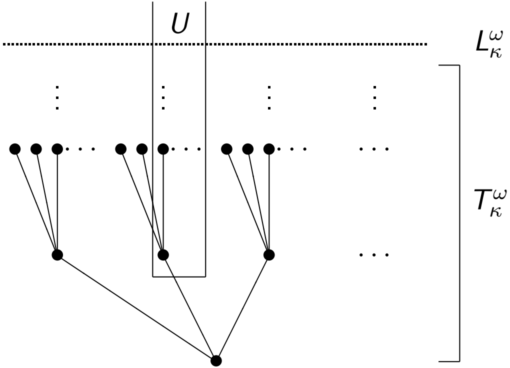


(for countable arity: $\kappa \leq \omega$)

Observations about $(L_\kappa, \tau^\#)$

Theorem

A basis for $(L_\kappa, \tau^\#)$ consists of sets of the form $\uparrow\sigma \setminus \bigcup_{i=0}^{n} \uparrow\sigma_i$ where $n \in \omega$, σ is finite, and each σ_i is a child of σ **(finitely many)**



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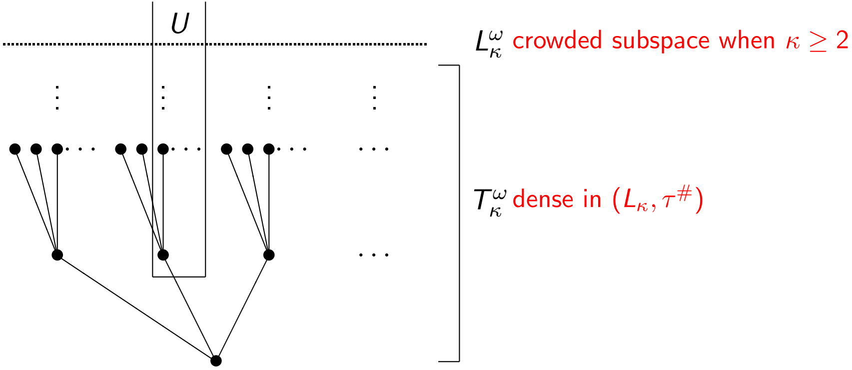

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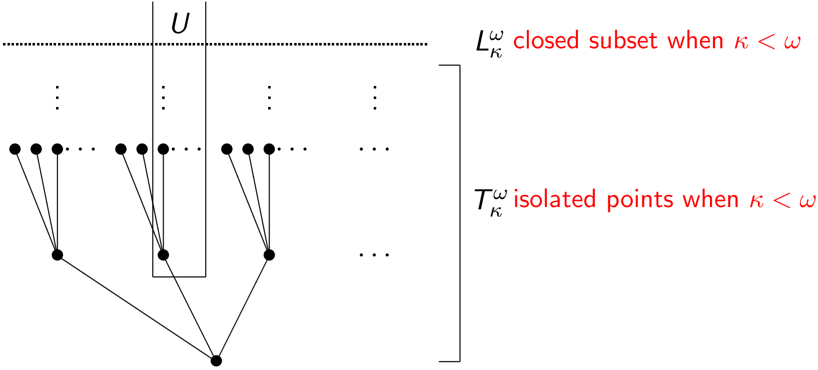

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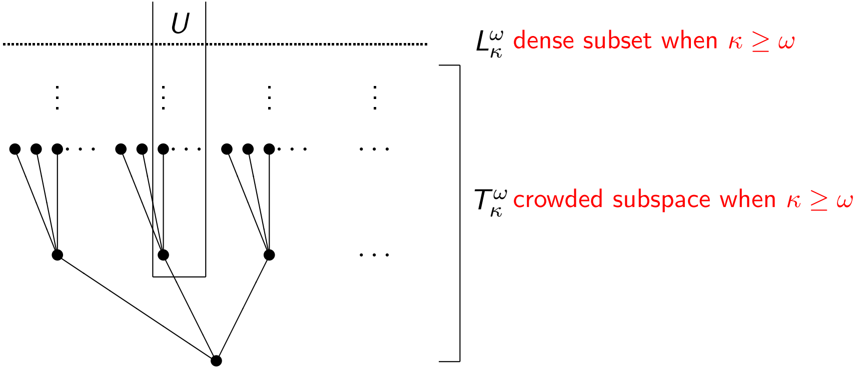

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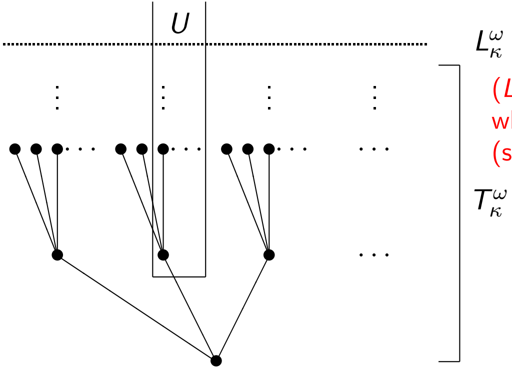

$(L_\kappa, \tau^\#)$ is metrizable when κ is countable (second-countable and regular)

Observations about $(L_\kappa, \tau^{\#})$

Theorem

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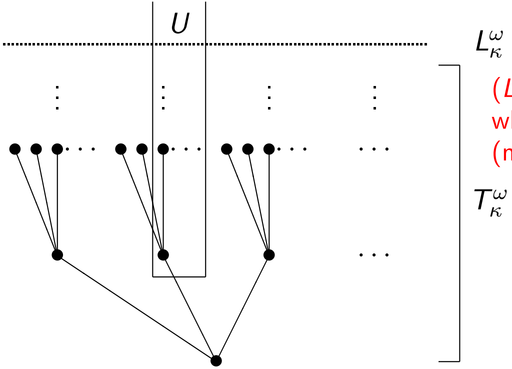

$(L_\kappa, \tau^{\#})$ is not metrizable when κ is uncountable (not first-countable)

$\kappa = 1$

$$(L_1, \tau^{\#}) \approx \omega + 1$$

$\kappa = 1$

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$\mathsf{Log}(L_1, \tau^{\#}) = \mathsf{Grz}_2$

