MacNeille transferability of finite lattices

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joint work with
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2. Quasi-equations and universal clauses to a lesser extent.
3. We will look at special universal clauses $\rho(\mathbf{L})$ associated with finite lattices $\mathbf{L}$.
4. We determine conditions on $\mathbf{L}$ ensuring that $\rho(\mathbf{L})$ is preserved by ideal and MacNeille completions of different types of lattices.

Ideal transferability

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A (finite) lattice $\mathbf{L}$ is *ideal transferable* if for all lattice $\mathbf{K}$,

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Remark

Grätzer was interested in finding first-order sentences in the language of lattices preserved and reflected by the operation $\mathbf{K} \mapsto \operatorname{Idl}(\mathbf{K})$.

Universal sentences and forbidden configurations

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Let $\tau \subseteq \{0, 1, \wedge, \vee\}$ and let $\mathbf{L}$ be a finite lattice. Then there exist a universal sentence $\rho_\tau(\mathbf{L})$ such that

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5. Any universal class of locally finite lattices can be axiomatised by such clauses.

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New results for ideal transferability of distributive lattices with respect to certain classes of modular lattices Wehrung 2018.

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2. $\mathbf{L}$ is *sharply τ-MacNeille transferable for $\mathcal{K}$* if for all $\mathbf{K} \in \mathcal{K}$:

$$\forall\, h\colon \mathbf{L} \hookrightarrow_\tau \overline{\mathbf{K}}\ \exists\, k\colon \mathbf{L} \hookrightarrow_\tau \mathbf{K}\ (x \leq y \iff k(x) \leq h(y)).$$

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For $\mathbf{K}$ a bounded lattice we have that

$$\mathrm{Idl}(\mathbf{K}) \hookrightarrow_{\wedge,\vee} \mathbf{K}^{\delta} \hookrightarrow_{\wedge,\vee} \overline{\mathbf{K}^{X}/U},$$

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Gehrke, Harding & Venema 2006. So if $\mathbf{L} \hookrightarrow_{\wedge,\vee} \mathrm{Idl}(\mathbf{K})$, then $\mathbf{L} \hookrightarrow_{\wedge,\vee} \mathbf{K}$, by Łos' Theorem. □

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Corollary

Any finite lattice $\{\wedge, \vee\}$-MacNeille transferable for the class of all lattices must be a linear sum of lattices isomorphic to:

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Problem

Does this exactly characterise the lattices $\{\wedge, \vee\}$-MacNeille transferable for the class of all lattices?

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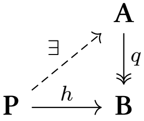
An object $\mathbf{P}$ in a concrete category $\mathscr{C}$ is *(weakly) projective* if for any arrow $h\colon \mathbf{P} \to \mathbf{B}$ and any surjection $q\colon \mathbf{A} \twoheadrightarrow \mathbf{B}$ in $\mathscr{C}$, there exist an arrow $\mathbf{P} \to \mathbf{A}$ making the following diagram commute



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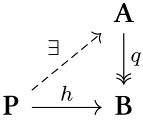

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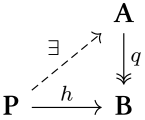

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1. *Every finite distributive lattice (reduct) is projective in the category of meet-semilattices (Horn & Kimura 1971),*
2. *A finite distributive lattice $\mathbf{L}$ is projective in the category of distributive lattices iff $J_0(\mathbf{L})$ is closed under meets (Balbes & Horn 1970).*

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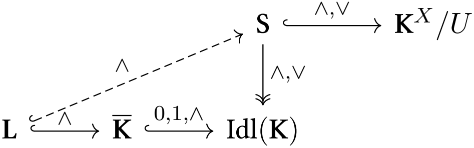

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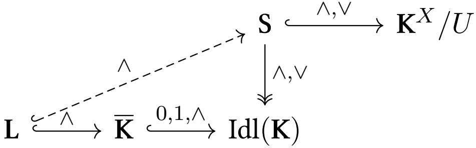

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Remark

This entails that any class of HAs axiomatised by $\{0, 1, \wedge\}$-clauses is closed under MacNeille completions Ciabattoni et al. 2011.

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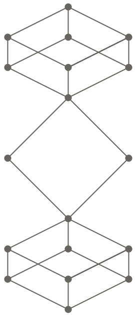

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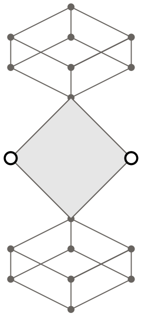

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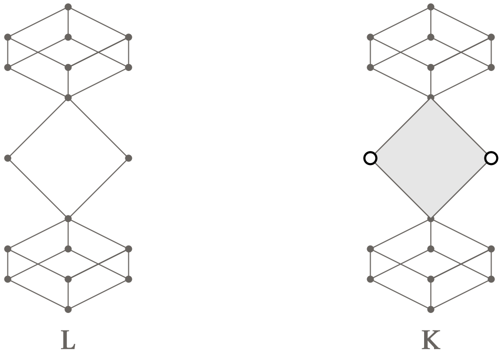

Note that **K** is *not* a Heyting algebra.

