Duality and Bounded Bisimulations: old and new applications

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- then, *taking equivalence classes under provable bi-implication in* (IPC), *the sequence* $\{ [A^i(x, \underline{y})] \}_{i \geq 1}$ *is ultimately periodic with period 2.*
- The latter means that there is N such that

$$\vdash_{IPC} A^{N+2} \leftrightarrow A^N \quad . \tag{2}$$

- An interesting consequence of this result is that *least (and greatest) fixpoints of monotonic formulae are definable in* (IPC) [Mardaev93].

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  `https://git8.cs.fau.de/redmine/projects/ruitenburg1984`
- We supply a semantic proof, using duality and bounded bisimulations machinery.

The Algebraic Reformulation

In classical propositional calculus (CPC), Ruitenburg Theorem holds with index 1 and period 2, namely given a formula $A(x, \underline{y})$, we have that

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The first step is to re-interpret this statement in the category of finitely presented Boolean algebras (actually, finitely generated free algebras would suffice).

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A morphism $\mu : \mathcal{F}_B(x_1, \ldots, x_n) \longrightarrow \mathcal{F}_B(\underline{z})$ associates with the equivalence class of $B(x_1, \ldots, x_n)$ in $\mathcal{F}_B(x_1, \ldots, x_n)$ the equivalence class of $B(A_1/x_1, \ldots, A_n/x_n)$ in $\mathcal{F}_B(\underline{z})$ (for some tuple $A_1, \ldots, A_n$: we say that μ is induced by this tuple).

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Composition is substitution: if μ is induced by $A_1(\underline{z}), \dots, A_n(\underline{z})$ and ν is induced by $C_1(x_1, \dots, x_n), \dots, C_m(x_1, \dots, x_n)$, then

$$\mu \circ \nu : \mathcal{F}_B(y_1, \dots, y_m) \longrightarrow \mathcal{F}_B(x_1, \dots, x_n) \longrightarrow \mathcal{F}_B(\underline{z})$$

is induced by the m-tuple
$C_1(A_1/x_1, \dots, A_n/x_n), \dots, C_m(A_1/x_1, \dots, A_n/x_n)$.

The Algebraic Reformulation

Consider the map $\mu_A : \mathcal{F}_B(x, \underline{y}) \longrightarrow \mathcal{F}_B(x, \underline{y})$ induced by the tuple $A, \underline{y}$; then, the statement (3) is equivalent to

$$\mu_A^3 = \mu_A \quad . \tag{4}$$

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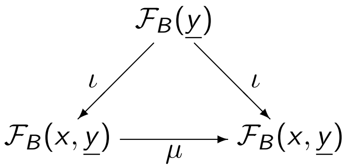
This raises the question: which endomorphisms of $\mathcal{F}_B(x, \underline{y})$ are of the kind μ_A for some $A(x, \underline{y})$? The answer is simple: they are the maps such that the triangle



commutes, where ι is the 'inclusion' map induced by the tuple $\underline{y}$.

The Algebraic Reformulation

Let us denote by $\mathcal{A}[x]$ the *algebra of polynomials* over $\mathcal{A}$, i.e. the coproduct of the Boolean algebra $\mathcal{A}$ with the free algebra on one generator (thus $\mathcal{F}_B(x, \underline{y})$ is equal to $\mathcal{F}_B(\underline{y})[x]$).

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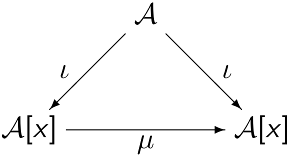
A slight generalization of statement (4) now reads as follows:

- let $\mathcal{A}$ be a finitely presented Boolean algebra and let the map $\mu : \mathcal{A}[x] \longrightarrow \mathcal{A}[x]$ commute with the coproduct injection $\iota : \mathcal{A} \longrightarrow \mathcal{A}[x]$



Then we have

$$\mu^3 = \mu \quad . \tag{5}$$

Dualization

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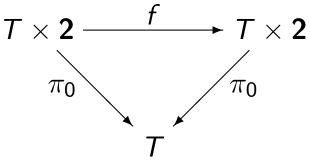
Thus statement (5) now becomes the following trivial exercise:

- Let T be a finite set and let the function $f : T \times \mathbf{2} \longrightarrow T \times \mathbf{2}$ commute with the product projection $\pi_0 : T \times \mathbf{2} \longrightarrow T$



Then we have

$$f^3 = f \quad . \tag{6}$$

Duality ingredients

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We may view an arbitrary algebra as a Lindenbaum algebra of a theory (in the given logic); in this sense, a finitely presented algebra is the Lindenbaum algebra of a *finitely axiomatized* theory.