$\kappa = 1$

$(L_1, \tau^{\#}) \approx \omega + 1$

$\text{Log}(L_1, \tau^{\#}) = \text{Grz}_2$

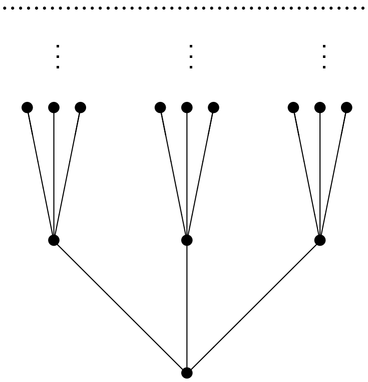
Subspaces L_1^{ω}, T_1^{ω}, and T_1^{n} are discrete

$\kappa = 1$

$(L_1, \tau^{\#}) \approx \omega + 1$
$\mathrm{Log}(L_1, \tau^{\#}) = \mathrm{Grz}_2$

Subspaces L_1^{ω}, T_1^{ω}, and T_1^{n} are discrete
$\mathrm{Log}(L_1^{\omega}) = \mathrm{Log}(T_1^{\omega}) = \mathrm{Log}(T_1^{n}) = \mathrm{Grz}_1$

$2 \leq \kappa < \omega$



$L^{\omega}_{\kappa} \approx \mathbf{C}$ (Brouwer's Theorem)

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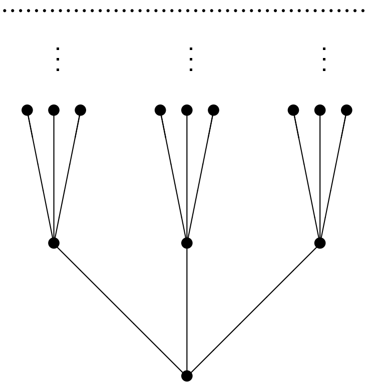
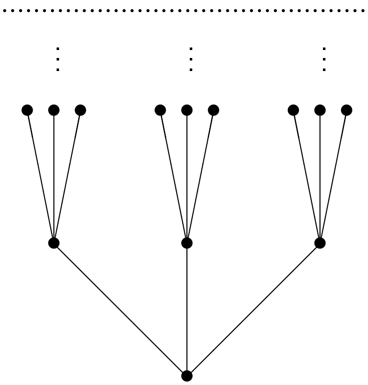


$L^{\omega}_{\kappa} \approx \mathbf{C}$ (Brouwer's Theorem)

$\mathrm{Log}(L^{\omega}_{\kappa}) = \mathrm{S4}$

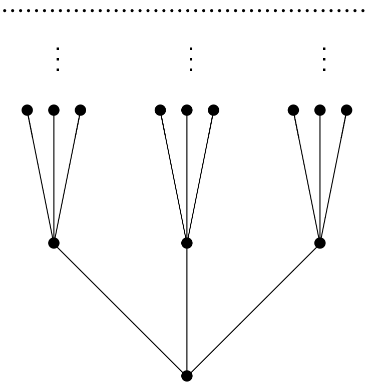
$2 \leq \kappa < \omega$



$L_\kappa^\omega \approx \mathbf{C}$ (Brouwer's Theorem)
$\mathrm{Log}(L_\kappa^\omega) = \mathrm{S4}$

Subspaces T_κ^ω and T_κ^n are discrete
$\mathrm{Log}(T_\kappa^\omega) = \mathrm{Log}(T_\kappa^n) = \mathrm{Grz}_1$

$2 \leq \kappa < \omega$



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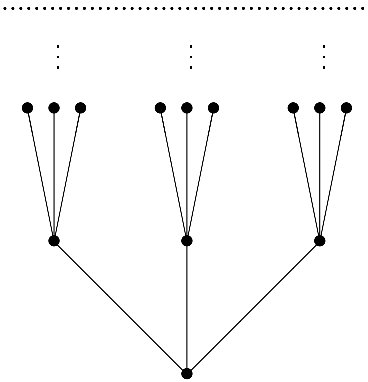
$(L_\kappa, \tau^\#)$ is homeomorphic to the Pelczynsky compactification of ω

$2 \leq \kappa < \omega$



$L_\kappa^\omega \approx \mathbf{C}$ (Brouwer's Theorem)
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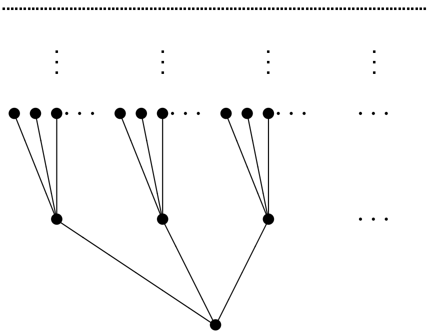
Subspaces T_κ^ω and T_κ^n are discrete
$\text{Log}(T_\kappa^\omega) = \text{Log}(T_\kappa^n) = \text{Grz}_1$

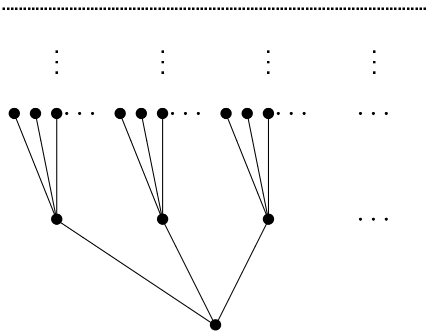
$(L_\kappa, \tau^\#)$ is homeomorphic to the Pelczynsky compactification of ω
$\text{Log}(L_\kappa, \tau^\#) = \text{S4.1}$

$\kappa = \omega$



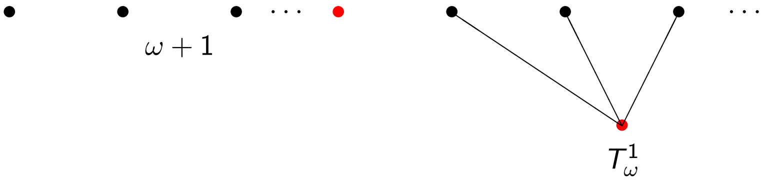
subspaces L_ω, L_ω^ω, and T_ω^ω are crowded

