Modal logics of polytopes – what we know so far

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in collaboration with

Members of Esakia Seminar

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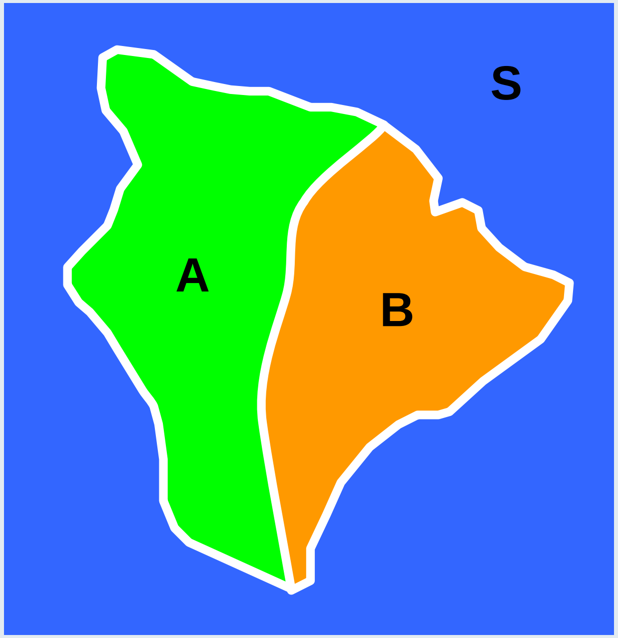
Topology and modal logic

- McKinsey and Tarski 1944
  - Interpret propositions as subsets of a topological space
  - Interpret Boolean operations as their set-theoretic counterparts
  - Interpret the modal diamond as closure, or as derivative
  - **S4** is the modal logic of any crowded, separable, metrizable space
- Rasiowa and Sikorski 1963
  - **S4** is the modal logic of any crowded, metrizable space
- So any $\mathbb{R}^n$ generates **S4**

Mapping a map

Map of an Island



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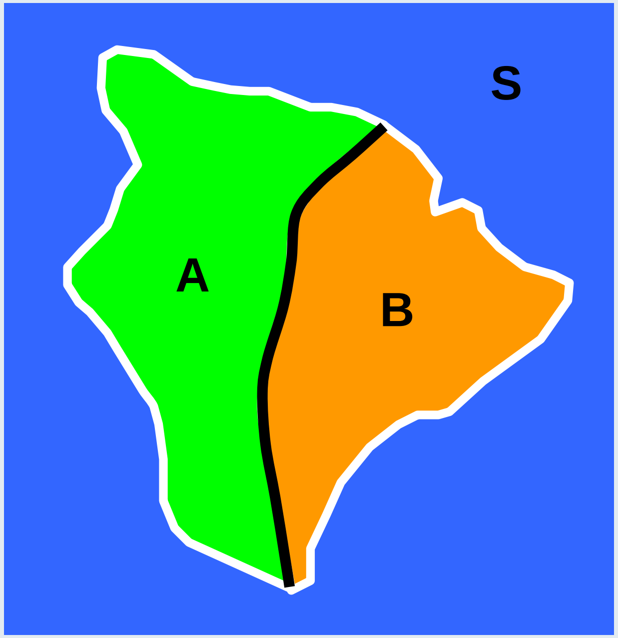

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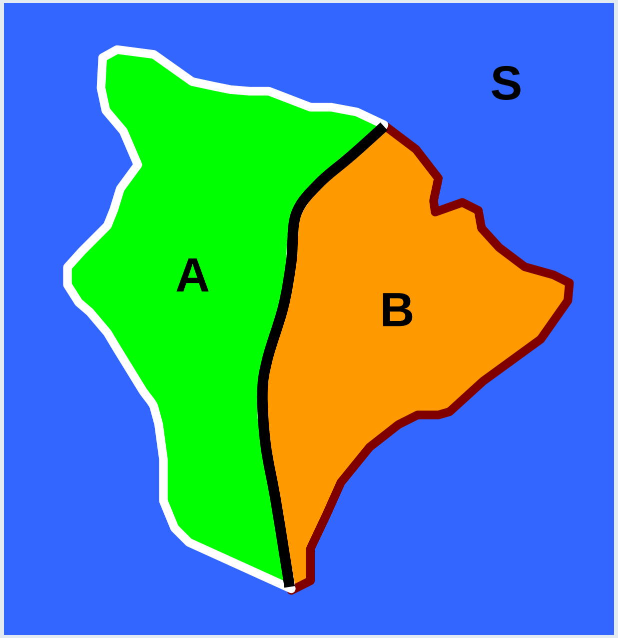

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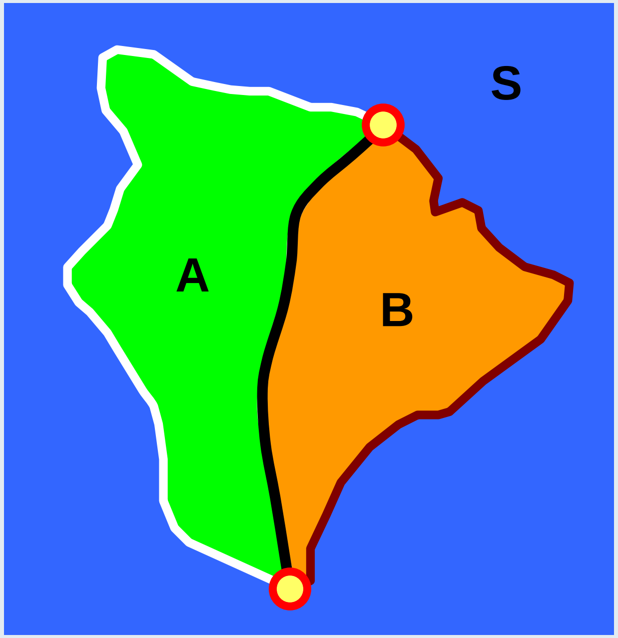
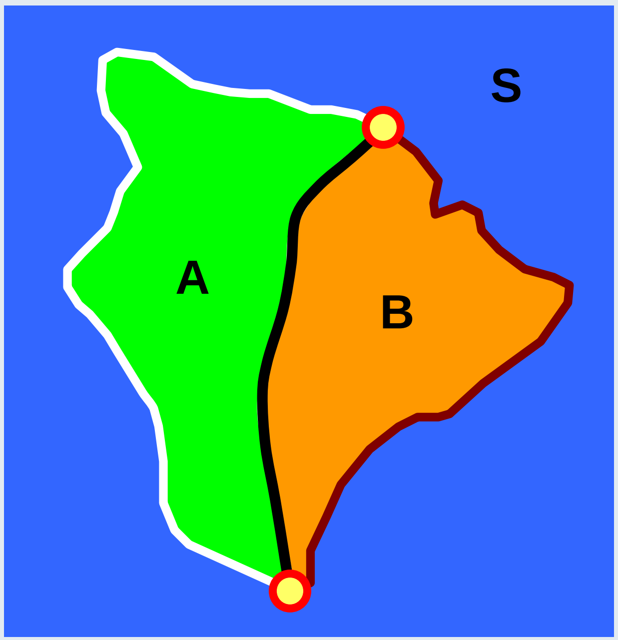
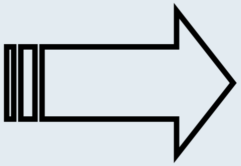


Mapping a map

Map of an Island

Mapping f





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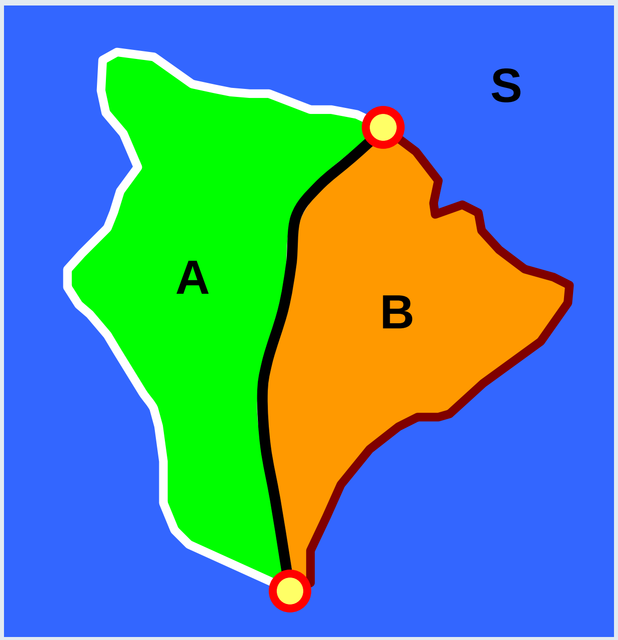
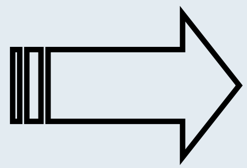


