Infinitary logic and basically disconnected compact Hausdorff spaces
ToLo VI

Serafina Lapenta

it includes joint works with Antonio Di Nola and Ioana Leuştean

University of Salerno

Summary

1. some ~~boring~~ preliminary notions
2. convergence in logic and deductive systems closed to limits
3. an infinitary logic that admits $C(X)$, with X BDKHaus-space, as models.

MV-algebras with product

MV-algebras with product

MV-algebras D endowed with a scalar multiplication with scalars in $[0,1]_Q = [0,1] \cap \mathbb{Q}$.

MV-algebras with product

MV-algebras D endowed with a scalar multiplication with scalars in $[0,1]_Q = [0,1] \cap \mathbb{Q}$.

DMV-algebras (B. Gerla, 2001, S.L. and Leuştean, 2016)

- they form a variety, $\mathbb{DMV} = HSP([0,1] \cap \mathbb{Q})$.
- categorical equivalence with divisible ℓu-groups and with $\mathbb{Q}$-vector lattices with s.u..

MV-algebras with product

MV-algebras D endowed with a scalar multiplication with scalars in $[0,1]_Q = [0,1] \cap \mathbb{Q}$.

DMV-algebras (B. Gerla, 2001, S.L. and Leuştean, 2016)

- they form a variety, $\mathbb{DMV} = HSP([0,1] \cap \mathbb{Q})$.
- categorical equivalence with divisible ℓu-groups and with $\mathbb{Q}$-vector lattices with s.u..

MV-algebras R endowed with a scalar multiplication with scalars in $[0,1]$.

MV-algebras with product

MV-algebras D endowed with a scalar multiplication with scalars in $[0,1]_Q = [0,1] \cap \mathbb{Q}$.

DMV-algebras (B. Gerla, 2001, S.L. and Leuştean, 2016)

- they form a variety, $\mathbb{DMV} = HSP([0,1] \cap \mathbb{Q})$.
- categorical equivalence with divisible ℓu-groups and with $\mathbb{Q}$-vector lattices with s.u..

MV-algebras R endowed with a scalar multiplication with scalars in $[0,1]$.

Riesz MV-algebras (Di Nola, Leuştean, 2014)

- they form a variety, $\mathbb{RMV} = HSP([0,1]_{RMV})$
- categorical equivalence with Riesz Spaces (vector lattices) with strong unit.

Logics

Logic	Algebra	Completeness
$\mathcal{L}$	$Lind_{\mathcal{L}}$ is an MV-algebra	$[0,1]_{MV}$
$\mathbb{Q}\mathcal{L}$	$Lind_{\mathbb{Q}\mathcal{L}}$ is a DMV-algebra	$[0,1] \cap \mathbb{Q}$
$\mathbb{R}\mathcal{L}$	$Lind_{\mathbb{R}\mathcal{L}}$ is a Riesz MV-algebra	$[0,1]_{RMV}$

Logics

Logic	Algebra	Completeness
$\mathcal{L}$	$Lind_{\mathcal{L}}$ is an MV-algebra	$[0,1]_{MV}$
$\mathbb{Q}\mathcal{L}$	$Lind_{\mathbb{Q}\mathcal{L}}$ is a DMV-algebra	$[0,1] \cap \mathbb{Q}$
$\mathbb{R}\mathcal{L}$	$Lind_{\mathbb{R}\mathcal{L}}$ is a Riesz MV-algebra	$[0,1]_{RMV}$

Functional representation

Let $R \subseteq \mathbb{R}$ be a ring. $f : [0,1]^n \to [0,1]$ is a $\mathrm{PWL}_u(R)$ function if it is continuous and there is a finite set of affine functions $p_1, \ldots, p_k : \mathbb{R}^n \to \mathbb{R}$ with coefficients in R such that for any $(a_1, \ldots, a_n) \in [0,1]^n$ there exists $i \in \{1, \ldots, k\}$ with $f(a_1, \ldots, a_n) = p_i(a_1, \ldots, a_n)$.

Functional representations

Functional representations

Free MV-algebra $MV_n \simeq Lind_{\mathcal{L},n}$ [R. McNaughton, 1951]

$MV_n = \{f_\varphi \colon [0,1]^n \to [0,1] \mid \varphi \text{ formula of } \mathcal{L}\} = \mathrm{PWL}_u(\mathbb{Z})$

Functional representations

Free MV-algebra $\boldsymbol{MV_n} \simeq \boldsymbol{Lind_{\mathcal{L},n}}$ [R. McNaughton, 1951]

$MV_n = \{f_\varphi \colon [0,1]^n \to [0,1] \mid \varphi \text{ formula of } \mathcal{L}\} = \mathrm{PWL}_u(\mathbb{Z})$

Free DMV-algebra $\boldsymbol{DMV_n} \simeq \boldsymbol{Lind_{\mathbb{Q}\mathcal{L},n}}$ [B.Gerla, 2001]

$DMV_n = \{f_\varphi \colon [0,1]^n \to [0,1] \mid \varphi \text{ formula of } \mathbb{Q}\mathcal{L}\} = \mathrm{PWL}_u(\mathbb{Q})$

Functional representations

Free MV-algebra $MV_n \simeq Lind_{\mathcal{L},n}$ [R. McNaughton, 1951]

$MV_n = \{f_\varphi \colon [0,1]^n \to [0,1] \mid \varphi \text{ formula of } \mathcal{L}\} = \mathrm{PWL}_u(\mathbb{Z})$

Free DMV-algebra $DMV_n \simeq Lind_{\mathbb{Q}\mathcal{L},n}$ [B.Gerla, 2001]

