Bands, Skew lattices, and sheaves

Based on joint work with Andrej Bauer, Karin Cvetko-Vah, Sam van Gool, and Ganna Kudryavtseva and on ongoing work with Clemens Berger

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Plan

- Bands are idempotent semigroups. We review the **basic theory of (regular) bands**
- **Comprehension factorization systems** (CFSs) come from categorical logic. We show that the Street-Walters CFS for small categories restricted to partial orders lifts to regular bands [ongoing work with Clemens Berger]
- Restricting the CFS for regular bands to normal bands: **normal bands are presheaves over meet-semilattices** [Kimura'58]
- Adding the **sheaf condition**: distributive and Boolean bands
- Boolean bands are algebraic: **Skew Boolean algebras** [Bauer-CvetkoVah 2013 and Kudryatseva 2013]
- Strongly distributive skew lattices (SDSLs) and **non-commutative Priestley duality** [BCVGvGK 2013]
- SDSLs and **the patch monad** [ongoing with Clemens Berger]

Bands

A **band** $(X, \cdot)$ is an *idempotent semigroup*

Eventually we will consider bands with a **zero**: $\quad 0 \cdot x = x \cdot 0 = 0$

Induced **Partial order:**

$$x \leqslant y \quad \Longleftrightarrow \quad x = yxy \quad \Longleftrightarrow \quad x = yx = xy$$

Induced **Quasi-order:**

$$x \preceq y \quad \Longleftrightarrow \quad x = xyx \quad \Longleftrightarrow \quad \exists s, t \in X^1 \quad x = syt$$

Corresponding equivalence relation:

$$x\mathcal{D}y \quad \Longleftrightarrow \quad x \preceq y \quad \text{and} \quad y \preceq x$$

Note that $\leqslant$ is contained in $\preceq$ and that $X \to X/\mathcal{D}$ preserves both and makes them equal. However, much more is true

Fundamental facts about bands

For any band X

(i) The quotient map $(X, \leqslant) \to (X/\mathcal{D}, \preceq /\mathcal{D})$ is order preserving. That is,

$$x \leqslant y \implies x \preceq y \iff [x]_{\mathcal{D}} \, (\preceq /\mathcal{D}) \, [y]_{\mathcal{D}} \text{ denoted } [x]_{\mathcal{D}} \leqslant [y]_{\mathcal{D}}$$

(ii) The quasi-order $\preceq$ is *compatible* with the band operation and thus $\mathcal{D}$ is a *semigroup congruence* on X

(iii) The quotient $X/\mathcal{D}$ is the *universal semilattice quotient* of X

(iv) For $x, y \in X$, if $x \leqslant y$ and $y \preceq x$ then $x = y$

(v) $(X, \leqslant)$ is a partially ordered set in which $\mathcal{D}$-classes are order-discrete

Regular bands

For any $z \in X$, the **order-ideal** generated by z is given by

$$\downarrow z = \{x \in X \mid x \leqslant z\} = zXz$$

It is always a subband of X but the map $p_z : X \to \downarrow z$ given by $x \mapsto zxz$ is not necessarily a homomorphism

X is said to be **regular** provided each p_z is a homomorphism

The one-sided Green's relations are defined by $x\mathcal{L}y$ if and only if $Xx = Xy$ and $x\mathcal{R}y$ if and only if $xX = xY$. Clearly $\mathcal{L}$ is a right congruence, while $\mathcal{R}$ is a left congruence. However, for bands, the following conditions are equivalent

- p_z is a homomorphism for each $z \in X$
- $zxzyz = zxyz$ for all $x, y, z \in X$
- $\mathcal{L}$ and $\mathcal{R}$ are semigroup congruences

Fundamental fact about regular bands

For any regular band X the following commutative diagram is a *pullback square* in the category of semigroups

$$\begin{array}{ccc} X & \longrightarrow & X/\mathcal{L} \\ \downarrow & & \downarrow \\ X/\mathcal{R} & \longrightarrow & X/\mathcal{D} \end{array}$$