Mapping f



Mapping a map

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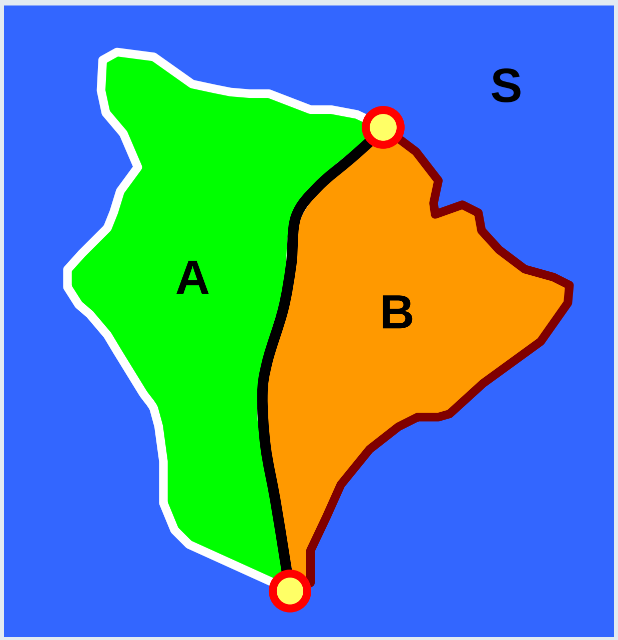
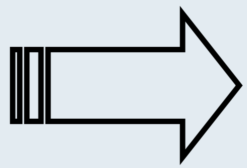
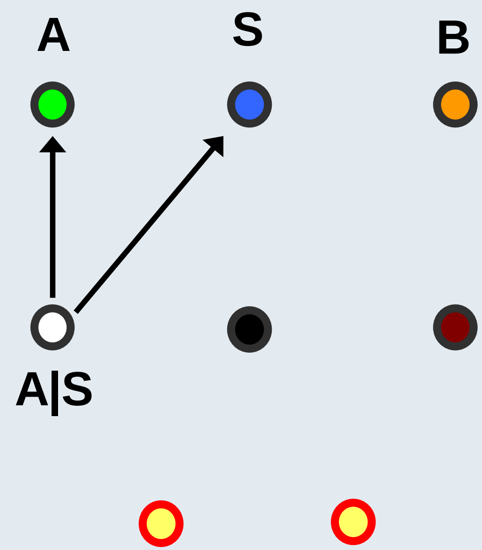
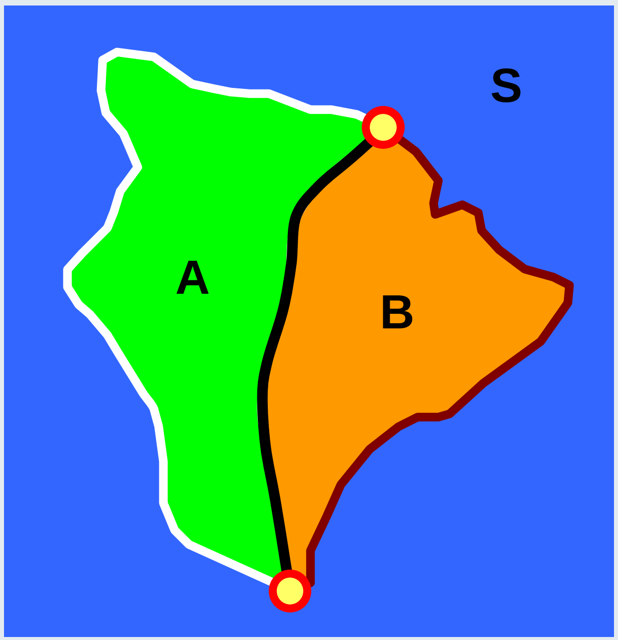


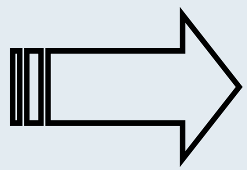


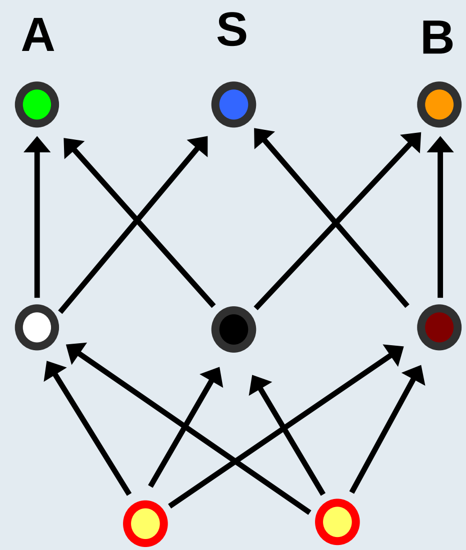


Mapping a map

Map of an Island | **Mapping f** | **Kripke frame**







(M)Any subsets – wild logics

- Any finite connected quasiorder (**S4**-frame) is an interior image of $\mathbb{R}^n$

[G. Bezhanishvili and Gehrke, 2002]

(M)Any subsets – wild logics

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- The subalgebras of the closure algebra $(\wp(\mathbb{R}^n), \mathbf{C})$ generate all connected extensions of **S4**

(M)Any subsets – wild logics

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[G. Bezhanishvili and Gehrke, 2002]

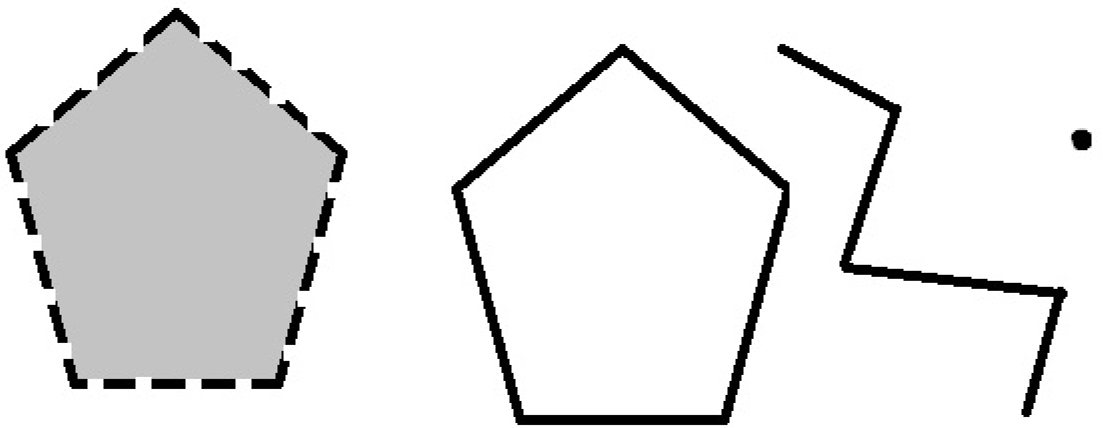
- The subalgebras of the closure algebra $(\wp(\mathbb{R}^n), \mathbf{C})$ generate all connected extensions of **S4**
- The subalgebras of the closure algebra $(\wp(\mathbb{Q}), \mathbf{C})$ generate all normal extensions of **S4**

[G. Bezhanishvili, DG and Lucero-Bryan, 2015]

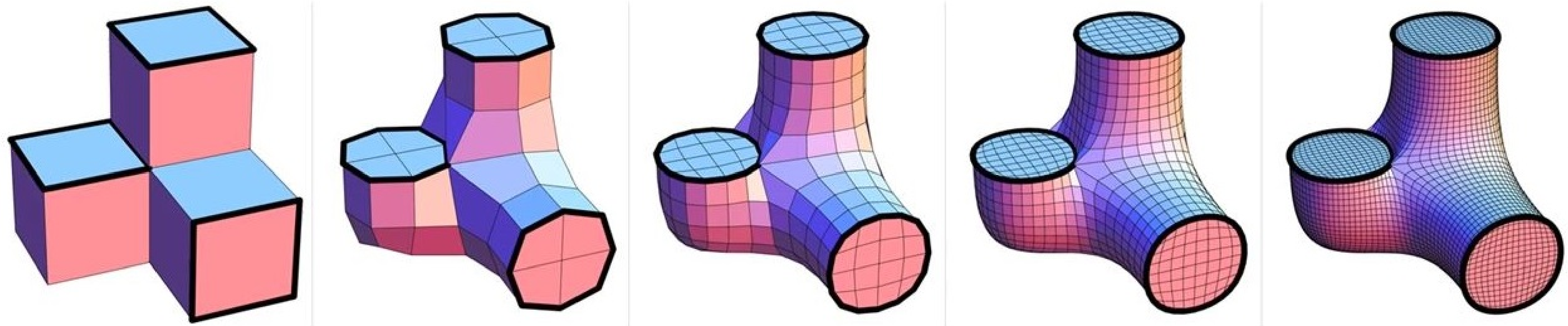
- Too many subsets!