$DMV_n = \{f_\varphi \colon [0,1]^n \to [0,1] \mid \varphi \text{ formula of } \mathbb{Q}\mathcal{L}\} = \mathrm{PWL}_u(\mathbb{Q})$

Free Riesz MV-algebra $RMV_n \simeq Lind_{\mathbb{R}\mathcal{L},n}$ [Di Nola, Leuştean 2014]

$RMV_n = \{f_\varphi \colon [0,1]^n \to [0,1] \mid \varphi \text{ formula of } \mathbb{R}\mathcal{L}\} = \mathrm{PWL}_u(\mathbb{R})$

Tensor product of semisimple MV-algebras. Mundici, 1999

Tensor product of semisimple MV-algebras. Mundici, 1999

$A \subseteq C(X)$,$B \subseteq C(Y)$ semisimple,

$$A \otimes_{ss} B = \langle a \cdot b \mid a \in A, B \in B \rangle_{MV} \subseteq C(X \times Y)$$

$(a \cdot b)(x, y) = a(x) \cdot b(y)$ for any $x \in X,\ y \in Y$.

It enjoys a suitable universal property and each factor embeds in the product.

Tensor product of semisimple MV-algebras. Mundici, 1999

$A \subseteq C(X)$,$B \subseteq C(Y)$ semisimple,

$$A \otimes_{ss} B = \langle a \cdot b \mid a \in A, B \in B \rangle_{MV} \subseteq C(X \times Y)$$

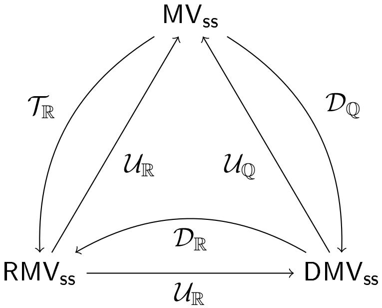
$(a \cdot b)(x, y) = a(x) \cdot b(y)$ for any $x \in X, \ y \in Y$.

It enjoys a suitable universal property and each factor embeds in the product.

Scalar extension properties [S.L., I. Leuştean, 2016, 2017]

- If R is a semisimple Riesz MV-algebra and A is a semisimple MV-algebra, then $R \otimes A$ is a semisimple Riesz MV-algebra.
- If D is a semisimple DMV-algebra and A is a semisimple MV-algebra, then $D \otimes A$ is a semisimple DMV-algebra.

Semisimple algebras and tensor product



$\mathsf{MV_{ss}} \xrightarrow{\mathcal{D}_{\mathbb{Q}}} \mathsf{DMV_{ss}}$ $\quad$ $\mathcal{D}_{\mathbb{Q}}(A) = [0,1]_{\mathbb{Q}} \otimes A$

$\mathsf{MV_{ss}} \xrightarrow{\mathcal{T}_{\mathbb{R}}} \mathsf{RMV_{ss}}$ $\quad$ $\mathcal{T}_{\mathbb{R}}(A) = [0,1] \otimes A$

$\mathsf{DMV_{ss}} \xrightarrow{\mathcal{D}_{\mathbb{R}}} \mathsf{RMV_{ss}}$ $\quad$ $\mathcal{D}_{\mathbb{R}}(A) = \mathcal{T}_{\mathbb{R}}(\mathcal{U}_{\mathbb{R}}(A)) = [0,1] \otimes A.$

Convergence in $\mathbb{R}\mathcal{L}$

Di Nola A., Lapenta S., Leuştean I., *An analysis of the logic of Riesz Spaces with strong unit*, Annals of Pure ans Applied Logic (2018), 169(3) 216–234.

Convergence in $\mathbb{R}\mathcal{L}$

Convergence in $\mathbb{R}\mathcal{L}$

Uniform Limit of formulas

A formula φ is the uniform limit of the sequence $(\varphi_m)_{m\in\mathbb{N}}$ in $\mathbb{R}\mathcal{L}$ if for any $r < 1$ there is k such that for any $m \geq k$: $\vdash \mathrm{r} \to (\varphi \leftrightarrow \varphi_m)$. We write $\lim_m \varphi_m = \varphi$.

Convergence in $\mathbb{R}\mathcal{L}$

Uniform Limit of formulas

A formula φ is the uniform limit of the sequence $(\varphi_m)_{m\in\mathbb{N}}$ in $\mathbb{R}\mathcal{L}$ if for any $r < 1$ there is k such that for any $m \geq k$: $\vdash \mathrm{r} \to (\varphi \leftrightarrow \varphi_m)$. We write $\lim_m \varphi_m = \varphi$.

TFAE:

1. $\lim_m \varphi_m = \varphi$,
2. $\lim_m f_{\varphi_m} = f_\varphi$ (uniform convergence),
3. there exists $(f_{\psi_m})_{m\in\mathbb{N}}$ such that $\inf_{m\in\mathbb{N}}(f_{\psi_m}(x)) = 0$ for all $x \in [0,1]^n$ and $\mid f_{\varphi_m}(x) - f_\varphi(x) \mid \leq f_{\psi_m}(x)$ in $Lind_{\mathbb{R}\mathcal{L},n}$ (strong order convergence)

A remark

A remark

- Riesz MV-algebras are equivalent with Riesz Spaces with a strong unit,

A remark

- Riesz MV-algebras are equivalent with Riesz Spaces with a strong unit,
- In a Riesz space we have three notions of convergence: uniform, in order and in norm,

A remark

- Riesz MV-algebras are equivalent with Riesz Spaces with a strong unit,
- In a Riesz space we have three notions of convergence: uniform, in order and in norm,
- $(s_m)_{m\in\mathbb{N}}$ converges in order to s if there exists $(r_m)_{m\in\mathbb{N}}$ such that $\bigwedge_m r_m = 0$ and $\mid s - s_m \mid \leq r_m$ for any $m \in \mathbb{N}$,

A remark

- Riesz MV-algebras are equivalent with Riesz Spaces with a strong unit,
- In a Riesz space we have three notions of convergence: uniform, in order and in norm,
- $(s_m)_{m\in\mathbb{N}}$ converges in order to s if there exists $(r_m)_{m\in\mathbb{N}}$ such that $\bigwedge_m r_m = 0$ and $\mid s - s_m \mid \leq r_m$ for any $m \in \mathbb{N}$,
- Order converge does not imply uniform convergence nor norm-convergence, because in spaces of functions, the pointwise infimum and the infimum do not need to coincide. This is why we called 3. strong order convergence.