$\kappa = \omega$

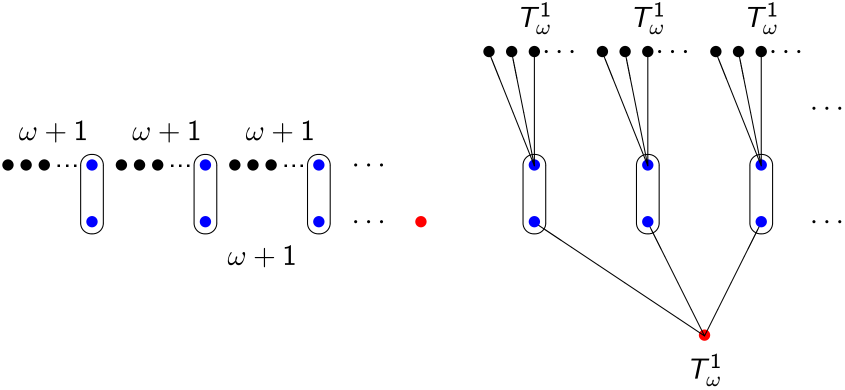


subspaces L_ω, L_ω^ω, and T_ω^ω are crowded
$\text{Log}(L_\omega) = \text{Log}(L_\omega^\omega) = \text{Log}(T_\omega^\omega) = \text{S4}$

$\kappa = \omega$ AND T_ω^n

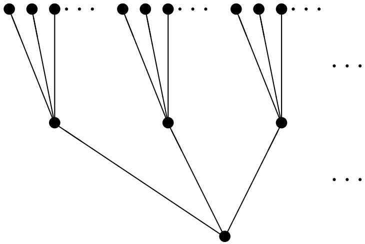


$\kappa = \omega$ AND T^n_ω



$\kappa = \omega$ AND T_ω^n

$\omega^2 + 1$



T_ω^2

$\kappa = \omega$ AND T^n_ω

THEOREM

- $T^n_\omega \approx \omega^n + 1$
- $\mathrm{Log}(T^n_\omega) = \mathrm{Grz}_{n+1}$ for $n \geq 0$
- $\mathrm{Log}\left(\bigoplus_{n\geq 0} T^n_\omega\right) = \bigcap_{n\geq 0} \mathrm{Grz}_{n+1} = \mathrm{Grz}$

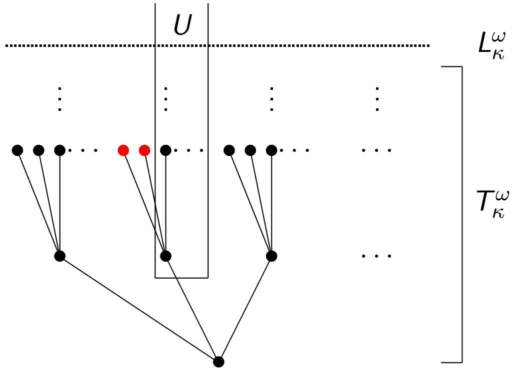
REMARK

This merely a new perspective of some known results...
Moving to the ED setting requires leaving the metric setting...
To do so, our current machinery requires an increase in cardinality

Observations about (L_κ, τ^*)

Theorem

A basis for (L_κ, τ^*) consists of sets of the form $\uparrow\sigma \setminus \bigcup_{i\in\omega} \uparrow\sigma_i$ where $n \in \omega$, σ is finite, and each σ_i is a child of σ **(countably many)**



Observations about (L_κ, τ^*)

Theorem

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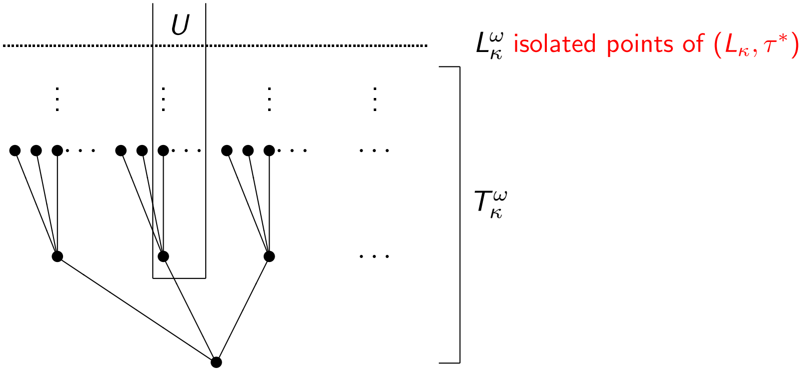

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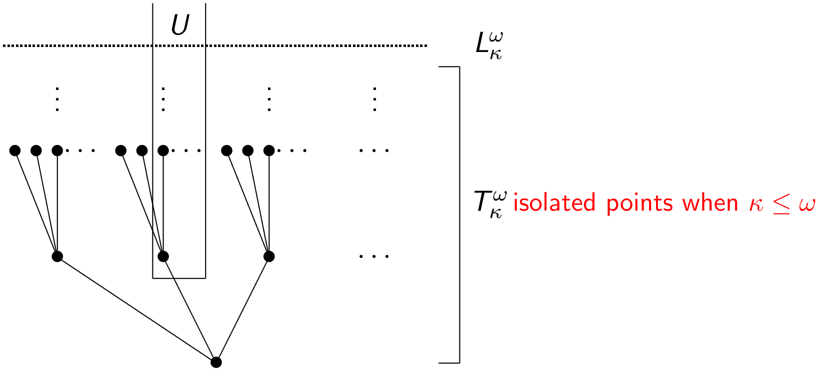

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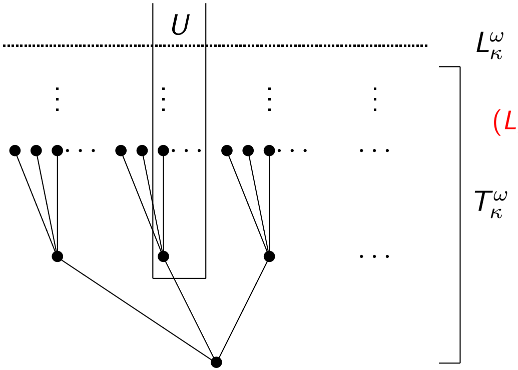
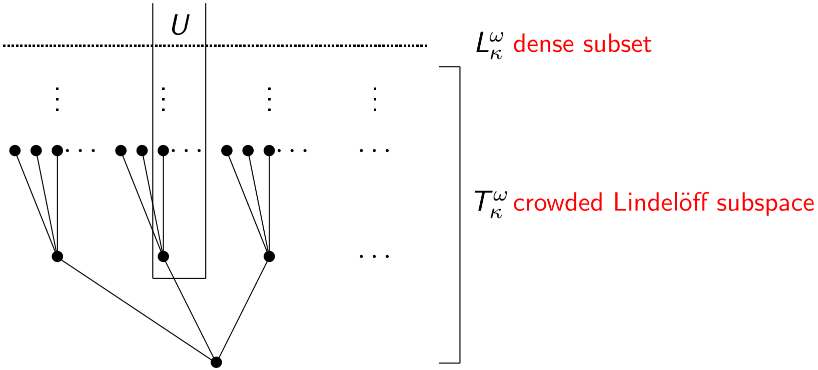