Nice subsets – tame logics?

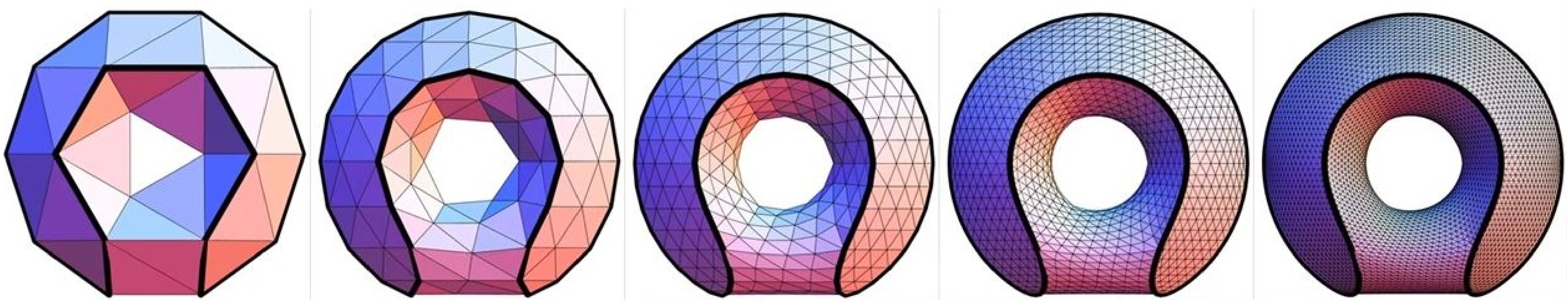
- Piecewise linear subsets = polytopes

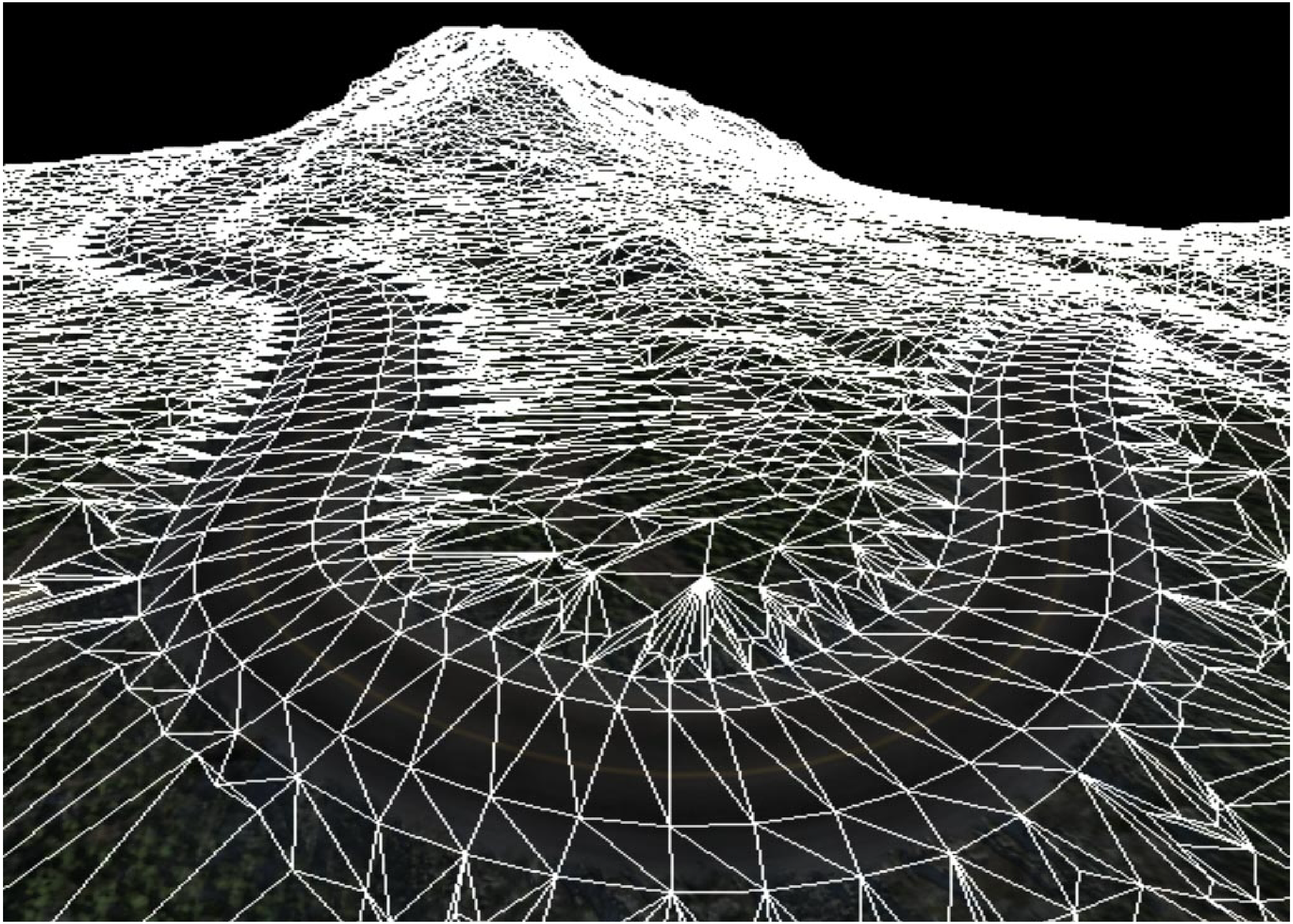


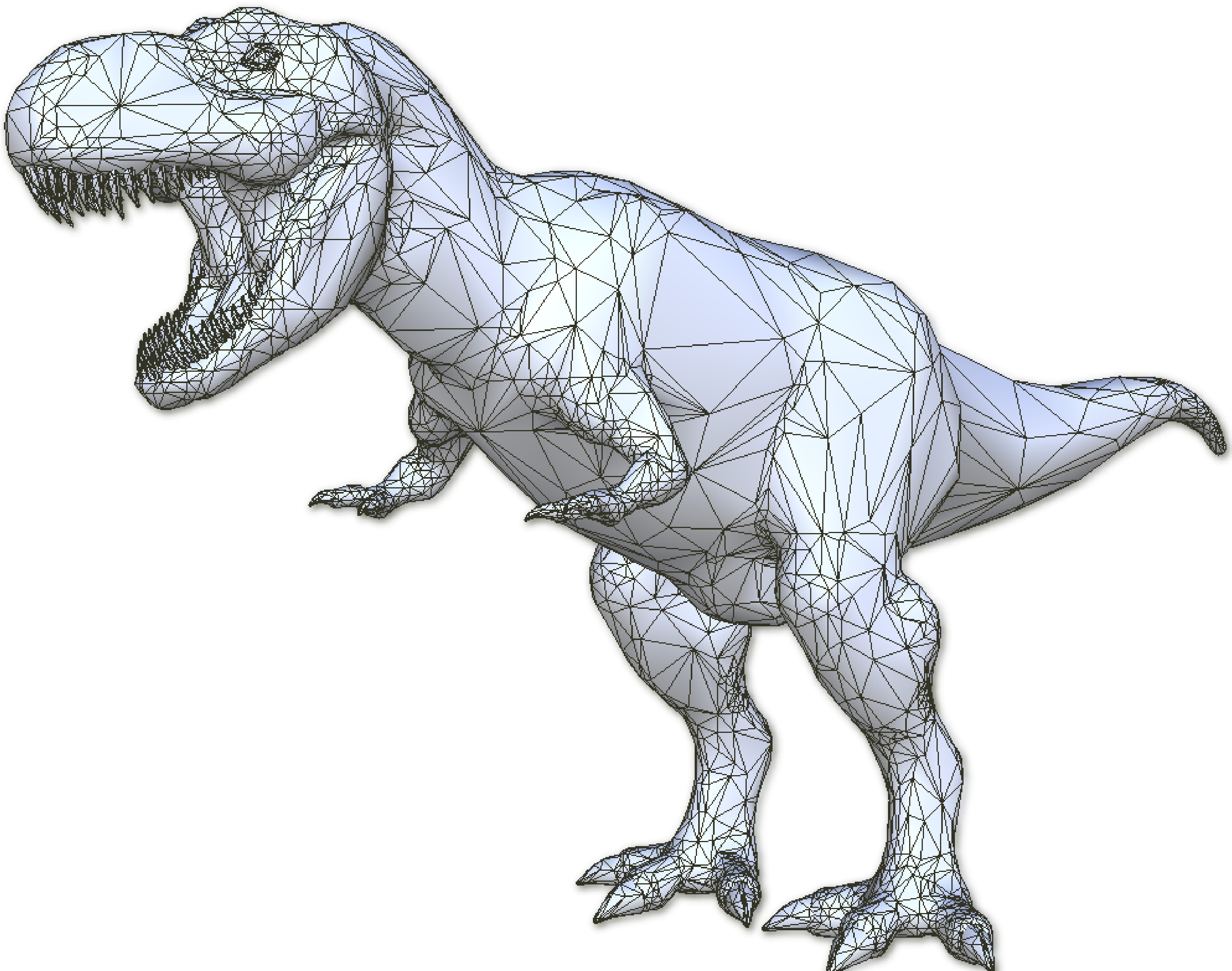
Catmull-Clark

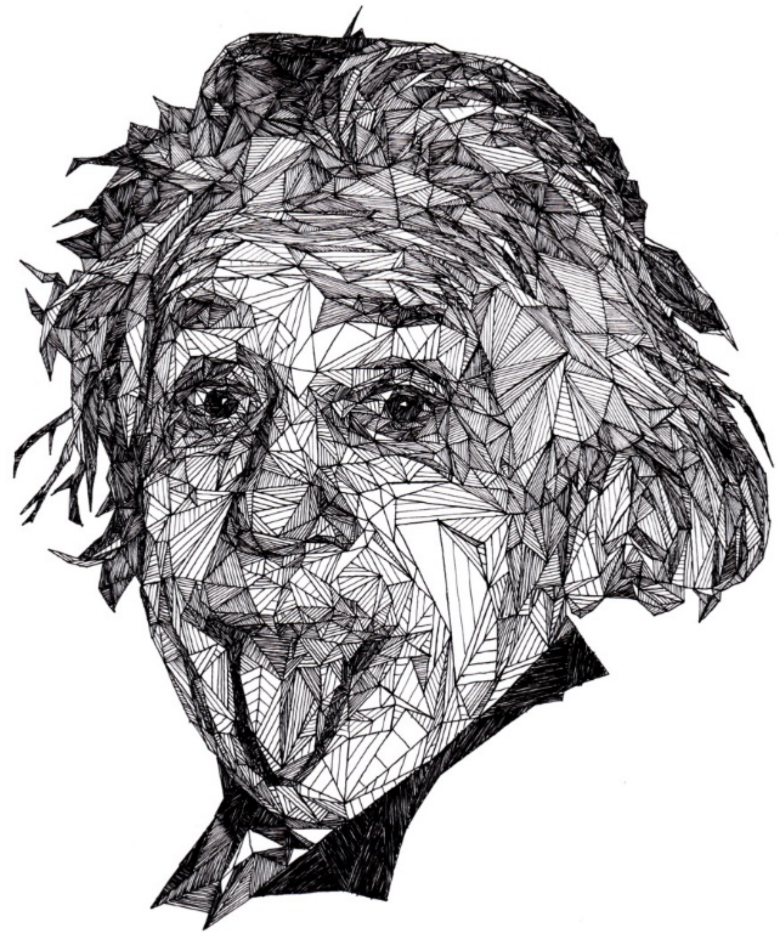


Loop









Nice subsets – tame logics?

- Piecewise linear subsets = polytopes

$\mathbf{PC}^n$ = C-logic of all polytopal subsets of $\mathbb{R}^n$

$\mathbf{PD}^n$ = d-logic of all polytopal subsets of $\mathbb{R}^n$