Norm of formulas: the unit-norm

Norm of formulas: the unit-norm

φ formula in $\mathbb{R}\mathcal{L}$, setting $\|[\varphi]\|_u = \|f_\varphi\|_\infty$.

$(Lind_{\mathbb{R}\mathcal{L},n}, \|\cdot\|_u)$ becomes a normed space.

Norm of formulas: the unit-norm

φ formula in $\mathbb{R}\mathcal{L}$, setting $\|[\varphi]\|_u = \|f_\varphi\|_\infty$.

$$(Lind_{\mathbb{R}\mathcal{L},n}, \|\cdot\|_u) \text{ becomes a normed space.}$$

Completion

The norm-completion of the normed space $(Lind_{\mathbb{R}\mathcal{L},n}, \|\cdot\|_u)$ is isometrically isomorphic with $(C([0,1]^n), \|\cdot\|_\infty)$.

Norm of formulas: the unit-norm

φ formula in $\mathbb{R}\mathcal{L}$, setting $\|[\varphi]\|_u = \|f_\varphi\|_\infty$.

$$(Lind_{\mathbb{R}\mathcal{L},n}, \|\cdot\|_u) \text{ becomes a normed space.}$$

Completion

The norm-completion of the normed space $(Lind_{\mathbb{R}\mathcal{L},n}, \|\cdot\|_u)$ is isometrically isomorphic with $(C([0,1]^n), \|\cdot\|_\infty)$.

Norm of formulas: the integral norm

It is possible to define an integral norm on $Lind_{\mathbb{R}\mathcal{L},n}$. With respect to this norm, the completion of $Lind_{\mathbb{R}\mathcal{L},n}$ is a suitable space of integrable function and it is connected to the theory of L-spaces.

We now have an appropriate notion of syntactical limit, which is compatible with the semantic notion.

We now have an appropriate notion of syntactical limit, which is compatible with the semantic notion.

- analyze deductive systems closed to limits,
- discuss norm completions in logic,
- axiomatize a logic whose models are $C(X)$, for basically disconnected $X \in \mathsf{KHausd}$.

From $\mathbb{Q}\mathcal{L}$ to $\mathbb{R}\mathcal{L}$

Monotone sequences of formulas

A sequence $(\varphi_n)_n$ of formulas is

1. increasing if $\vdash \varphi_n \to \varphi_{n+1}$
2. decreasing if $\vdash \varphi_{n-1} \to \varphi_n$.

From $\mathbb{Q}\mathcal{L}$ to $\mathbb{R}\mathcal{L}$

Monotone sequences of formulas

A sequence $(\varphi_n)_n$ of formulas is

1. increasing if $\vdash \varphi_n \to \varphi_{n+1}$
2. decreasing if $\vdash \varphi_{n-1} \to \varphi_n$.

Rational approximation

For any formula φ in $\mathbb{R}\mathcal{L}$ there exist an increasing sequence of formulas $\{\psi_n\}_{n\in\mathbb{N}}$ and a decreasing sequence of formulas $\{\chi_n\}_{n\in\mathbb{N}}$, both in $\mathbb{Q}\mathcal{L}$, such that $\lim_n \psi_n = \varphi$ and $\lim_n \chi_n = \varphi$.

Deductive systems

Clearly, the three logical system we are considering are entangled with each other.

Each formula in $\mathbb{R}\mathcal{L}$ can be approximated by sequences in $\mathbb{Q}\mathcal{L}$.

Deductive systems

Clearly, the three logical system we are considering are entangled with each other.

Each formula in $\mathbb{R}\mathcal{L}$ can be approximated by sequences in $\mathbb{Q}\mathcal{L}$.

Can be said the same for formulas in $\mathbb{Q}\mathcal{L}$ wrt formulas in Łukasiewicz logic?

Deductive systems

Clearly, the three logical system we are considering are entangled with each other.

Each formula in $\mathbb{R}\mathcal{L}$ can be approximated by sequences in $\mathbb{Q}\mathcal{L}$.

Can be said the same for formulas in $\mathbb{Q}\mathcal{L}$ wrt formulas in Łukasiewicz logic?

If not, how these consideration are reflected on the deductive systems of these logics?

Deductive systems

Recall that $\mathcal{L}$ denotes Łukasiewicz logic.

$\Theta \subseteq Form_{\mathcal{L}}$, we denote

$$Thm(\Theta, \mathcal{L}) = \{\varphi \in Form_{\mathcal{L}} \mid \Theta \vdash_{\mathcal{L}} \varphi\}$$

the theory determined by Θ in $\mathcal{L}$.

Deductive systems

Recall that $\mathcal{L}$ denotes Łukasiewicz logic.

$\Theta \subseteq Form_{\mathcal{L}}$, we denote

$$Thm(\Theta, \mathcal{L}) = \{\varphi \in Form_{\mathcal{L}} \mid \Theta \vdash_{\mathcal{L}} \varphi\}$$

the theory determined by Θ in $\mathcal{L}$.