That is, $X \cong X/\mathcal{R} \times_{X/\mathcal{D}} X/\mathcal{L}$

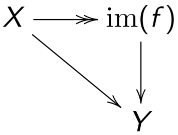
A band X is called **left regular** provided $xyx = xy$ for all x, y in X. The quotient band $X/\mathcal{R}$ is the *universal left regular reflection* of X

Similarly, X is called **right regular** provided $xyx = yx$ for all x, y in X and the quotient band $X/\mathcal{L}$ is the universal *right regular reflection* of X

From hereon, we focus on the categories LRB of left regular bands

Comprehension schemes

The **comprehension axiom** in set theory governs the formation of sets out of properties: if X is a set and $\varphi(x)$ a property, then $\{x \in X \mid \varphi(x) \text{ holds}\}$ is again a set. In Lawvere's *categorical approach to logic* the comprehension axiom is modeled as (the existence of) a **right adjoint** and gives rise to a **factorisation system**. E.g. in the category of sets, it corresponds to the well-known epi-mono factorisation of a set mapping



where Lawvere's adjointness boils down to the property that the image $\mathrm{im}(f)$ can be characterised as the smallest subset of the target Y through which f factors. In 1973 Street and Walters provided a *comprehensive factorisation of any functor between small categories*

Comprehensive factorisation systems as a tool in topology

Recently, Berger and Kaufmann have established an equivalence between so-called consistent comprehension schemes and complete orthogonal factorisation systems obtaining applications in topological covering theory

Our work with Clemens Berger starts from the comprehensive factorisation of Street-Walters in the special case of **posets** as it is treated in Berger-Kaufmann

<u>Theorem</u> [Street-Walters 1973, Berger-Kaufmann 2017]
Every order preserving map factors essentially uniquely as a *connected map* followed by a *covering*

Coverings

Let X and Y be posets. An order preserving map

$$f : X \to Y$$

is said to be a **covering** provided, for each $x \in X$ and $y' \leqslant f(x)$ there exists a unique $x' \leqslant x$ such that $f(x') = y'$

The best way to understand covering maps is on the basis of **presheaves**. In fact, any covering $f : X \to Y$ is isomorphic (over Y) to the projection map on the *poset of elements* $\mathrm{el}_Y(F)$ of an, up to isomorphism, uniquely determined set-valued presheaf

$$F : Y^{\mathrm{op}} \to \mathrm{Set}$$

Coverings and presheaves

Let Y be a poset and $F : Y^{\mathrm{op}} \to \mathrm{Set}$ a set-valued presheaf on Y

$$\mathrm{el}_Y(F) = \{(y, s) \mid y \in Y \text{ and } s \in F(y)\}$$

with the partial order

$$(y, s) \leqslant (z, t) \iff y \leqslant z \text{ and } t|^z_y = s$$

Then $\pi_F : \mathrm{el}_Y(F) \to Y, (y, s) \mapsto y$ *is a covering*

Futhermore, for any covering $f : X \to Y$, define

$$F_f : Y^{\mathrm{op}} \to \mathrm{Set}, y \mapsto f^{-1}(y)$$

and if $y' \leqslant y$, and $f(x) = y$, then $x|^y_{y'} = x'$ as given by the covering property. Then F_f is a presheaf and f is isomorphic to π_{F_f} over Y

Interlude on presheaves

Let PX be the category of set-valued presheaves on X. This category has a **terminal element**, $\star_{PX}$, which sends each element of X to a singleton set with the obvious restriction maps

An order preserving map $f : X \to Y$ induces an adjunction

$$f_! : PY \leftrightarrows PX : f^*$$

where $f^*F(x) = F(f(x))$ with the restriction maps of F. We are interested in $f_!(\star_{PX})$, which is given by

$$f_!(\star_{PX}) : Y^{\mathrm{op}} \to \mathrm{Set}, y \mapsto \pi_0(y \downarrow f)$$

where $y \downarrow f = \{x \in X \mid y \leqslant f(x)\}$ and π_0 takes the connected components of the underlying undirected graph of a poset. For $y_1 \leqslant y_2$ the restriction map sends a connected component of $y_2 \downarrow f$ to the connected component of $y_1 \downarrow f$ in which it lies