Our aim is to investigate these modal systems

- In this talk - $\mathbf{PC}^n$

General observations

If $A \cap B = \varnothing$ and $A \subseteq \mathbf{C}B$

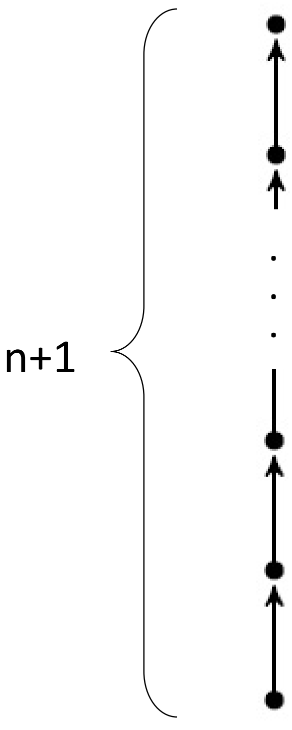
Then $\dim(A) < \dim(B)$

Put $\beta A \equiv \mathbf{C}A \backslash A$ (boundary of A)

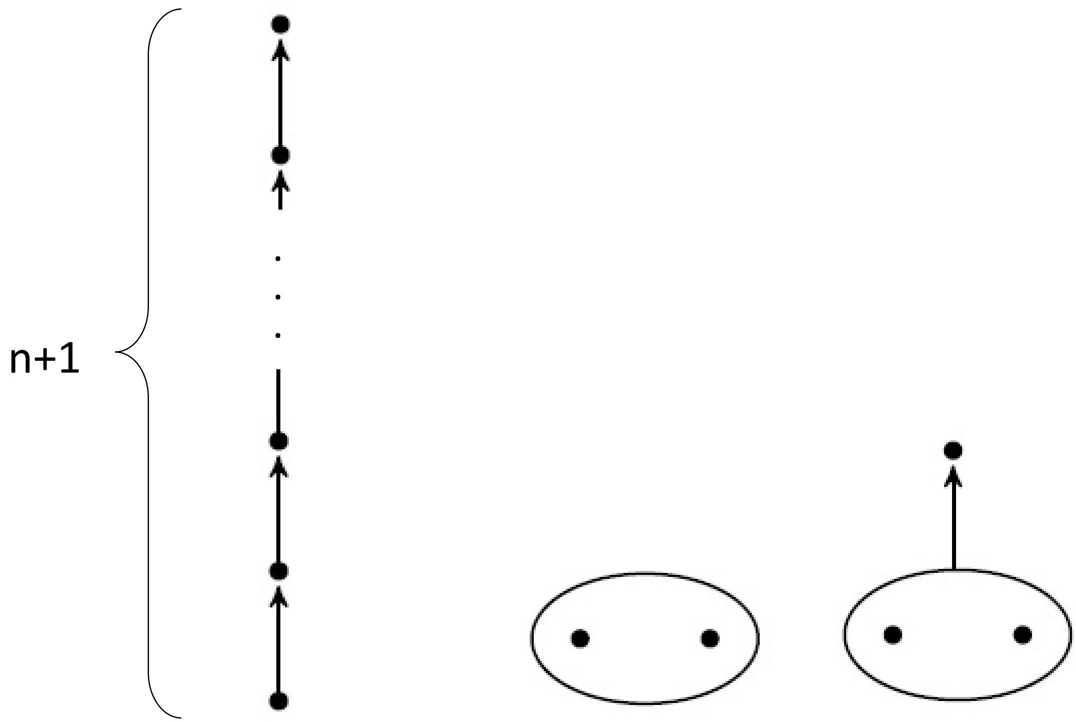
Then $\beta^n A = \varnothing$ iff $\dim(A) < n$

It follows that each $\mathbf{PC}^n$ is a logic of finite height.