Analogously for $\mathbb{Q}\mathcal{L}$ and $\mathbb{R}\mathcal{L}$, we get

$$Thm(\Theta, \mathbb{Q}\mathcal{L}) = \{\varphi \in Form_{L} \mid \Theta \vdash_{\mathbb{Q}\mathcal{L}} \varphi\}$$

$$Thm(\Theta, \mathbb{R}\mathcal{L}) = \{\varphi \in Form_{L} \mid \Theta \vdash_{\mathbb{R}\mathcal{L}} \varphi\}$$

Ł-generated theories in $\mathbb{Q}\mathcal{L}$

Ł-generated theories in $\mathbb{Q}\mathcal{L}$

It is easy to check that, for any $f \in DMV_n$ there exist $\overline{f} \in MV_n$ such that

$$\langle f \rangle_{DMV} = \langle \overline{f} \rangle_{DMV}.$$

Ł-generated theories in $\mathbb{Q}\mathcal{L}$

It is easy to check that, for any $f \in DMV_n$ there exist $\overline{f} \in MV_n$ such that

$$\langle f \rangle_{DMV} = \langle \overline{f} \rangle_{DMV}.$$

Thus, via the usual corresponded between filters and deductive systems,

Let φ be a formula of $\mathbb{Q}\mathcal{L}$. There exists a formula β of $\mathcal{L}$ such that $Thm(\varphi, \mathbb{Q}\mathcal{L}) = Thm(\beta, \mathbb{Q}\mathcal{L})$.

Ł-generated theories in $\mathbb{R}\mathcal{L}$

Ł-generated theories in $\mathbb{R}\mathcal{L}$

An ideal I of RMV_n, $n \in \mathbb{N}$, is said to be norm-closed if, whenever $f_1, f_2, \ldots, f_m, \ldots$ is a sequence of elements of I and $\{f_m\}_{m\in\mathbb{N}}$ uniformly converges to f, then $f \in I$.

Ł-generated theories in $\mathbb{R}\mathcal{L}$

An ideal I of RMV_n, $n \in \mathbb{N}$, is said to be norm-closed if, whenever $f_1, f_2, \ldots, f_m, \ldots$ is a sequence of elements of I and $\{f_m\}_{m \in \mathbb{N}}$ uniformly converges to f, then $f \in I$.

For example, any σ-ideal is norm-closed.

Ł-generated theories in $\mathbb{R}\mathcal{L}$

An ideal I of RMV_n, $n \in \mathbb{N}$, is said to be norm-closed if, whenever $f_1, f_2, \ldots, f_m, \ldots$ is a sequence of elements of I and $\{f_m\}_{m\in\mathbb{N}}$ uniformly converges to f, then $f \in I$.

For example, any σ-ideal is norm-closed.

An infinitary deduction rule

$$(\star) \qquad \text{if } \varphi = \lim_m \varphi_m \quad \text{then} \quad \frac{\varphi_1, \varphi_2, \ldots, \varphi_m, \ldots}{\varphi}$$

Ł-generated theories in $\mathbb{R}\mathcal{L}$

The logic $\mathbb{R}\mathcal{L}\star$

It is the logic obtained from $\mathbb{R}\mathcal{L}$ adding the rule $(\star)$.

Ł-generated theories in $\mathbb{R}\mathcal{L}$

The logic $\mathbb{R}\mathcal{L}\star$

It is the logic obtained from $\mathbb{R}\mathcal{L}$ adding the rule $(\star)$.

A consequence

The deductive systems of $\mathbb{R}\mathcal{L}\star$ are in correspondence with norm-closed ideals of the Lindenbaum-Tarki algebra of $\mathbb{R}\mathcal{L}$.

Ł-generated theories in $\mathbb{R}\mathcal{L}$

The logic $\mathbb{R}\mathcal{L}\star$

It is the logic obtained from $\mathbb{R}\mathcal{L}$ adding the rule $(\star)$.

A consequence

The deductive systems of $\mathbb{R}\mathcal{L}\star$ are in correspondence with norm-closed ideals of the Lindenbaum-Tarki algebra of $\mathbb{R}\mathcal{L}$.

Let φ be a formula of $\mathbb{R}\mathcal{L}$. There exists a sequence of formulas $\Theta = \{\varphi_n\}_{n\in\mathbb{N}} \subseteq \mathit{Form}_{Ł}$ such that $\mathit{Thm}(\varphi, \mathbb{R}\mathcal{L}\star) = \mathit{Thm}(\Theta, \mathbb{R}\mathcal{L}\star)$.

How to get compact Hausdorff spaces from Riesz MV-algebras?

Di Nola A., Lapenta S., Leuştean I., *An infinitary logic for basically disconnected compact Hausdorff spaces*, accepted for publication on the Journal of Logic and Computation, arXiv:1709.08397 [math.LO]

Some approaches to KHausd

1. frames of opens $\rightarrow$ duality with compact regular frames
   (Isbell)
2. frame of regular opens with a proximity $\rightarrow$ duality with De Vries algebras
   (De Vries)

Some approaches to KHausd

1. frames of opens $\rightarrow$ duality with compact regular frames
   (Isbell)
2. frame of regular opens with a proximity $\rightarrow$ duality with De Vries algebras
   (De Vries)
3. algebras of continuous functions $\rightarrow$ duality with "norm-complete" lattices of functions
   (Gelfand, Neumark, Stone, Yosida, Kakutani, Banaschewski)

Algebras of continuous functions

- By Stone duality, a subcategory of **KHausd** is equivalent to the category whose objects are element of the finitary variety of Boolean Algebras;