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Theorem

Let $\tau \subseteq \{0, 1, \wedge, \vee\}$ *be such that* $\{0, 1\} \not\subseteq \tau$ *and let* $\mathbf{P}$ *be a finite projective distributive lattice.*

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Proof.

If $\mathbf{P} \hookrightarrow_{0,\wedge,\vee} \overline{\mathbf{K}}$ then $\mathbf{P} \hookrightarrow_{0,\wedge} \mathbf{K}$.

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Proof.

If $\mathbf{P} \hookrightarrow_{0,\wedge,\vee} \overline{\mathbf{K}}$ then $\mathbf{P} \hookrightarrow_{0,\wedge} \mathbf{K}$. Since $\mathbf{P}$ is a finite projective distributive lattice we have that

$$h\colon \mathbf{P} \hookrightarrow_{0,\wedge} \mathbf{K} \implies \hat{h}\colon \mathbf{P} \hookrightarrow_{0,\wedge,\vee} \mathbf{K},$$

for $\hat{h}(x) := \bigvee\{h(a) : a \in J_0(\mathbf{P}) \cap {\downarrow}x\}$ Balbes & Horn 1970. □

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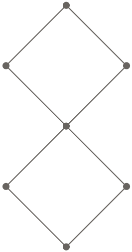
The lattice **D** *is* $\{\wedge, \vee\}$-MacNeille transferable for the class of distributive lattices but *not* projective in the category of distributive lattices.



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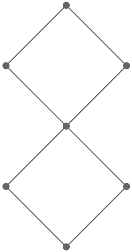

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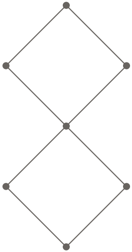

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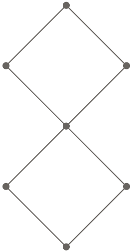

D

However, **D** is *not* sharply $\{\wedge, \vee\}$-MacNeille transferable for the class of distributive lattices. Not even for the class of Heyting algebras. Note: The lattice **D** also plays a central role in Wehrung 2018.

MacNeille transferability for Heyting algebras

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Lemma

Let $\mathcal{K}$ be a class of $(\tau \cup \{1\})$-lattices closed under principal ideals. If $\mathbf{L}$ *is* τ*-MacNeille transferable for* $\mathcal{K}$ *the* $\mathbf{L} \oplus \mathbf{1}$ *is* $(\tau \cup \{1\})$*-MacNeille transferable for* $\mathcal{K}$*. Similar, mutatis mutandis, for principal filters.*

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The following lattices are all $\{0, 1, \wedge, \vee\}$*-MacNeille transferable for the class of Heyting algebras:*

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The following lattices are all $\{0, 1, \wedge, \vee\}$-MacNeille transferable for the class of Heyting algebras:

$$\mathbf{1} \oplus \mathbf{P}, \quad \mathbf{P} \oplus \mathbf{1}, \quad \mathbf{1} \oplus \mathbf{P} \oplus \mathbf{1}, \quad \mathbf{1} \oplus \mathbf{D} \oplus \mathbf{1}, \quad \mathbf{1} \oplus \mathbf{D}, \quad \mathbf{D} \oplus \mathbf{1},$$

where $\mathbf{P}$ is a finite lattice projective in the category of distributive lattices, and $\mathbf{D}$ is the seven element distributive lattice with a doubly-reducible element.

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No *finite and directly decomposable distributive lattice is* $\{0, 1, \wedge, \vee\}$*-MacNeille transferable for the class of Heyting algebras.*

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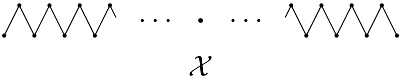

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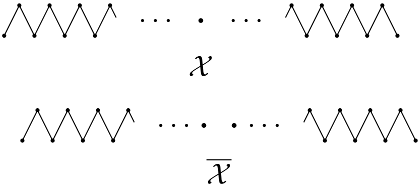

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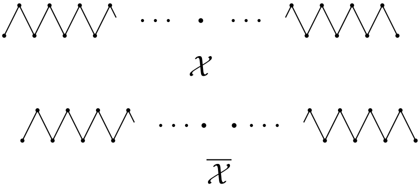

So for $\mathbf{A} := \mathsf{ClpUp}(\mathcal{X})$ we have that $\overline{\mathbf{A}} = \mathbf{B} \times \mathbf{B}$, with the property that $\mathbf{C} \hookrightarrow_{0,1,\wedge,\vee} \mathbf{B}$, for any finite directly indecomposable distributive lattice $\mathbf{C}$.

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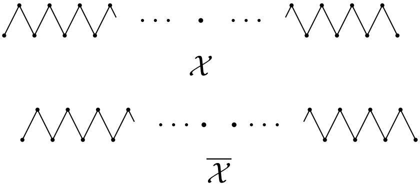

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Remark

Note that a positive answer to 3 will entail that every stable intermediate logic is canonical.

MacNeille transferability for bi-Heyting algebras

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Theorem

If $\mathbf{A}$ *is a bi-Heyting algebra of finite width then* $\overline{\mathbf{A}} \hookrightarrow_{0,1,\wedge,\vee} \mathrm{Idl}(\mathbf{A})$.

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4. Syntax? Cf., Grätzer 1966/1970, Baker & Hales 1974.

Thank you very much for your time and attention