Forbidden frames for $\mathbf{PC}^n$



Forbidden frames for $\mathbf{PC}^n$



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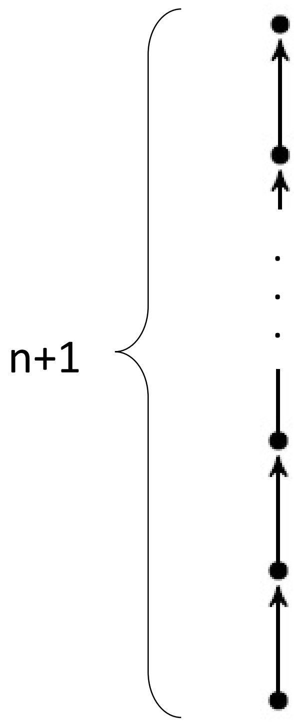
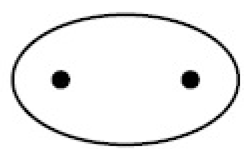
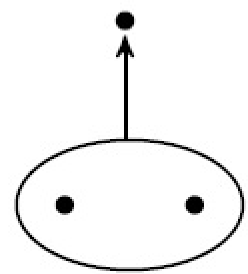


$\mathbf{PC}^n$ is an extension of $\mathbf{S4.Grz}_n$

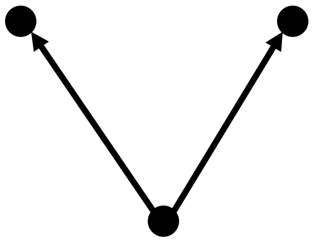




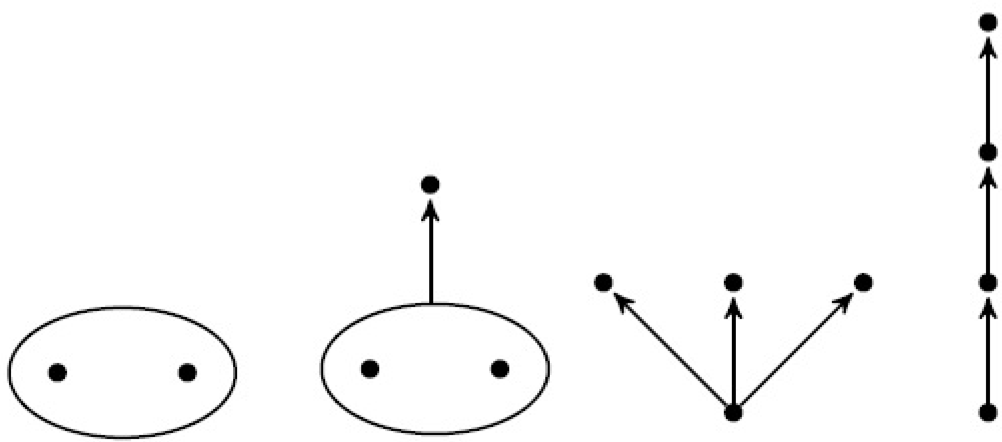
PC^1

- **PC^1** is the modal logic of a 2-fork

[van Benthem, G. Bezhanishvili and Gehrke, 2003]



PC^2 – forbidden frames



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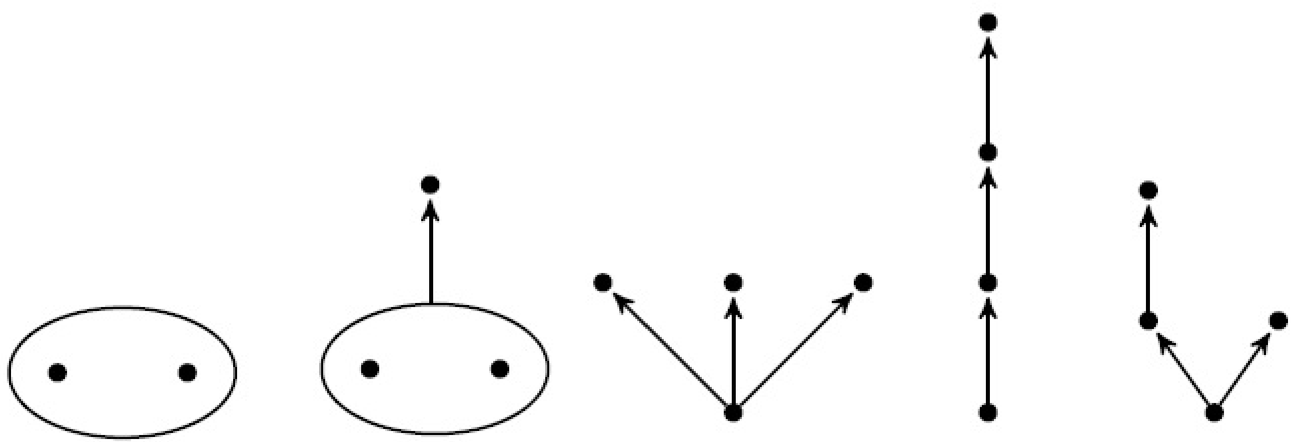

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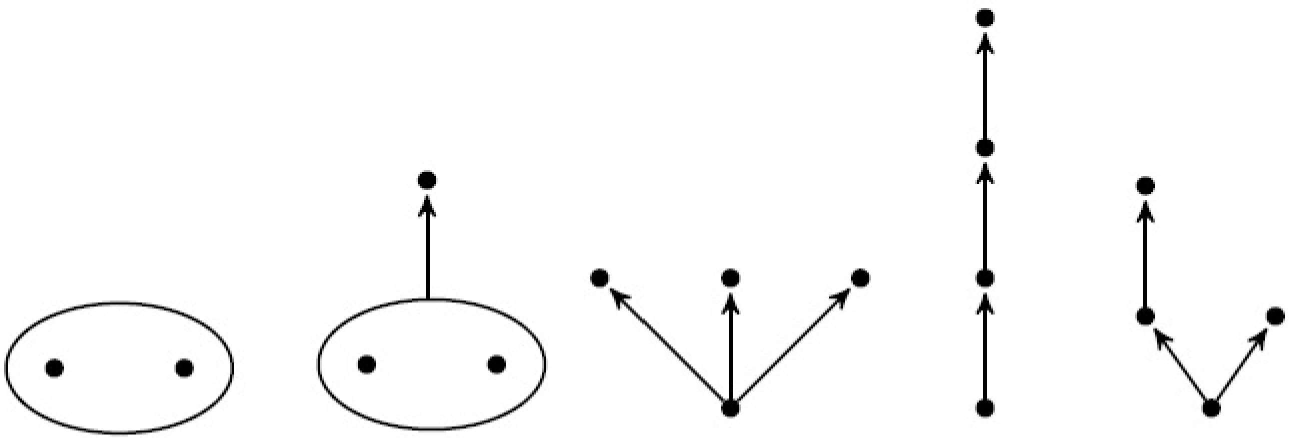
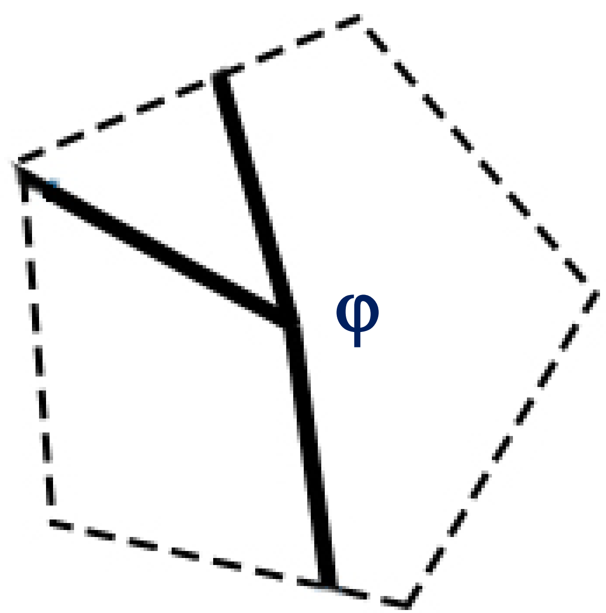


Any other forbidden configurations?