Algebras of continuous functions

- By Stone duality, a subcategory of **KHausd** is equivalent to the category whose objects are element of the finitary variety of Boolean Algebras;
- An analogous result is trickier for the whole **KHausd**: indeed, the dual of **KHausd** is an infinitary variety (Rosický, Banaschewski, Duskin);

Algebras of continuous functions

- By Stone duality, a subcategory of **KHausd** is equivalent to the category whose objects are element of the finitary variety of Boolean Algebras;
- An analogous result is trickier for the whole **KHausd**: indeed, the dual of **KHausd** is an infinitary variety (Rosický, Banaschewski, Duskin);
- Isbell actually proved that it is "enough" to have a variety in which every function has at most countable arity, and explicitly described this variety;

Algebras of continuous functions

- By Stone duality, a subcategory of **KHausd** is equivalent to the category whose objects are element of the finitary variety of Boolean Algebras;
- An analogous result is trickier for the whole **KHausd**: indeed, the dual of **KHausd** is an infinitary variety (Rosický, Banaschewski, Duskin);
- Isbell actually proved that it is "enough" to have a variety in which every function has at most countable arity, and explicitly described this variety;
- Marra and Reggio provided a finite axiomatization for a variety of MV-algebras with an infinitary operation δ: δ-algebras are a finitary variety of infinitary algebras that is dual to **KHausd**. On $C(X)$, their operator coincides with Isbell's.

How to get compact Hausdorff spaces from Riesz MV-algebras?

How to get compact Hausdorff spaces from Riesz MV-algebras?

Norm-complete Riesz MV-algebras

$R \in \mathsf{RMV}$ semisimple, $\| \cdot \|_u : R \to [0,1]$

$$\|x\|_u = \min\{r \in [0,1] \mid x \leq r1\}$$

How to get compact Hausdorff spaces from Riesz MV-algebras?

Norm-complete Riesz MV-algebras

$R \in \mathsf{RMV}$ semisimple, $\| \cdot \|_u : R \to [0,1]$

$$\|x\|_u = \min\{r \in [0,1] \mid x \leq r1\}$$

A Riesz MV-algebra is norm-complete if it is a complete normed space wrt to $\| \cdot \|_u$.

How to get compact Hausdorff spaces from Riesz MV-algebras?

Norm-complete Riesz MV-algebras

$R \in \mathsf{RMV}$ semisimple, $\| \cdot \|_u : R \to [0,1]$

$$\|x\|_u = \min\{r \in [0,1] \mid x \leq r1\}$$

A Riesz MV-algebra is norm-complete if it is a complete normed space wrt to $\| \cdot \|_u$.

M-spaces

An M-space is a Banach lattice (norm-complete Riesz Space) endowed with a norm $\|\cdot\|$ such that $\|x \vee y\| = \max(\|x\|, \|y\|)$.

How to get compact Hausdorff spaces from Riesz MV-algebras?

Kakutani's duality

The category of M-spaces and suitable morphisms is dual to the category of compact Hausdorff spaces and continuous maps.

How to get compact Hausdorff spaces from Riesz MV-algebras?

Kakutani's duality

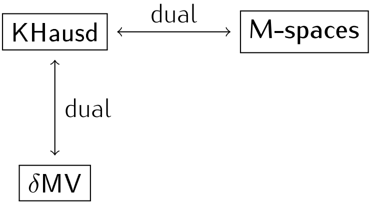
The category of M-spaces and suitable morphisms is dual to the category of compact Hausdorff spaces and continuous maps.

M-spaces and Riesz MV-algebras [A. Di Nola and I. Leuştean, 2014]

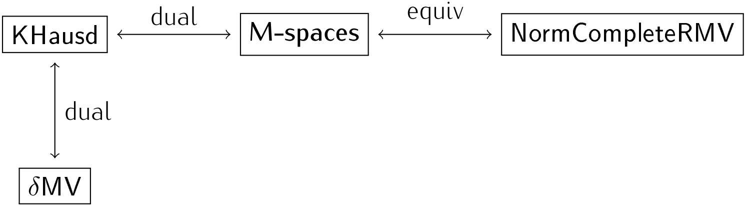
The category of M-spaces and suitable morphisms is equivalent to the full subcategory of norm-complete Riesz MV-algebras.

How to get compact Hausdorff spaces from Riesz MV-algebras?

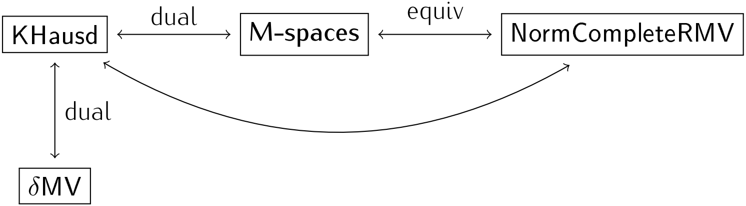
How to get compact Hausdorff spaces from Riesz MV-algebras?



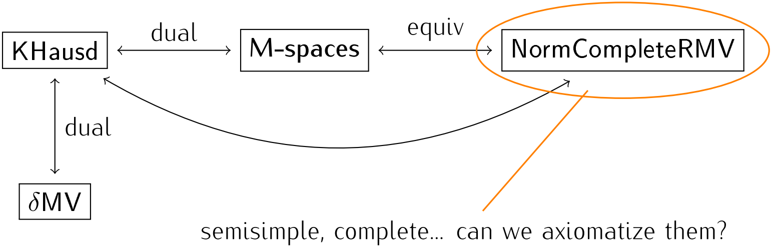
How to get compact Hausdorff spaces from Riesz MV-algebras?