Connected maps and the comprehensive factorisation

A map $f : X \to Y$ is **connected** provided $G = f_!(\star_{PX}) = \star_{PY}$. That is, if and only if $y \downarrow f$ is non-empty and connected for each $y \in Y$

Given an order preserving map $f : X \to Y$, the **comprehensive factorisation** is given by

$$X \xrightarrow{\alpha_f} \mathrm{el}_Y(G) \xrightarrow{\beta_f} Y$$

where $G = f_!(\star_{PX})$, $\alpha_f(x) = (f(x), [x]_{f(x)\downarrow f})$ where $[x]_{f(x)\downarrow f}$ denotes the connected component of x in the poset $f(x) \downarrow f$ and β_f is the projection given by $\beta_f((y,s)) = \pi_G((y,s)) = y$

What does this have to do with bands and skew lattices? In short, it lifts to left/right regular bands and the lifting specializes to the non-commutative Boolean and Priestley dualities for skew Boolean algebras and SDSLs

Lifting the factorisation to left regular bands

Lemma: Let Y be a left regular band and $F : Y^{\mathrm{op}} \to \mathrm{Set}$ a presheaf on the underlying poset. Then $\mathrm{el}_Y(F)$ carries a *unique left regular band structure* so that $\pi_F : \mathrm{el}_Y(F) \to Y$ is a homomorphism. Moreover, $\pi_F/\mathcal{D} : \mathrm{el}_Y(F)/\mathcal{D} \to Y/\mathcal{D}$ is an embedding (isomorphism if F is pointwise non-empty)

Proof sketch: In order for π_F to be a homomorphism, the first coordinate of $(x, s)(y, t)$ must be xy. Also, since Y is left regular, $x(xy)x = xy$ and thus $xy \leqslant x$ and thus we must have $(x, s)(y, t) = (xy, s|^x_{xy})$

A map of regular bands $f : X \to Y$ is said to be a **covering** (resp. **connected**) provided the underlying map of posets is so

Theorem: Any map of left (resp. right) regular bands factors essentially uniquely as a connected map followed by a covering. The resulting comprehensive factorisation system is orthogonal and stable under pullback along coverings

Normal bands

A band X is **normal** provided $\downarrow z = zXz$ is *commutative* for all z, that is, X satisfies the identity $zxyz = zyxz$. Clearly normal bands are regular. X is **left** (resp. **right**) **normal** provided it is simultaneously normal and left (resp. right) regular

Normal bands have a natural characterisation w.r.t. comprehensive factorisation

Proposition: [Kimura'58; Kudryatseva-Lawson'16; Berger-G] A regular band X is *normal* if and only if its semilattice reflection $X \to X/\mathcal{D}$ is a *covering*
In particular, if X is left normal

$$X \cong \mathrm{el}_{X/\mathcal{D}}(F_X)$$

where $F_X : X/\mathcal{D}^{\mathrm{op}} \to \mathrm{Set}, [x]_{\mathcal{D}} \mapsto [x]_{\mathcal{D}}$ and *left normal bands are precisely the elements of pointwise non-empty presheaves on meet semilattices*

Comprehensive factorisation for left normal bands

Theorem: [Berger-G] Let X and Y be left normal bands, $f : X \to Y$ a homomorphism, and $f_{\mathcal{D}} : X/\mathcal{D} \to Y/\mathcal{D}$ the induced meet semilattice morphism. The comprehensive factorisation of f can be obtained as

$$X \longrightarrow \mathrm{el}_{Y/\mathcal{D}}((f_{\mathcal{D}})_!(F_X)) \to Y$$

where $F_X : (X/\mathcal{D})^{\mathrm{op}} \to \mathrm{Set}$ is the presheaf corresponding to X