Example



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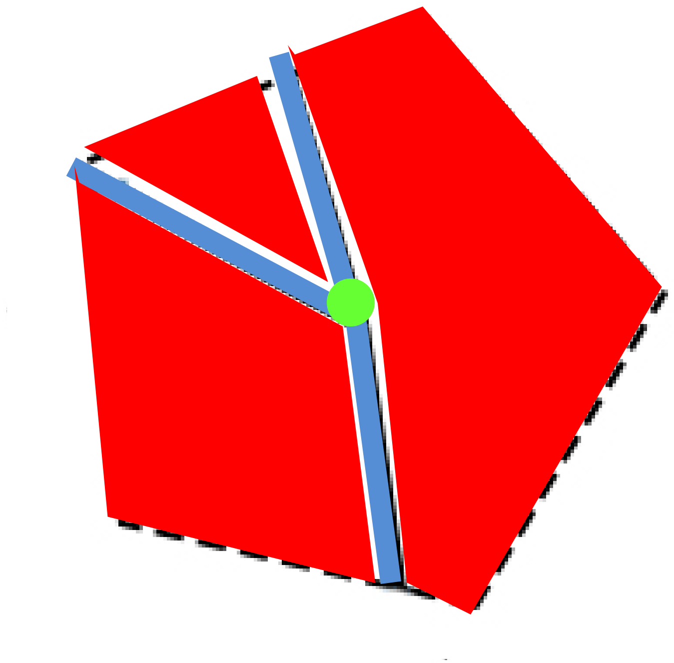

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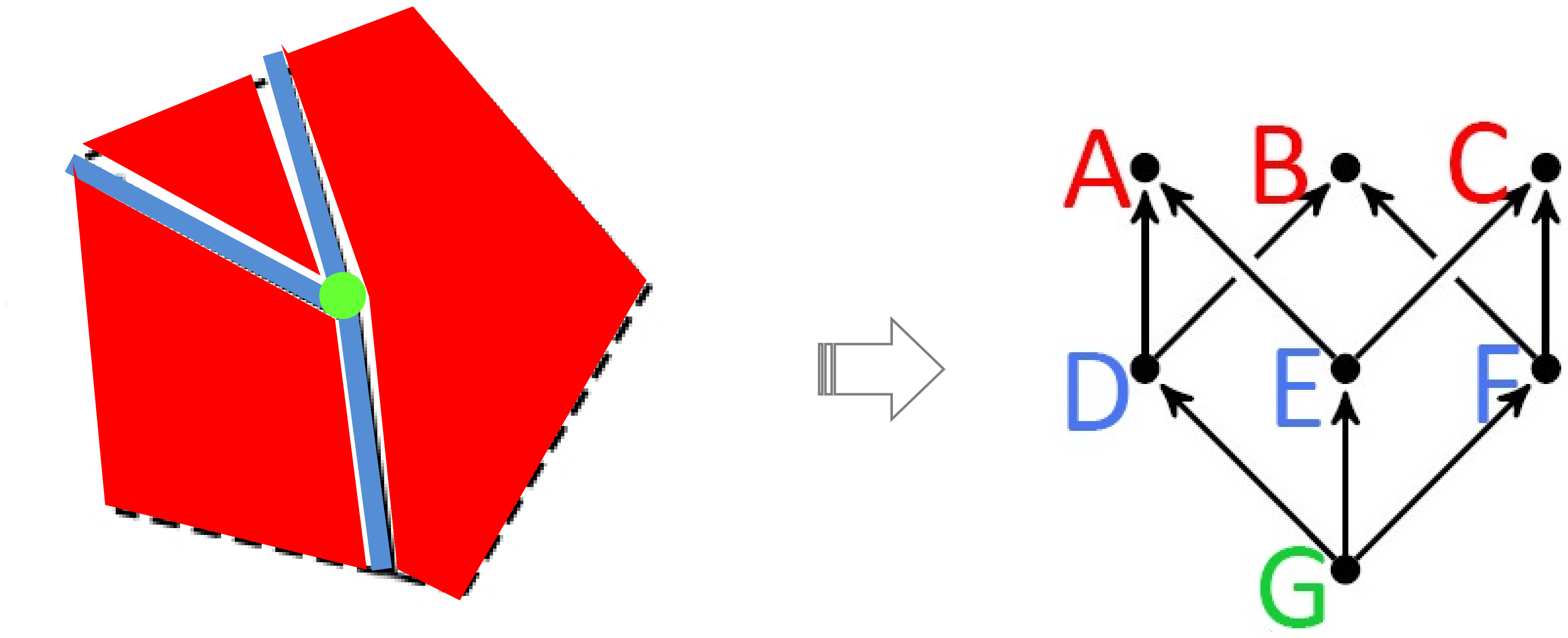
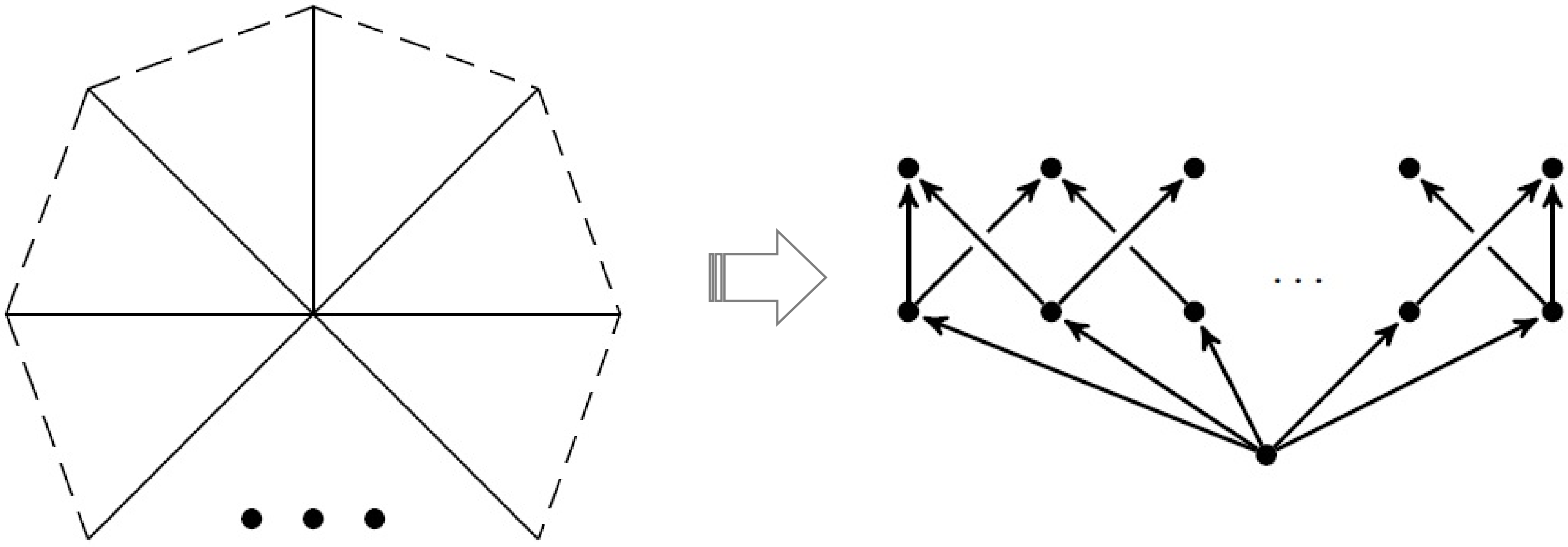


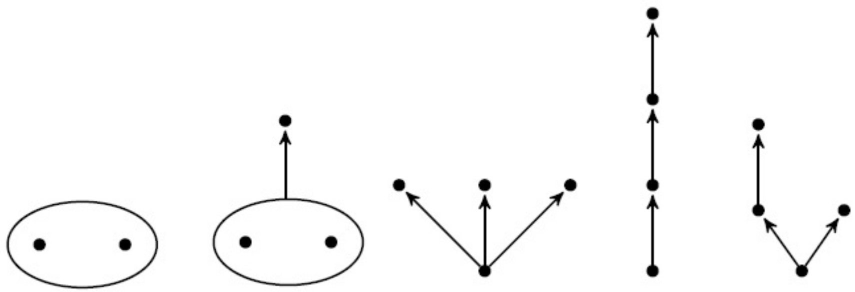
PC^2 – admitted frames

Lemma: Any crown frame is a partial polygonal interior image of the plane.