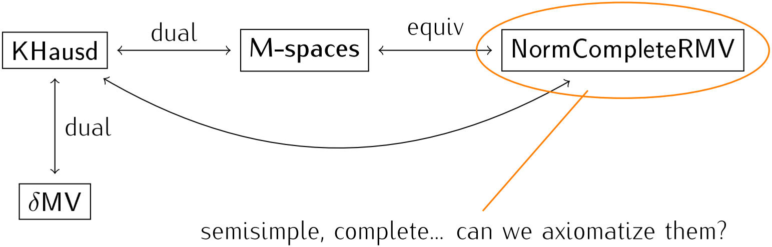
How to get compact Hausdorff spaces from Riesz MV-algebras?



How to get compact Hausdorff spaces from Riesz MV-algebras?



How to get compact Hausdorff spaces from Riesz MV-algebras?



Recalling that the uniform limit of formulas is equivalent to "strong order convergence"...

σ-complete algebras

σ-complete algebras

The category RMV_σ

σ-complete algebras

The category RMV_σ

objects: σ-complete Riesz MV-algebras (i.e. closed to countable suprema),

arrows: σ-homomorphisms of Riesz MV-algebras.

σ-complete algebras

The category $\mathbf{RMV}_\sigma$

objects: σ-complete Riesz MV-algebras (i.e. closed to countable suprema),
arrows: σ-homomorphisms of Riesz MV-algebras.

It follows from the general theory of Riesz spaces that:

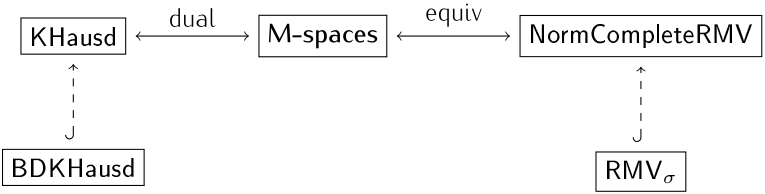
- Any σ-complete Riesz MV-algebra is norm-complete;
- for any $R \in \mathbf{RMV}_\sigma$ there exists a basically disconnected compact Hausdorff space X space such that $R \simeq C(X)$.

What we got:

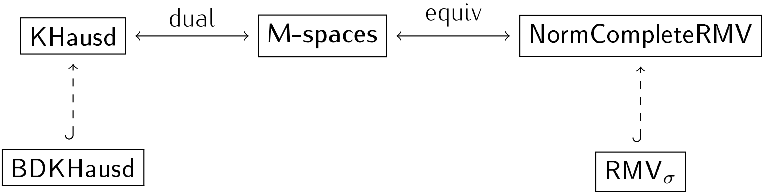
What we got:

KHausd $\xleftrightarrow{\text{dual}}$ M-spaces $\xleftrightarrow{\text{equiv}}$ NormCompleteRMV

What we got:


KHausd
dual
M-spaces
equiv
NormCompleteRMV
BDKHausd
RMV$_\sigma$

What we got:



BDKHausd

A compact Hausdorff space is basically disconnected if the closure of any open F_σ (i.e. countable union of closed sets) is open.

An important remark

An important remark

σ-complete Riesz MV-algebras are actually infinitary algebras in the sense of Słomiński.

Słomiński J., *The theory of abstract algebras with infinitary operations*, Instytut Matematyczny Polskiej Akademi Nauk, Warszawa (1959).

An important remark

σ-complete Riesz MV-algebras are actually infinitary algebras in the sense of Słomiński.

Spoiler: they are an infinitary variery!

Słomiński J., *The theory of abstract algebras with infinitary operations*, Instytut Matematyczny Polskiej Akademi Nauk, Warszawa (1959).

The logic $\mathcal{IRL}$

The logic $\mathcal{IRL}$

- Language: the one of $\mathbb{R}\mathcal{L}$ + $\bigvee$

- Axioms: the ones of $\mathbb{R}\mathcal{L}$ +

  (S1) $\varphi_k \to \bigvee_{n\in\mathbb{N}} \varphi_n$, for any $k \in \mathbb{N}$

- Deduction rules: Modus Ponens +

(SUP) $$\frac{(\varphi_1 \to \psi), \ldots, (\varphi_k \to \psi) \ldots}{\bigvee_{n\in\mathbb{N}} \varphi_n \to \psi}$$

The semantics of $\mathcal{IRL}$, main results:

- Models of the logic are objects in RMV_σ,

The semantics of $\mathcal{IRL}$, main results:

- Models of the logic are objects in $\mathbf{RMV}_\sigma$,
- *Lind*$_{\mathcal{IRL}}$ is the smallest σ-complete Riesz MV-algebra that contains *Lind*$_{\mathbb{R}\mathcal{L}}$,

The semantics of $\mathcal{IRL}$, main results:

- Models of the logic are objects in $\mathbf{RMV}_\sigma$,
- $Lind_{\mathcal{IRL}}$ is the smallest σ-complete Riesz MV-algebra that contains $Lind_{\mathbb{RL}}$,
- alternatively, models are spaces $C(X)$, with X basically disconnected compact Hausdorff space.

The semantics of $\mathcal{IRL}$, main results:

- Models of the logic are objects in $\mathbf{RMV}_\sigma$,
- $Lind_{\mathcal{IRL}}$ is the smallest σ-complete Riesz MV-algebra that contains $Lind_{\mathbb{R}\mathcal{L}}$,
- alternatively, models are spaces $C(X)$, with X basically disconnected compact Hausdorff space.

Hence,

There exists a basically disconnected compact Hausdorff space X such that $Lind_{\mathcal{IRL}} \simeq C(X)$.

Functional representations for $\mathit{Lind}_{\mathcal{IRL}}$

Functional representations for $\mathit{Lind}_{\mathcal{IRL}}$

On the one end,

$\mathit{Lind}_{\mathcal{IRL}} \simeq C(X)$, for some basically disconnected $X \in \mathsf{KHausd}$.