Duality ingredients

The dual of an algebra/theory is the space of its models (in the Boolean case, the dual of B is the set $Hom[B, \mathbf{2}]$ of the homomorphisms of B into the truth value algebra - this is nothing but the set of models of B, viewed as a theory).

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However, going beyond the classical case, the situation becomes more involved: models must be structured!

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There is however a deep difference between bounded and unbounded bisimulations: unbounded bisimulation has to be ascribed to a geometric structure (typically, a sheaf structure), whereas bounded bisimulation retains specific combinatorial features related to definability aspects.

Most logical problems are analyzed in G.-Zawadowski book "Sheaf, games and model completions" taking into account the role of both aspects (the geometric and the combinatorial aspects).

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- existence of uniform interpolants is shown to be equivalent to existence of images in the dual of the category of finitely presented algebras (algebraization step);
- as models are structured as sheaves, if such images exists, they must be sheaf-theoretic images;
- sheaf theoretic images are in fact ‘definable’ because they are closed under bounded (sufficiently high bounded!) bisimulation.

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A similar strategy has been used for many other questions, for positive and negative results (definability of dual difference operators, regularity of epis, characterization of projectivity, effectiveness of equivalence relations, etc.).

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We are applying the same strategy for Ruitenburg Theorem: dual morphisms are seen as natural transformations, 2-periodicity is verified for them and finally made uniform using bounded bisimulation ranks.

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Recall that a finite distributive lattice is isomorphic to the set of downward closed subsets $\downarrow L$ of a *finite poset* $(L, \leq)$.

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As geometric environment, we consider the category $\mathbf{P_0}$ of finite rooted posets (with *p*-morphisms) and the category of sheaves over them with the canonical (Grothendieck) topology.

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$f : Q \longrightarrow P$ is a p-morphism iff it is order-preserving and moreover satisfies the following condition forall $q \in Q, p \in P$

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A presheaf is a contravariant functor

$$F : \mathbf{P_0}^{op} \longrightarrow \mathbf{Set}$$

into the category of sets.

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The typical (pre)sheaf we use is the sheaf of L-evaluations

$$h_L \simeq Hom(-, L)$$

(the *Hom* is taken into the category of posets) for a finite poset $(L, \leq)$: in case L is the powerset of a finite set ordered by reverse inclusion, this is the sheaf of finite Kripke models (over a finite propositional language).

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The easy but crucial fact we use is that product in presheaves (and sheaves) is pointwise: i.e. $[F \times G](P) \simeq F(P) \times G(P)$, etc.

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Thus, for definability issues (i.e. for a full duality), we need another ingredient, of a more combinatorial nature: bounded bisimulations.

Bounded bisimulations can be introduced either via a recursive definition of via Ehrenfeucht-Fraissé games.

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Player 1 can choose either a point in P or a point in Q and Player 2 must answer by choosing a point in the other poset; the only rule of the game is that, if $\langle p \in P, q \in Q \rangle$ is the last move played so far, then in the successive move the two players can only choose points $\langle p', q' \rangle$ such that $p' \leq p$ and $q' \leq q$.

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If $\langle p_1, q_1 \rangle, \ldots, \langle p_i, q_i \rangle, \ldots$ are the points chosen in the game, Player 2 wins iff for every $i = 1, 2, \ldots,$ we have that $u(p_i) = v(q_i)$.

Games and Bounded Bisimulations

We say that

- $u \sim_\infty v$ iff *Player 2 has a winning strategy* in the above game with infinitely many moves;
- $u \sim_n v$ (for $n > 0$) iff *Player 2 has a winning strategy* in the above game with n moves, i.e. he has a winning strategy provided we stipulate that the game terminates after n moves;
- $u \sim_0 v$ iff $u(\rho(P)) = v(\rho(Q))$ (recall that $\rho(P), \rho(Q)$ denote the roots of P, Q).

