Choice-free Stone duality

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Note: of course we won't prove that every BA is isomorphic to *a field of sets*, since this implies the Boolean Prime Filter Theorem.

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- “a mix of Stone and Tarski, connected by Vietoris”;
- “possibility semantics (further) topologized”.
- “the hyperspace approach, in contrast to the pointfree approach”.

Stone Representation of BAs

Theorem (Stone 1936). Every Boolean algebra is isomorphic to the BA of clopen sets of some topological (Stone) space.

Marshall Stone (1903 - 1989)

Stone Representation of DLs

Theorem (Stone 1937). Every distributive lattice is isomorphic to the distributive lattice of compact open sets of some topological (spectral) space.

Marshall Stone (1903 - 1989)

Boolean algebra of regular open sets

Theorem (Tarski 1937). For every topological space X, the set $\mathsf{RO}(X)$ of regular open subsets of X forms a Boolean algebra.

Alfred Tarski (1901 - 1983)

Regular open sets

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For $U, V \in \mathsf{RO}(X)$ we put

$$U \wedge V = U \cap V,$$

$$U \vee V = \mathrm{Int}(\mathrm{Cl}(U \cup V)),$$

$$\neg U = \mathrm{Int}(X \setminus U).$$

Vietoris space

Theorem (Vietoris 1922, Stone version). For every Stone space X its Vietoris space, i.e., the space of closed sets equipped with the hit-and-miss topology, is again a Stone space.

Leopold Vietoris (1891 - 2002)

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The upper Vietoris topology has the basis

$$[U] = \{F \in VX : F \subseteq U\}, \ U \in \mathsf{Clop}(X).$$

The lower Vietoris topology has the subbasis

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The Vietoris topology is the join of the upper and lower Vietoris topologies.

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This isomorphism $\varphi : A \to \mathsf{Clop}(X_A)$ is given by $\varphi(a) = \widehat{a}$.

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This will resemble Stone's representation of distributive lattices.

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- A subset of X is $\leqslant$-regular open if it is regular open in the upset topology induced by $\leqslant$.
- Then $(X_A, \leqslant)$ is a separative poset, i.e., every principal upset is regular open.

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Then $\mathsf{CO}\mathcal{RO}(X_A)$ is a Boolean algebra, where

$$U \wedge V = U \cap V,$$

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What kind of space is X_A?

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3. every proper filter in $\mathsf{CO}\mathcal{RO}(X)$ is $\mathsf{CO}\mathcal{RO}(x)$ for some $x \in X$.

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Proposition. Every UV-space is a spectral space.

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Theorem (Choice-free representation of BAs) For each Boolean algebra A there is a UV-space X such that A is isomorphic to $\mathsf{C}\mathcal{O}\mathcal{R}\mathcal{O}(X)$.

This correspondence can be extended to a full duality of the corresponding categories.

This duality is the topological version of the duality between BAs and (filter-descriptive) possibility frames (Holliday 2015).

Examples of UV-spaces

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Assuming the PFT, every UV-space is homeomorphic to $UV(X)$ for some Stone space X.

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A UV-map between UV-spaces X and X' is a spectral map $f : X \to X'$ that is also a p-morphism:

$$\text{if } f(x) \leqslant' y', \text{ then } \exists y : x \leqslant y \text{ and } f(y) = y'.$$

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Theorem. The category of UV-spaces with UV-maps is dually equivalent to the category of Boolean algebras with Boolean homomorphisms.

Duality dictionary

BA	**UV**	**Stone**
BA	UV-space	Stone space
homomorphism	UV-map	continuous map
filter	$\uparrow x$, $x \in X$	closed set
ideal	$U \in \mathsf{O}\mathcal{RO}(X)$	open set
principal filter	$U \in \mathsf{C}\mathsf{O}\mathcal{RO}(X)$	clopen set
principal ideal	$U \in \mathsf{C}\mathsf{O}\mathcal{RO}(X)$	clopen set
maximal filter	$\{x\}$, $x \in \mathrm{Max}_{\leqslant}(X)$	$\{x\}$, $x \in X$
maximal ideal	$X \setminus \downarrow x$, $x \in \mathrm{Max}_{\leqslant}(X)$	$X \setminus \{x\}$, $x \in X$
relativization	subspace $U \in \mathsf{C}\mathsf{O}\mathcal{RO}(X)$	subspace $U \in \mathsf{Clop}(X)$
complete algebra	complete UV-space	ED Stone space
atom	isolated point	isolated point
atomic algebra	$\mathrm{Cl}(X_{\mathrm{iso}}) = X$	$\mathrm{Cl}(X_{\mathrm{iso}}) = X$
atomless algebra	$X_{\mathrm{iso}} = \varnothing$	$X_{\mathrm{iso}} = \varnothing$
homomorphic image	subspace induced by $\uparrow x$, $x \in X$	closed set
subalgebra	image under UV-map	image under continuous map
direct product	UV-sum	disjoint union
canonical completion	$\mathcal{RO}(X)$	$\wp(X)$
MacNeille completion	$\mathcal{RO}(\{x \in X \mid \uparrow x \in \mathsf{C}\mathsf{O}\mathcal{RO}(X)\})$	$\mathsf{RO}(X)$

Table: Dictionary for **BA**, **UV**, and **Stone**.

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The standard Stone duality proof uses the fact that if X is an infinite set and $U \subseteq X$, then either U is infinite or $X \setminus U$ is infinite.