Functional representations for $\mathit{Lind}_{\mathcal{IRL}}$

On the one end,

$\mathit{Lind}_{\mathcal{IRL}} \simeq C(X)$, for some basically disconnected $X \in$ **KHausd**.

We tried to get an analogous of the Gleason cover, but for a general space X the construction is very complicated (Jayne, Zakherov and Kuldonov, Vermeer)

Functional representations for $Lind_{\mathcal{IRL}}$

On the one end,

$Lind_{\mathcal{IRL}} \simeq C(X)$, for some basically disconnected $X \in$ KHausd.

We tried to get an analogous of the Gleason cover, but for a general space X the construction is very complicated (Jayne, Zakherov and Kuldonov, Vermeer)

On the other end,

we can prove that $Lind_{\mathbb{R}\mathcal{L},n} \subseteq C([0,1]^n) \subseteq Lind_{\mathcal{IRL},n}$

$\Rightarrow$ $Lind_{\mathcal{IRL},n}$ is also isomorphic to some class of non-continuous $[0,1]^n$-valued functions! Can we characterize them?

Let's start with Riesz tribes...

Let's start with Riesz tribes...

A Riesz tribe over X is a Riesz MV-algebra of $[0,1]$-valued functions over X that are closed under pointwise countable suprema.

Let's start with Riesz tribes...

A Riesz tribe over X is a Riesz MV-algebra of $[0,1]$-valued functions over X that are closed under pointwise countable suprema.

The Loomis-Sikorski theorem for Riesz MV-algebras

Any σ-complete R Riesz MV-algebra is an homomorphic image of a Riesz tribe $\mathcal{T}$.

Let's start with Riesz tribes...

A Riesz tribe over X is a Riesz MV-algebra of $[0,1]$-valued functions over X that are closed under pointwise countable suprema.

The Loomis-Sikorski theorem for Riesz MV-algebras

Any σ-complete R Riesz MV-algebra is an homomorphic image of a Riesz tribe $\mathcal{T}$.

$R = C(X)$ and we say that $f \backsim g$ iff $\{x \in X \mid f(x) \neq g(x)\}$ is meager. Then R is homomorphic image of:

$$\mathcal{T} = \{f \in [0,1]^X \mid \text{ there exists } g \in R \colon f \backsim g\}$$

A completeness theorem

The class of Dedekind σ-complete Riesz MV-algebras is $HSP([0,1])$, the infinitary variety generated by $[0,1]$.

by the Loomis-Sikorski theorem.

A completeness theorem

The class of Dedekind σ-complete Riesz MV-algebras is $HSP([0,1])$, the infinitary variety generated by $[0,1]$.

by the Loomis-Sikorski theorem.

Corollary:

$\mathcal{IRL}$ is $[0,1]$-complete.

Term functions in σ-complete Riesz MV-algebras

Term functions in σ-complete Riesz MV-algebras

Absolutely free algebras

- $Term_{RMV\sigma}$, the set of terms in the language of $\mathbf{RMV}_{\sigma}$, is the absolutely free algebra in the same language, denoted by $Term_{RMV\sigma}(n)$ when only n variables occur.

Term functions in σ-complete Riesz MV-algebras

Absolutely free algebras

- $Term_{RMV_\sigma}$, the set of terms in the language of RMV_σ, is the absolutely free algebra in the same language, denoted by $Term_{RMV_\sigma}(n)$ when only n variables occur.
- for $A \in \mathsf{RMV}_\sigma$, we get

$$\tau \in Term_{RMV_\sigma}(n) \mapsto f_\tau^A : A^n \to A$$

Term functions in σ-complete Riesz MV-algebras

Absolutely free algebras

- $Term_{RMV\sigma}$, the set of terms in the language of $\mathbf{RMV}_\sigma$, is the absolutely free algebra in the same language, denoted by $Term_{RMV\sigma}(n)$ when only n variables occur.
- for $A \in \mathbf{RMV}_\sigma$, we get

$$\tau \in Term_{RMV\sigma}(n) \mapsto f_\tau^A : A^n \to A$$

- $\mathcal{RT}_n = \{f_\tau : [0,1]^n \to [0,1] \mid \tau \in Term_{RMV\sigma}(n)\}$ is a Riesz tribe.

Free algebras

The following hold

1. $\mathcal{RT}_n$ is the smallest Riesz tribe that contains the projections.

Free algebras

The following hold

1. $\mathcal{RT}_n$ is the smallest Riesz tribe that contains the projections.
2. $\mathcal{RT}_n$ is the free n-generated Riesz σ-algebra in RMV_σ.

Free algebras

The following hold

1. $\mathcal{RT}_n$ is the smallest Riesz tribe that contains the projections.
2. $\mathcal{RT}_n$ is the free n-generated Riesz σ-algebra in RMV_σ.
3. By standard arguments, the free algebra in RMV_σ is $Lind_{\mathcal{IRL},n}$.

Free algebras

The following hold

1. $\mathcal{RT}_n$ is the smallest Riesz tribe that contains the projections.
2. $\mathcal{RT}_n$ is the free n-generated Riesz σ-algebra in RMV_σ.
3. By standard arguments, the free algebra in RMV_σ is $Lind_{\mathcal{IRL},n}$.
4. Whence, $\mathcal{RT}_n \simeq Lind_{\mathcal{IRL},n}$.

Free algebras

The following hold

1. $\mathcal{RT}_n$ is the smallest Riesz tribe that contains the projections.
2. $\mathcal{RT}_n$ is the free n-generated Riesz σ-algebra in RMV_σ.
3. By standard arguments, the free algebra in RMV_σ is $Lind_{\mathcal{IRL},n}$.
4. Whence, $\mathcal{RT}_n \simeq Lind_{\mathcal{IRL},n}$.