We shall use the notation $[v]_n$ for the equivalence class of an L-valuation v via the equivalence relation $\sim_n$.

The Duality Statement

We say that a natural transformation $\psi : h_L \longrightarrow h_{L'}$ *has b-index n* iff for every $v : P \longrightarrow L$ and $v' : P' \longrightarrow L$, we have that $v \sim_n v'$ implies $\psi(v) \sim_0 \psi(v')$.

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Theorem

The category of Heyting algebras freely generated by a finite bounded distributive lattice is dual to the subcategory of (pre)sheaves having as objects the evaluations sheaves and as arrows the natural transformations having a b-index.

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A natural transformation f has a b-index iff the inverse image along f of a definable sub(pre)sheaf is definable. Such a map is the dual of a substitution.

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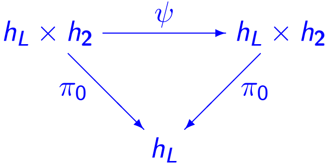
All natural transformations from $h_L \times h_{\mathbf{2}}$ into itself, commuting over the first projection π_0 and having a b-index, are ultimately periodic with period 2.

Spelling this out, this means the following. Fix a natural transformation $\psi = \langle \pi_0, \chi \rangle : h_L \times h_{\mathbf{2}} \longrightarrow h_L \times h_{\mathbf{2}}$ having a b-index such that the diagram



commutes; we have to find an N such that $\psi^{N+2} = \psi^N$.

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Let $\psi = \langle \pi_0, \chi \rangle : h_L \times h_\mathbf{2} \longrightarrow h_L \times h_\mathbf{2}$ *be a natural transformation. Then for all rooted finite poset* P *there is* N_P *such that* $\psi^{N_P+2}(P) = \psi^{N_P}(P)$

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The proof is a moderate complication of what happens in the classical logic case (one can take N_P to be the height of P).

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As usual, for such problems, one needs an appropriate notion of rank. Ranks were used by various authors (Fine, Visser, G., etc.), the variant we use here is explained below. First, we need some definitions.

Call $(v, u) \in h_{L \times \mathbf{2}}(P)$ *2-periodic* (or just *periodic*) iff we have $\psi^2(v, u) = (v, u)$; a point $q \in P$ is similarly said periodic in (v, u) iff $(v, u)_q$ is periodic (here $(v, u)_q$ is (v, u) restricted to the points below q).

Ranks

Let $\psi = \langle \pi_0, \chi \rangle$ have b-index $n \geq 1$. and let $(v, u) \in h_L(P)$ be given. The *type* of a periodic point $p \in P$ is the pair of equivalence classes

$$\langle [(v_p, u_p)]_{n-1}, [\psi(v_p, u_p)]_{n-1} \rangle. \tag{7}$$

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A non-periodic point $p \in P$ has *minimal rank* iff we have $rk(p) = rk(q)$ for all non-periodic $q \leq p$.

The final step

The first trick is to show that the periodicity number N_P of the above Lemma can be taken to depend not on the height of a finite poset, but on the height of $v(\rho_P)$ in the (fixed) finite poset L. Thus one can make an induction on this height.

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The base of the induction is the classical logic case. So, one can suppose that, in a given $L \times \mathbf{2}$-evaluation (v, u), all points whose v-values have L-height less than the induction parameter l become periodic after applying our ψ a sufficiently number of times, namely N_l-times.

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After such iterations, suppose that p has v-value of L-height l, but it is not yet periodic. We let r be the minimum rank of points $q \leq p$ which are not periodic.

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It is shown that after *two more iterations*, all points $p_0 \leq p$ having rank r become periodic or increase their rank, thus causing the overall minimum rank below p to increase.

This means that after at most $2(R - r) \leq 2R$ iterations of ψ, all points below p (p itself included!) become periodic (here $R := R(n, L)$, see above).

A question

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QUESTION: *is it possible to refine the above arguments and get a better bound, still within a semantic approach?*

THANKS FOR ATTENTION !