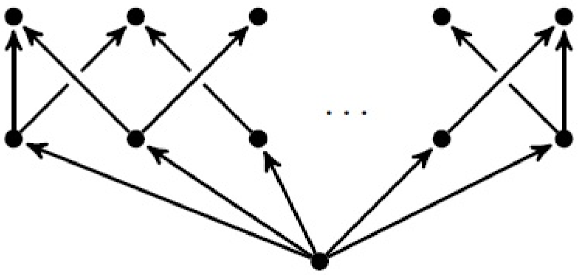


PC^2 – Axiomatization

Bad, but almost good guys



Very nice guys

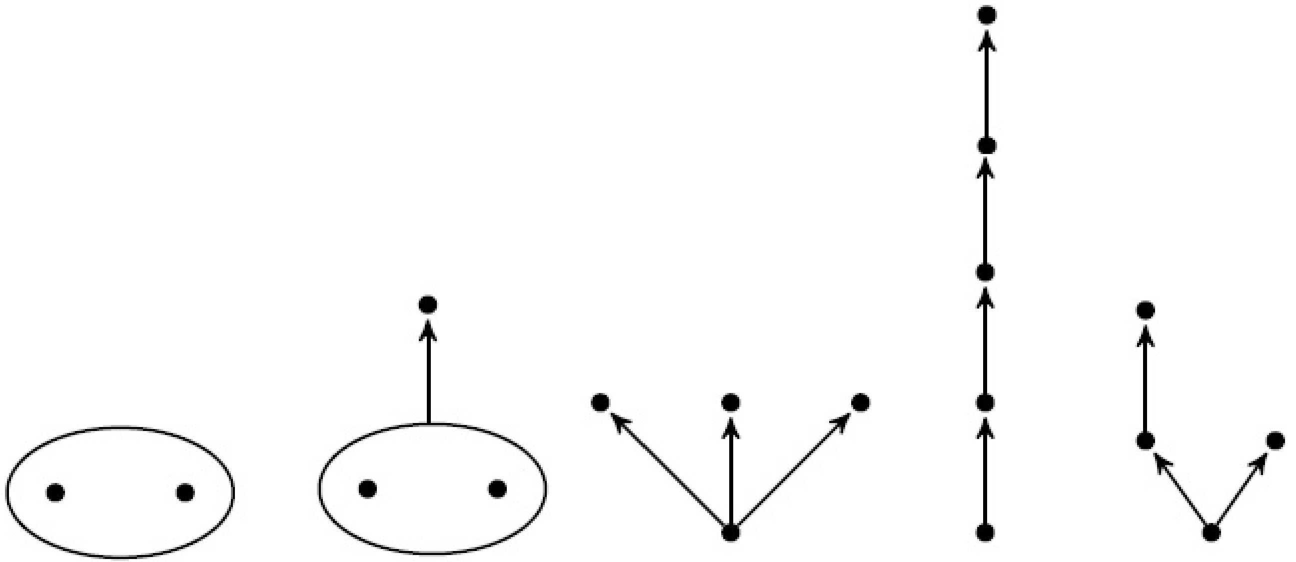


PC^2 – admitted frames

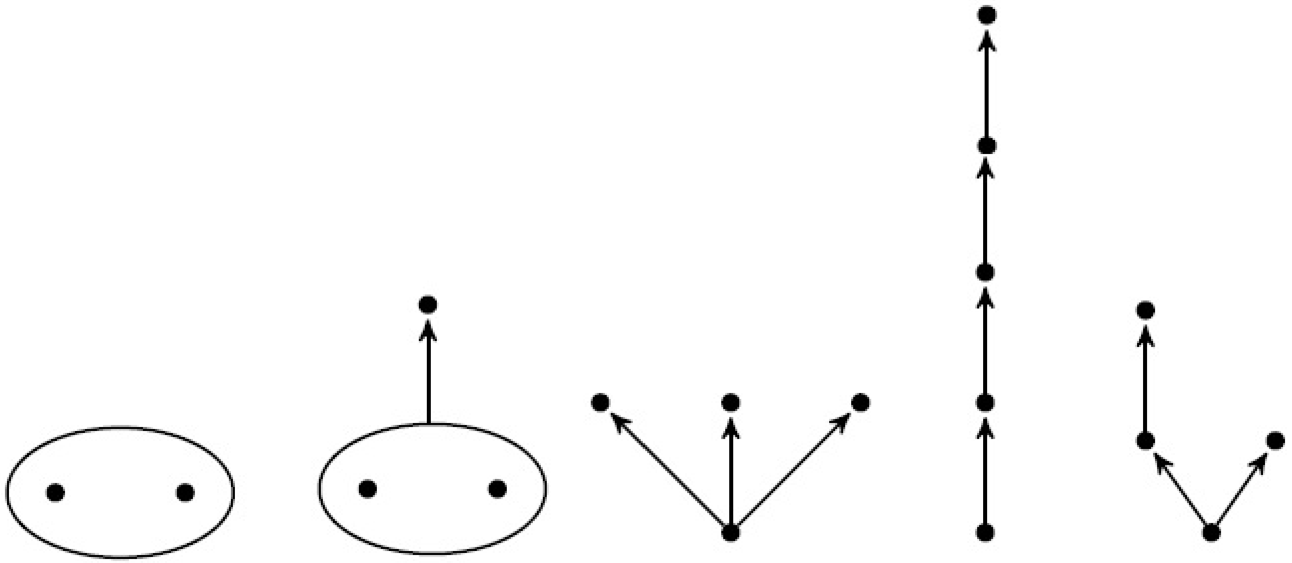
Lemma: Any rooted poset not reducible to any of the forbidden frames is a p-morphic image of a crown frame.

Theorem: The logic $\mathbf{PC^2}$ is axiomatizable by Jankov-Fine axioms of the five forbidden frames.