(L_κ, τ^*) is discrete when $\kappa \leq \omega$

Observations about (L_κ, τ^*) for $\kappa > \omega$

Theorem

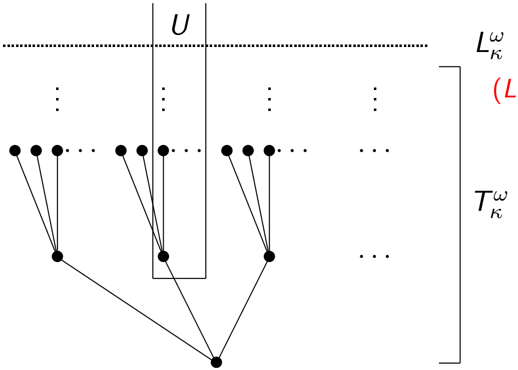
A basis for (L_κ, τ^*) consists of sets of the form $\uparrow\sigma \setminus \bigcup_{i\in\omega} \uparrow\sigma_i$ where $n \in \omega$, σ is finite, and each σ_i is a child of σ **(countably many)**



Observations about (L_κ, τ^*) for $\kappa > \omega$

Theorem

A basis for (L_κ, τ^*) consists of sets of the form $\uparrow\sigma \setminus \bigcup_{i\in\omega} \uparrow\sigma_i$ where $n \in \omega$, σ is finite, and each σ_i is a child of σ **(countably many)**

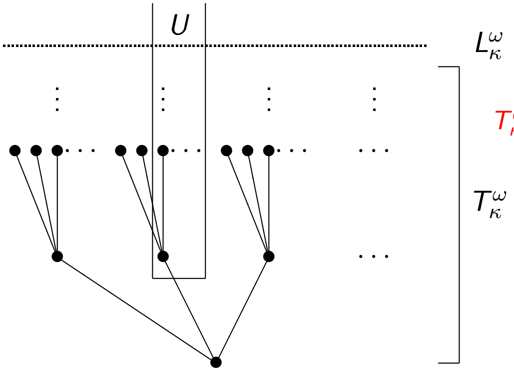


(L_κ, τ^*) is a non-scattered
weakly scattered
non-Lindellöff P-space

Observations about (L_κ, τ^*) for $\kappa > \omega$

Theorem

A basis for (L_κ, τ^*) consists of sets of the form $\uparrow\sigma \setminus \bigcup_{i\in\omega} \uparrow\sigma_i$ where $n \in \omega$, σ is finite, and each σ_i is a child of σ **(countably many)**



T_κ^ω is a crowded Lindellöff P-space

Logics of (L_κ, τ^*) and T_κ^ω

Theorem

- $\mathrm{Log}(T_\kappa^\omega) = \mathrm{S4}$
- $\mathrm{Log}(L_\kappa, \tau^*) = \mathrm{S4.1}$

Proof sketch

Soundness:

- $\mathrm{S4} \subseteq \mathrm{Log}(T_\kappa^\omega)$ always
- $\mathrm{S4.1} \subseteq \mathrm{Log}(L_\kappa, \tau^*)$ because (L_κ, τ^*) is weakly scattered

Logics of (L_κ, τ^*) and T_κ^ω

Theorem

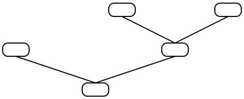
- $\mathrm{Log}(T_\kappa^\omega) = \mathrm{S4}$
- $\mathrm{Log}(L_\kappa, \tau^*) = \mathrm{S4.1}$

Proof sketch

Completeness:

- Each ‘good’ S4-frame is an interior image of T_κ^ω
- Each ‘good’ S4.1-frame is an interior image of (L_κ, τ^*)

‘good’ S4-frame



Logics of (L_κ, τ^*) and T^ω_κ

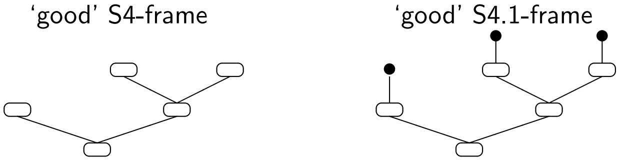
Theorem

- $\mathrm{Log}(T^\omega_\kappa) = \mathrm{S4}$
- $\mathrm{Log}(L_\kappa, \tau^*) = \mathrm{S4.1}$

Proof sketch

Completeness:

- Each ‘good’ S4-frame is an interior image of T^ω_κ
- Each ‘good’ S4.1-frame is an interior image of (L_κ, τ^*)



Logics of (L_κ, τ^*) and T_κ^ω

Theorem

- $\mathrm{Log}(T_\kappa^\omega) = \mathrm{S4}$
- $\mathrm{Log}(L_\kappa, \tau^*) = \mathrm{S4.1}$

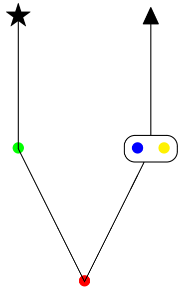
Proof sketch

Completeness:

- Example via picture: defining interior surjection $f : L_\kappa \to \mathfrak{F}$
- Restriction of $f : T_\kappa^\omega \to \mathfrak{F} \setminus \max(\mathfrak{F})$ is interior surjection