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Our proof is very similar, but we use the fact that if X is an infinite separative poset and $U \in \mathcal{RO}(X)$, then either U is infinite or $\neg U = \mathrm{Int}_{\leqslant}(X \setminus U) = \{x \in X \mid \forall y \geqslant x \;\; y \notin U\}$ is infinite.

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Proof. By duality, it suffices to show that in any infinite UV-space X,

- there is an infinite descending chain $U_0 \supsetneq U_1 \supsetneq \dots$ of sets from $\mathsf{CO}\mathcal{RO}(X)$, as well as
- an infinite family of pairwise disjoint sets from $\mathsf{CO}\mathcal{RO}(X)$.

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For this it suffices to show that

($\star$) for any $n \in \mathbb{N}$, there is a descending chain $U_0 \supseteq U_1 \supseteq \cdots \supseteq U_n$ of infinite sets from $\mathsf{CO}\mathcal{RO}(X)$ such that $U_i \cap \neg U_{i+1} \neq \varnothing$ for $i \in n$.

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For then by DC,

- there is an infinite descending chain $U_0 \supseteq U_1 \supseteq \dots$ of sets from $\mathsf{CO}\mathcal{RO}(X)$ with $U_i \cap \neg U_{i+1} \neq \varnothing$ for each $i \in \mathbb{N}$, in which case $\{U_0 \cap \neg U_1, U_1 \cap \neg U_2, \dots\}$ is our antichain.

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- Since U_n is infinite and X is T_0, there are $x, y \in U_n$ such that $x \nleqslant y$.
- Then by the separation property of UV-spaces, there is a $V \in \mathsf{CO}\mathcal{RO}(X)$ such that $x \in V$ and $y \notin V$, which with $y \in U_n$ and $U_n, V \in \mathcal{RO}(X)$ implies that there is a $z \geqslant y$ such that $z \in U_n \cap \neg V$.

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- Since $U_n, V \in \mathsf{CO}\mathcal{RO}(X)$, we have $U_n \cap V, U_n \cap \neg V \in \mathsf{CO}\mathcal{RO}(X)$ by the definition of a UV-space;

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- Since $U_n, V \in \mathsf{CO}\mathcal{RO}(X)$, we have $U_n \cap V, U_n \cap \neg V \in \mathsf{CO}\mathcal{RO}(X)$ by the definition of a UV-space; and since $z \in U_n \cap \neg V$ and $x \in U_n \cap V$, we have $z \in U_n \cap \neg(U_n \cap V) \neq \varnothing$ and $x \in U_n \cap \neg(U_n \cap \neg V) \neq \varnothing$.

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We prove ($\star$) by induction. Let $U_0 = X$. For the inductive step:

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- Then by the separation property of UV-spaces, there is a $V \in \mathsf{CO}\mathcal{RO}(X)$ such that $x \in V$ and $y \notin V$, which with $y \in U_n$ and $U_n, V \in \mathcal{RO}(X)$ implies that there is a $z \geqslant y$ such that $z \in U_n \cap \neg V$.
- Since $U_n, V \in \mathsf{CO}\mathcal{RO}(X)$, we have $U_n \cap V, U_n \cap \neg V \in \mathsf{CO}\mathcal{RO}(X)$ by the definition of a UV-space; and since $z \in U_n \cap \neg V$ and $x \in U_n \cap V$, we have $z \in U_n \cap \neg(U_n \cap V) \neq \varnothing$ and $x \in U_n \cap \neg(U_n \cap \neg V) \neq \varnothing$. Thus, if $U_n \cap V$ is infinite, then we can set $U_{n+1} := U_n \cap V$, and otherwise we claim that $U_n \cap \neg V$ is infinite, in which case we can set $U_{n+1} := U_n \cap \neg V$.

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We prove ($\star$) by induction. Let $U_0 = X$. For the inductive step:

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- Since $U_n \in \mathcal{RO}(X)$, we may regard U_n as a separative poset. Given $V \in \mathcal{RO}(X)$, we have $U_n \cap V, U_n \cap \neg V \in \mathcal{RO}(U_n)$ and $U_n \cap \neg V = \neg_n(U_n \cap V)$, where $\neg_n$ is the negation in $\mathcal{RO}(U_n)$. Then since U_n is infinite, either $U_n \cap V$ or $\neg_n(U_n \cap V)$ is infinite.

Priestley-like duality

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Then $(X_A, \leqslant)$ is a Priestley space.

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In addition, if $\mathsf{Clop}\mathcal{RO}(X) = \{\text{clopen } \leqslant\text{-regular open sets}\}$:

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2. $x \nleqslant y \Rightarrow$ there is a $U \in \mathrm{Clop}\mathcal{RO}(X)$ s.t. $x \in U$ and $y \notin U$;
3. every proper filter in $\mathrm{Clop}\mathcal{RO}(X)$ is $\mathrm{Clop}\mathcal{RO}(x)$ for some $x \in X$.

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Such spaces are order-homeomorphic to $(VX, \subseteq)$ for some Stone space X.

Conclusions and further directions

- We developed choice-free topological duality for Boolean algebras.
- With choice this can be converted into a Priestley-like order-topological duality.
- We also have extensions of this duality to modal algebras (modal logic) in connection with possibility semantics.
- It should also be possible to give choice-free dualities for distributive lattices and Heyting algebras (cf. Massas 2016).

Thank you!