PC^3 – forbidden frames

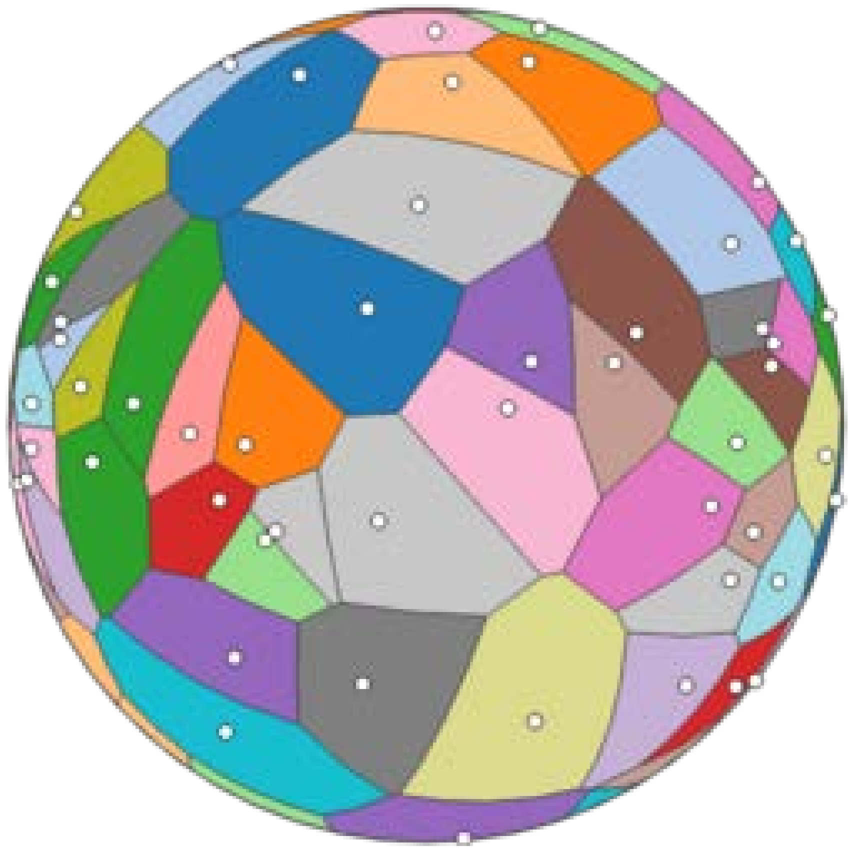


PC^3 – forbidden frames



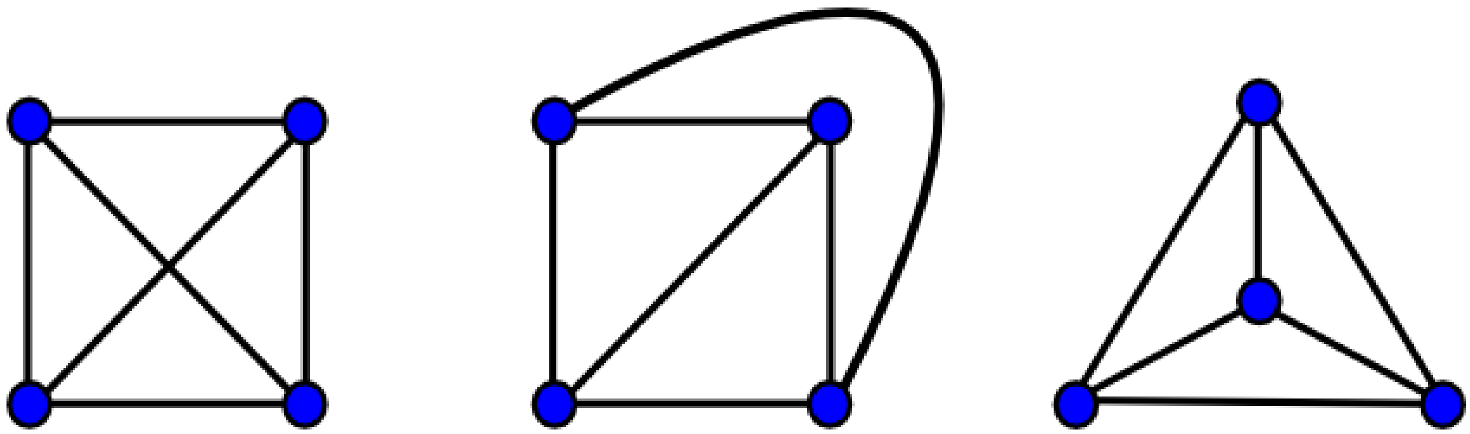
Any other forbidden configurations?

PC^3 – Spherical (open) polyhedra

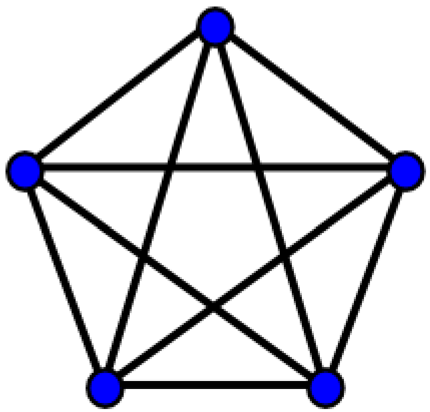


Planar graphs

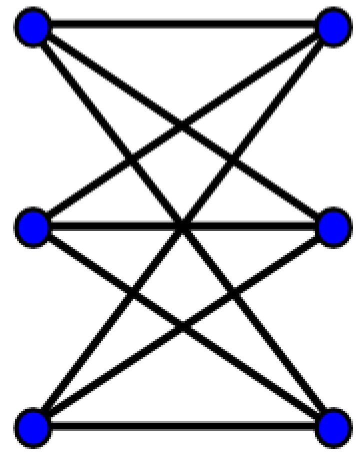
- A graph is planar if it can be drawn on the plane (=on a surface of a sphere) without intersecting edges



Non-planar graphs

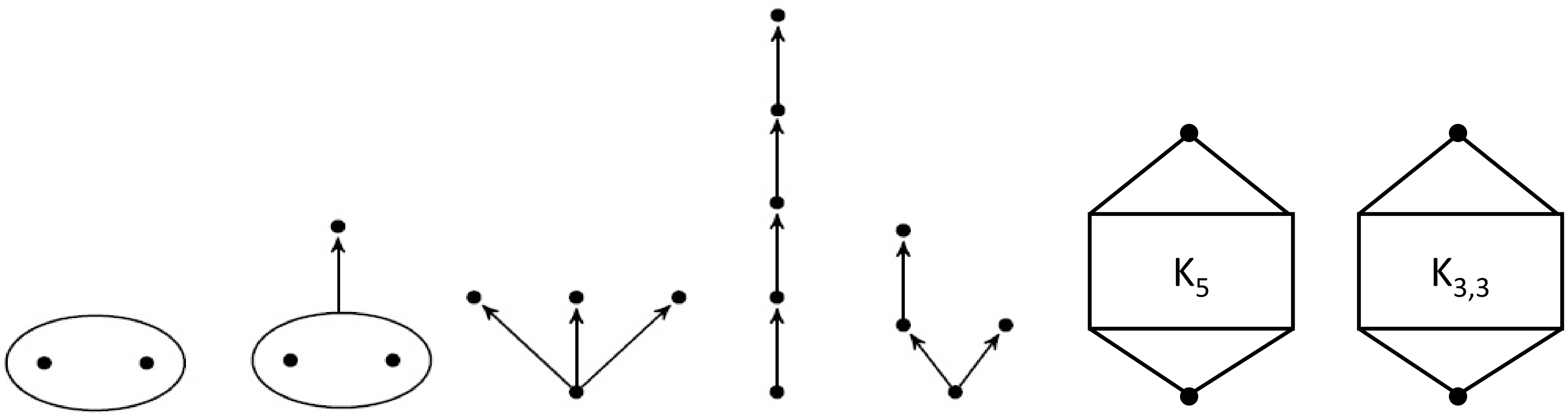


K_5

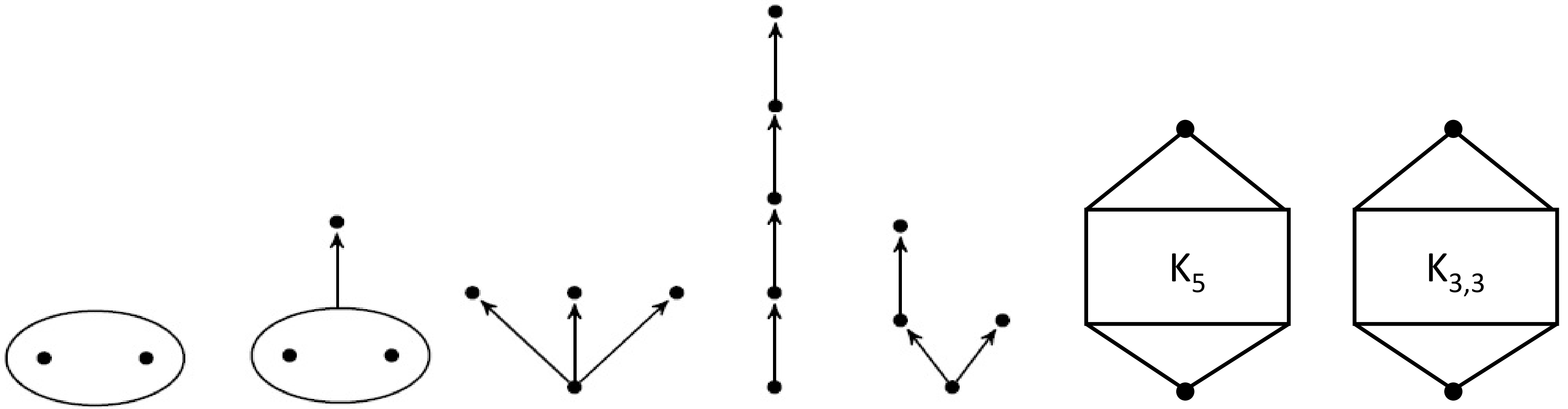


$K_{3,3}$

PC^3 – forbidden frames



PC³ – forbidden frames



Anything else?

Face posets of sphere triangulations