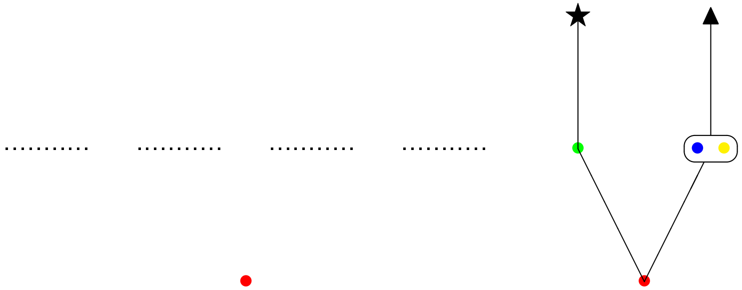
Example via Picture

(L_κ, τ^*)



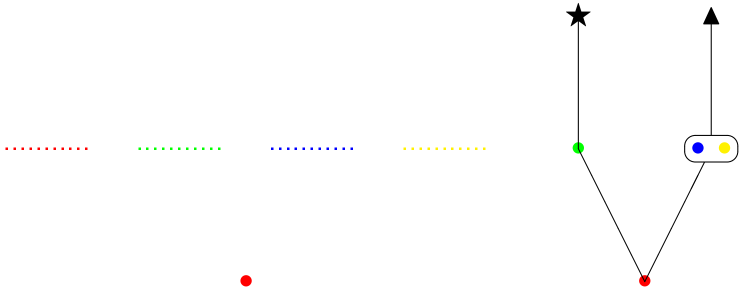
‘good’ S4.1-frame

Example via Picture



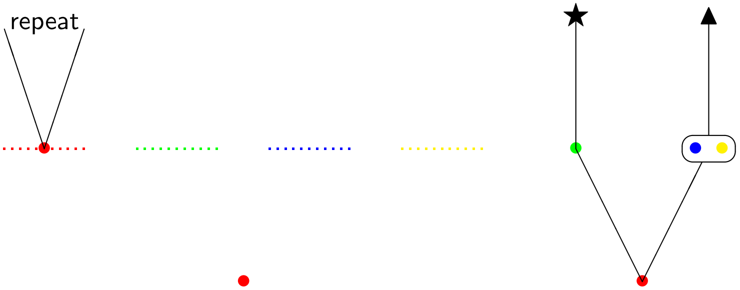
(L_κ, τ^*) 'good' S4.1-frame

Example via Picture



(L_κ, τ^*) ‘good’ S4.1-frame

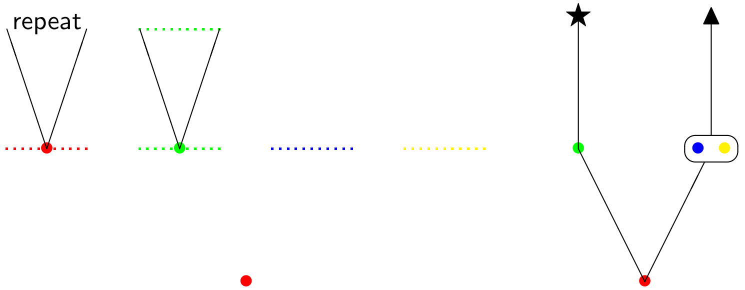
Example via Picture



(L_κ, τ^*)

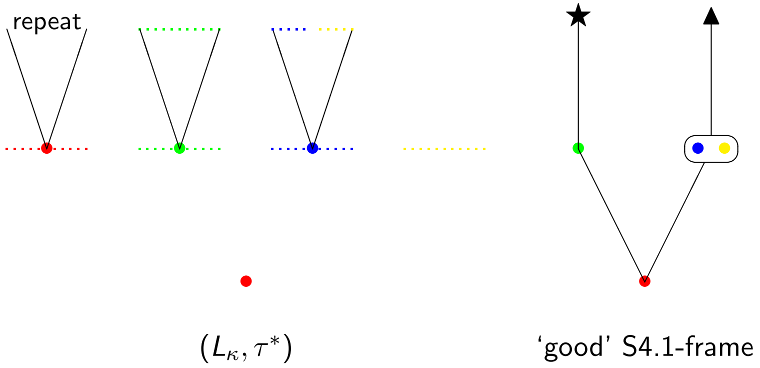
'good' S4.1-frame

Example via Picture

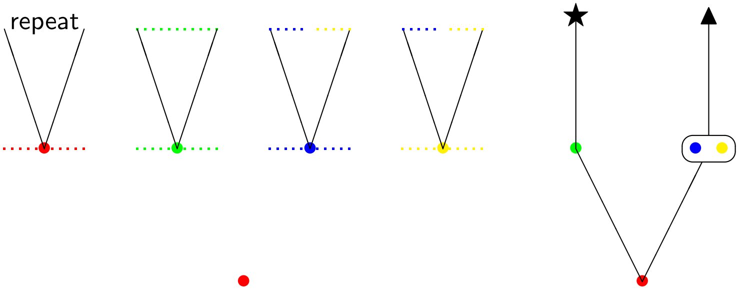


(L_κ, τ^*) ‘good’ S4.1-frame

Example via Picture



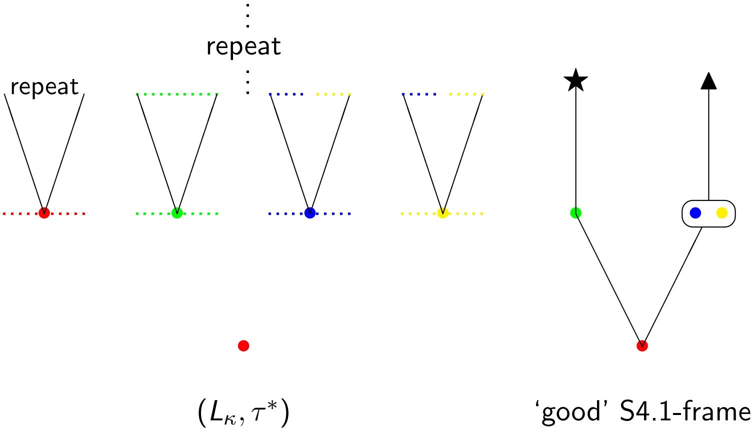
Example via Picture



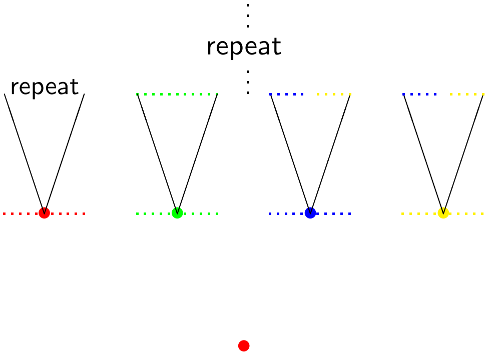
(L_κ, τ^*)