A topologist's notion of distributive band

A band X is said to be **distributive** (resp. **Boolean**) provided

- X is normal;
- $X/\mathcal{D}$ is a distributive lattice (resp. Boolean) lattice;
- for any finite subset S of X consisting of pairwise commuting elements the join $\bigvee S$ in $(X, \leqslant)$ exists

Proposition: [Berger-G] There is a duality between the category of spectral (resp. Boolean) sheaves and the category of distributive (resp. Boolean) bands

In particular, left distributive (resp. Boolean) bands are the **algebras of partial sections** over compact opens (resp. clopens) of a sheaf over a spectral (resp. Boolean) space with the operation of *left restriction*

$$x \cdot y = x|^{[x]_{\mathcal{D}}}_{[x]_{\mathcal{D}} \cap [y]_{\mathcal{D}}}$$

Boolean bands admit override and relative complement

Let X be a left Boolean band and $x, y \in X$ with domains $U = [x]_{\mathcal{D}}$ and $V = [y]_{\mathcal{D}}$, respectively. Then U and V are clopen and we may define operations of **override** and **relative complement**

$$x \vee y = \text{“}x \text{ on } (U - V)\text{''} \sqcup \text{“}y \text{ on } V\text{''} \qquad x \backslash y = \text{“}x \text{ on } (U - V)\text{''}$$

Denoting the basic operation of X by $\wedge$ we get a **skew Boolean algebra** [Cornish 1980]. That is, $(X, 0, \wedge, \vee, \backslash)$ satisfying

$$0 \wedge x = 0 = x \wedge 0$$
$$x \wedge (x \vee y) = x = (y \vee x) \wedge x \text{ and } x \vee (x \wedge y) = x = (y \wedge x) \vee x$$
$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \text{ and } (y \vee z) \wedge x = (y \wedge x) \vee (z \wedge x)$$
$$(x \backslash y) \wedge (x \wedge y) = 0 \text{ and } (x \backslash y) \vee (x \wedge y) = x$$

Conversely, any skew Boolean algebra comes from a Boolean band [Bauer and Cvetko-Vah 2013, Kudryavtseva 2012]

So Boolean bands are algebraic over normal bands.

Is there a distributive analogue?

Theorem: [B-CV-G-vG-K'13] Distributive bands that admit a symmetric skew lattice structure are dual to sheaves on Priestley spaces

In particular, such 'strongly distributive skew lattices' (SDSLs) allow us to define a sheaf on a Boolean space: For each point $h : X \to 2$ we may define a congruence $\sim_h$ on X by

$$x \sim_h y$$

provided there are $a, b \in X$ with $h(a) = 1$, $h(b) = 0$, and

$$(x \wedge a) \vee b = (y \wedge a) \vee b$$

Then $X/\sim_h$ is primitive, and its non-zero $\mathcal{D}$ class is the stalk at the point h

Can we understand SDSLs relative to distributive bands?

Y a **Stone space** (i.e. Stone dual of a distributive lattice L), and Y^p the **patch** of Y (i.e. dual space of the Booleanization of L).

Then the identity

$$i\colon Y^p \to Y$$

is continuous and induces an **adjunction** between sheaf categories

$$i^* : \mathrm{Sh}(Y) \leftrightarrows \mathrm{Sh}(Y^p) : i_*$$

Consider the corresponding monad $T = i_* i^*$ on $\mathrm{Sh}(Y)$

Claim: [Berger-G] The SDSLs are precisely the T-algebras

Conclusion

- The comprehensive factorization system on small categories restricted to posets lifts to regular bands
- In the case of normal bands this shows that normal bands are presheaves on meet-semilattices
- Restricting this to Boolean bands yields skew Boolean algebras and places non-commutative Boolean duality in a wider setting
- SDSLs are part of a non-commutative Priestley duality. Relative to distributive bands they are (almost surely :)) the algebras for the patch monad

THANK YOU