Borel functions on (X, τ)

$\mathcal{B}(X) = \langle \mathcal{O}(X) \rangle_\sigma$ is the Borel sigma algebra of X.

$f : X \to Y$ is Borel function if $f^{-1}(A) \in \mathcal{B}(X)$ for any $A \in \mathcal{B}(Y)$.

The smallest Riesz tribe that cointains all projections

$\mathcal{RT}_n \simeq \text{Borel}([0,1]^n, [0,1])$

The smallest Riesz tribe that cointains all projections

$\mathcal{RT}_n \simeq \text{Borel}([0,1]^n,[0,1])$

Sketch of proof

1. Since projections are Borel functions, $\mathcal{RT}_n \subseteq \text{Borel}([0,1]^n,[0,1])$.

The smallest Riesz tribe that cointains all projections

$\mathcal{RT}_n \simeq \text{Borel}([0,1]^n, [0,1])$

Sketch of proof

1. Since projections are Borel functions, $\mathcal{RT}_n \subseteq \text{Borel}([0,1]^n, [0,1])$. Viceversa,
2. Standard argument in literature. Any Borel function $f : [0,1]^n \to [0,1]$ is the uniform limit of an increasing sequence of simple functions $f_m : [0,1]^n \to [0,1]$, where $f_m = \sum_{i=1}^{k_m} \alpha_i \chi_{E_i}$ with $\alpha_i \in [0,1]$, k_m a suitable index that depends on m and E_i are Borel subsets of $[0,1]^n$.

The smallest Riesz tribe that cointains all projections

$\mathcal{RT}_n \simeq \text{Borel}([0,1]^n, [0,1])$

Sketch of proof

1. Since projections are Borel functions, $\mathcal{RT}_n \subseteq \text{Borel}([0,1]^n, [0,1])$. Viceversa,
2. Standard argument in literature. Any Borel function $f : [0,1]^n \to [0,1]$ is the uniform limit of an increasing sequence of simple functions $f_m : [0,1]^n \to [0,1]$, where $f_m = \sum_{i=1}^{k_m} \alpha_i \chi_{E_i}$ with $\alpha_i \in [0,1]$, k_m a suitable index that depends on m and E_i are Borel subsets of $[0,1]^n$. Then, we only have to prove that $\chi_E \in \mathcal{RT}_n$.

Sketch.

3. For $n = 1$ and $E = (r, 1]$, it is enough to note that $\chi_{(r,1]} = \bigwedge_m f_{m,r}$, where $f_{m,r}$ is the continuous piecewise linear function with real coefficients defined by

$$f_{m,r}(x) = \begin{cases} 0 & \text{if } x \leq r - \frac{r}{2^m} \\ \text{linear} & \text{if } r - \frac{r}{2^m} < x \leq r \\ 1 & \text{if } x > r \end{cases}$$

Sketch.

3. For $n = 1$ and $E = (r, 1]$, it is enough to note that $\chi_{(r,1]} = \bigwedge_m f_{m,r}$, where $f_{m,r}$ is the continuous piecewise linear function with real coefficients defined by

$$f_{m,r}(x) = \begin{cases} 0 & \text{if } x \leq r - \frac{r}{2^m} \\ \text{linear} & \text{if } r - \frac{r}{2^m} < x \leq r \\ 1 & \text{if } x > r \end{cases}$$

4. For $n > 1$, $E \in \mathcal{B}([0,1]^n)$ iff $E = \prod_{i=1}^n E_i$, with $E_i \in \mathcal{B}([0,1])$.

□

Another characterization

Baire functions

X, Y topological spaces.

$f : X \to Y$ is a Baire function if it belongs to the algebra of functions obtained by transfinite induction starting from the continuous functions and it is closed under pointwise limits of convergent sequences.

Another characterization

Baire functions

X, Y topological spaces.

$f : X \to Y$ is a Baire function if it belongs to the algebra of functions obtained by transfinite induction starting from the continuous functions and it is closed under pointwise limits of convergent sequences.

Lebesgue-Hausdorff theorem

If X is a metric space and $Y = [0,1]^n$, then

$$\mathrm{Baire}(X, Y) = \mathrm{Borel}(X, Y)$$

Finally,

Finally,

$$\mathcal{RT}_n \simeq \mathrm{Baire}([0,1]^n,[0,1]) \simeq \mathrm{Borel}([0,1]^n,[0,1]) \simeq Lind_{\mathcal{IRL},n}$$

Finally,

$$\mathcal{RT}_n \simeq \mathrm{Baire}([0,1]^n,[0,1]) \simeq \mathrm{Borel}([0,1]^n,[0,1]) \simeq \mathit{Lind}_{\mathcal{IRL},n}$$

Another way in:

- The isomorphism between $\mathcal{RT}$ and $\mathit{Baire}([0,1]^n,[0,1])$ can be also deduced as a straightforward consequence of the work of A. Dvurečenskij on the Loomis-Sikorski theorem for ℓ-groups.

A recap:

1. We have defined convergence in logic and characterized the norm-completion of $\boldsymbol{Lind}_{\mathbb{R}\mathcal{L},n}$,
2. We analyzed limits in deductive systems,
3. We have found a "nice" infinitary variety whose objects are in correspondence with basically disconnected compact Hausdorff spaces,
4. We have considered the logical system attached to such variety and have given different functional characterizations of its Lindenbaum-Tarski algebra,
5. We have proved the Loomis-Sikorski theorem for RMV-algebras and deduced $[0, 1]$-completeness of our logic.

Thank you!