'good' S4.1-frame

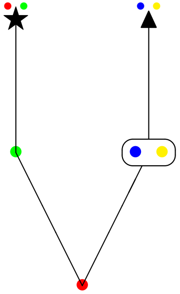
Example via Picture



Example via Picture

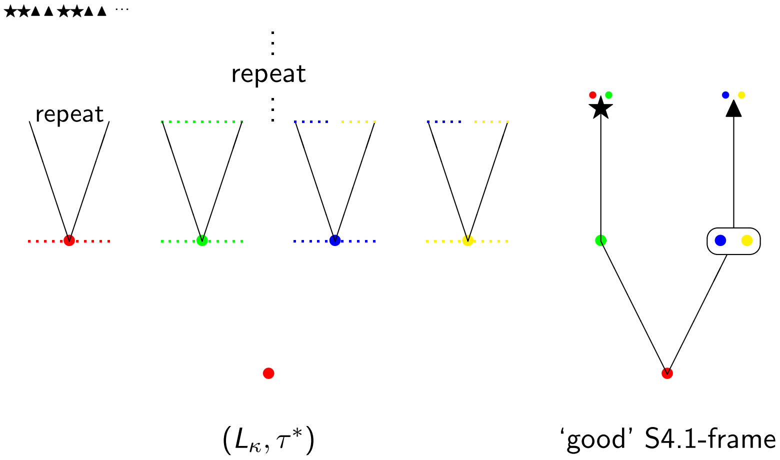


(L_κ, τ^*)



'good' S4.1-frame

Example via Picture



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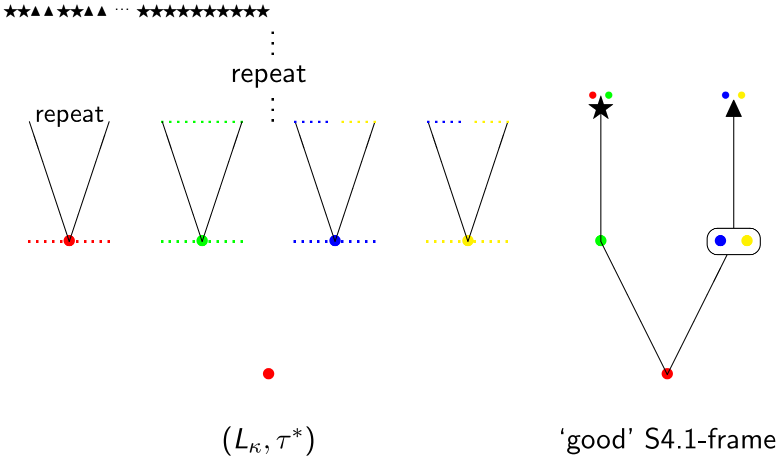

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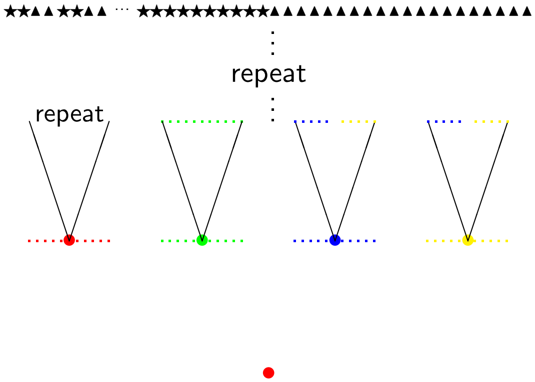
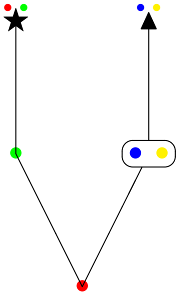


(L_κ, τ^*)



'good' S4.1-frame

Lindelöffication

Definition

View (uncountable) κ as a discrete space

The *one-point Lindellöffication* of κ is $\kappa \cup \{\infty\}$ with open subsets:

- each subset of κ
- any subset containing ∞ whose complement is countable

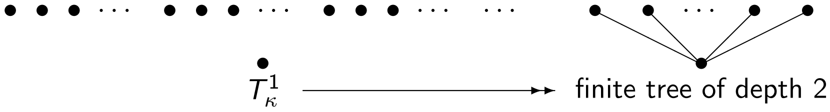
Observations

- Analogue of one-point compactification of a discrete space: 'countable' replaces 'finite'
- T^1_κ is homeomorphic to the one-point Lindellöffication of κ
- T^{n+1}_κ is obtained via appropriately gluing the roots/limit points of κ copies of T^1_κ to the leaves/isolated points of T^n_κ (similar to patch-setting where $T^n_\omega \approx \omega^n + 1$)

SCATTERED SUBSPACES OF (L_κ, τ^*)

THEOREM

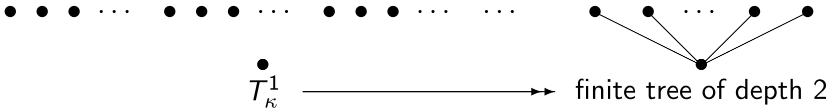
- T_κ^n is a scattered Lindellöff P-space of modal dimension n
- Any finite tree of depth $\leq n+1$ is an interior image of T_κ^n



Scattered subspaces of (L_κ, τ^*)

Theorem

- T_κ^n is a scattered Lindellöff P-space of modal dimension n
- Any finite tree of depth $\leq n+1$ is an interior image of T_κ^n
- $\mathrm{Log}(T_\kappa^n) = \mathrm{Grz}_{n+1}$
- $\mathrm{Log}\left(\bigoplus_{n\in\omega} T_\kappa^n\right) = \mathrm{Grz}$



Moving Beyond S4, S4.1, Grz, and Grz_n

Observations

- We have not yet realized any new logics...
- But we have realized the logics arising from metrizable spaces as also arising from non-metrizable spaces...
- How do we move to the ED setting?

Idea

Appropriately embed T_κ^ω (or T_κ^n) into an ED space

Theorem

A dense subspace of an ED space is ED

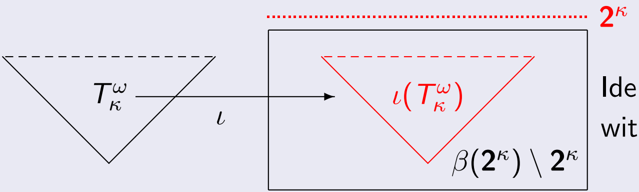
Picking up S4.1.2, Grz.2, and Grz.2_n ($n \geq 2$)

Dow and van Mill 1982 (within ZFC)

Any P-space of weight κ can be embedded into $\beta(\mathbf{2}^\kappa)$

Note

T^ω_κ is a crowded P-space of weight κ



Identify T^ω_κ with $\iota(T^\omega_\kappa)$

Definition

Set $X^\omega_\kappa = \iota(T^\omega_\kappa) \cup \mathbf{2}^\kappa = T^\omega_\kappa \cup \mathbf{2}^\kappa$

Picking up S4.1.2, Grz.2, and Grz.2_n ($n \geq 2$)

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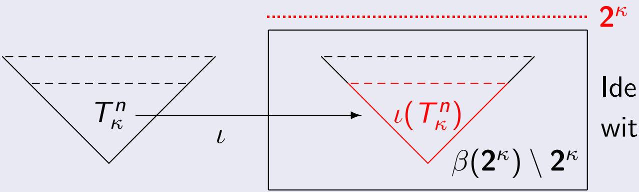


Identify T_κ^n with $\iota(T_\kappa^n)$

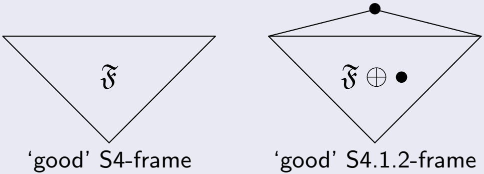
Definition

Set $X_\kappa^\omega = \iota(T_\kappa^\omega) \cup \mathbf{2}^\kappa = T_\kappa^\omega \cup \mathbf{2}^\kappa$ and $X_\kappa^n = \iota(T_\kappa^n) \cup \mathbf{2}^\kappa = T_\kappa^n \cup \mathbf{2}^\kappa$

PICKING UP S4.1.2, Grz.2, AND Grz.2_n $(n \geq 2)$

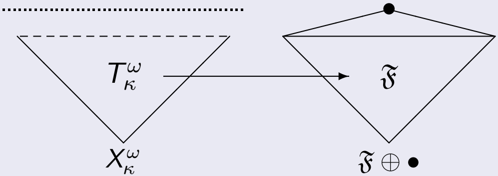
'GOOD' S4.1.2-FRAMES

Obtained by adding unique maximal point to a 'good' S4-frame



A MAPPING THEOREM

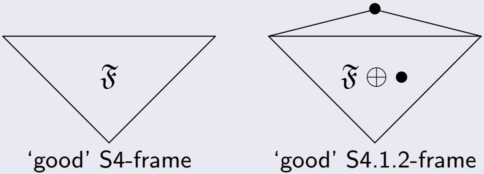
Every 'good' S4.1.2-frame is an interior image of X_κ^ω



Picking up S4.1.2, Grz.2, and Grz.2$_n$ ($n \geq 2$)

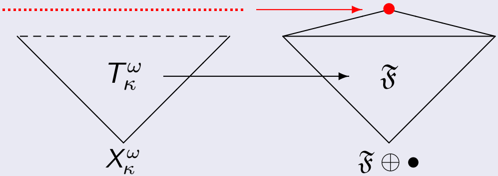
'Good' S4.1.2-frames

Obtained by adding unique maximal point to a 'good' S4-frame



A mapping theorem

Every 'good' S4.1.2-frame is an interior image of X_κ^ω



Picking up S4.1.2, Grz.2, and Grz.2_n $(n \geq 2)$

Theorem

X_κ^ω is a non-scattered weakly scattered ED space

Proof sketch

Weakly scattered: the isolated points of X_κ^ω, namely $\mathbf{2}^\kappa$, are dense
ED: X_κ^ω is dense in $\beta(\mathbf{2}^\kappa)$ since $\mathbf{2}^\kappa \subseteq X_\kappa^\omega$

Theorem

$\mathrm{Log}(X_\kappa^\omega) = \mathrm{S4.1.2}$

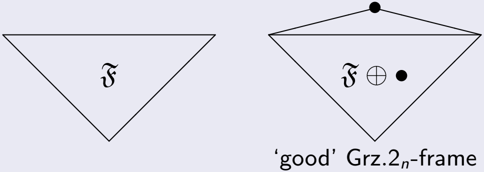
Proof sketch

Soundness: X_κ^ω is weakly scattered and ED
Completeness: follows from the previous mapping theorem

Picking up S4.1.2, Grz.2, and Grz.2_n ($n \geq 2$)

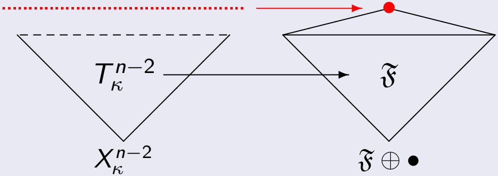
'Good' Grz.2_n-frames

Add a maximal point to a finite tree $\mathfrak{F}$ of depth $\leq n-1$



A mapping theorem

Every 'good' Grz.2_n-frame is an interior image of X_κ^{n-2}



Picking up S4.1.2, Grz.2, and $\mathsf{Grz.2}_n$ $(n \geq 2)$

Theorem

X_κ^{n-2} is a scattered ED space of modal dimension $n-1$

Partial idea of proof

Adding a dense 'layer' of isolated points to T_κ^{n-2} preserves scattered and increases modal dimension by one;
i.e. $\mathrm{mdim}(X_\kappa^{n-2}) = \mathrm{mdim}(T_\kappa^{n-2}) + 1$

Theorem

$\mathsf{Log}(X_\kappa^{n-2}) = \mathsf{Grz.2}_n$

Proof sketch

Soundness: follows from first theorem above
Completeness: follows from the previous mapping theorem

Picking up S4.1.2, Grz.2, and Grz.2_n $(n \geq 2)$

Theorem

$\text{Log}\left(\bigoplus_{n\in\omega} X_\kappa^n\right) = \bigcap_{n\in\omega} \text{Grz.2}_{n+2} = \text{Grz.2}$

Question

How do we obtain S4.2?

Balcar and Franek 1982 (within ZFC)

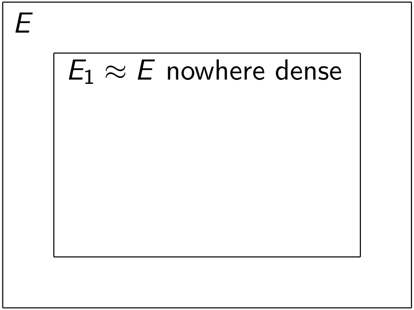
For infinite compact ED spaces X and Y

- if weight of Y $\leq$ weight of X, then Y embeds into X
- X contains a homeomorphic copy of itself as a nowhere dense subspace

Picking up S4.2

Known results

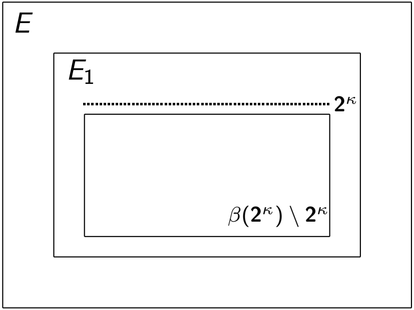
- $\beta(\mathbf{2}^\kappa)$ is an infinite compact ED-space of weight $\mathbf{2}^{\mathbf{2}^\kappa}$
- The Gleason cover $E := E\left([0,1]^{\mathbf{2}^{\mathbf{2}^\kappa}}\right)$ is an infinite compact ED-space of weight $\mathbf{2}^{\mathbf{2}^\kappa}$



Picking up S4.2

Known results

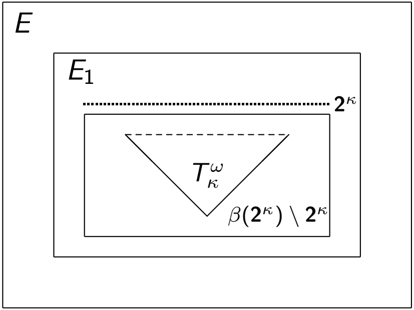
- $\beta(\mathbf{2}^\kappa)$ is an infinite compact ED-space of weight $\mathbf{2}^{\mathbf{2}^\kappa}$
- The Gleason cover $E := E\left([0,1]^{\mathbf{2}^{\mathbf{2}^\kappa}}\right)$ is an infinite compact ED-space of weight $\mathbf{2}^{\mathbf{2}^\kappa}$



Picking up S4.2

Known results

- $\beta(\mathbf{2}^\kappa)$ is an infinite compact ED-space of weight $\mathbf{2}^{\mathbf{2}^\kappa}$
- The Gleason cover $E := E\left([0,1]^{\mathbf{2}^{\mathbf{2}^\kappa}}\right)$ is an infinite compact ED-space of weight $\mathbf{2}^{\mathbf{2}^\kappa}$



Picking up S4.2

Known results

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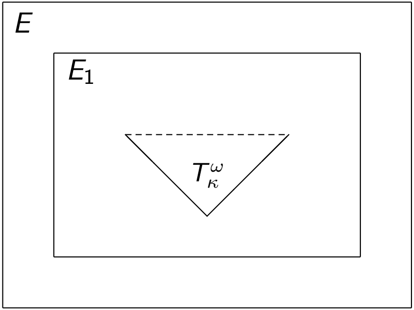
- The Gleason cover $E := E\left([0,1]^{\mathbf{2}^{\mathbf{2}^\kappa}}\right)$ is an infinite compact ED-space of weight $\mathbf{2}^{\mathbf{2}^\kappa}$



Set $X_\kappa = T^\omega_\kappa \cup (E \setminus E_1)$

Picking up S4.2

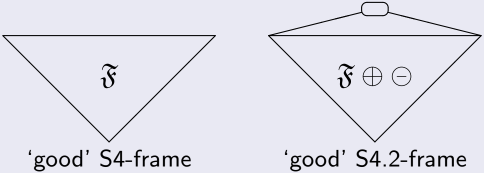
Observations

- $E \setminus E_1$ is open and dense in E
- X_κ is dense in E
- X_κ is ED
- T_κ^ω is closed and nowhere dense in X_κ
- $E \setminus E_1$ is n-resolvable for any $n \geq 2$

Picking up S4.2

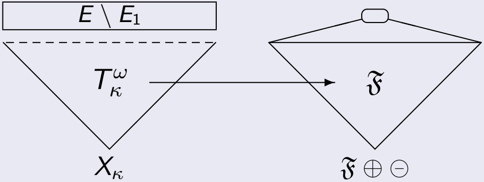
'Good' S4.2-frames

Obtained by adding finite maximal cluster to a 'good' S4-frame



A mapping theorem

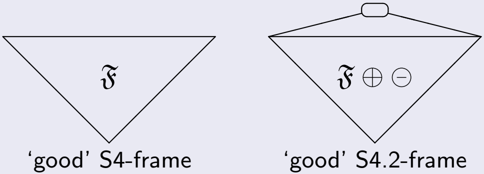
Every 'good' S4.2-frame is an interior image of X_κ



Picking up S4.2

‘Good’ S4.2-frames

Obtained by adding finite maximal cluster to a ‘good’ S4-frame



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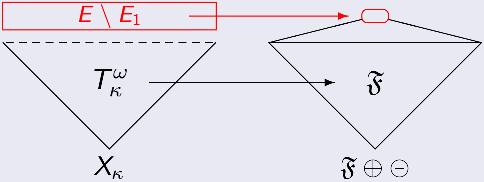
Every ‘good’ S4.2-frame is an interior image of X_κ



Picking up S4.2

Theorem

$\mathrm{Log}(X_\kappa) = \mathsf{S4.2}$

Proof sketch

Soundness: X_κ is ED

Completeness: mapping theorem

Recap

The logics S4.2, S4.1.2, Grz.2, and Grz.2_n ($n \geq 2$) arise from Tychonoff ED spaces built within ZFC

Thank You...

Organizers and Audience

Questions ...