Path categories
(jww Ieke Moerdijk)

Benno van den Berg
ILLC, University of Amsterdam

Tbilisi, 17 June 2016

What is homotopy type theory?

It introduces ideas from

HOMOTOPY THEORY

in

TYPE THEORY.

See:

The Univalent Foundations Program – *Homotopy type theory: univalent foundations of mathematics*. Institute of Advanced Studies, 2013.

Homotopy theory and type theory

Homotopy theory

Homotopy is a branch of algebraic topology: for a homotopy theorists a space consists of points, paths between these points, homotopies between these paths, homotopies between these homotopies, et cetera...

Type theory

- It is a foundation for constructive mathematics.
- It is a functional programming language.
- It is the basis for proof assistants like Coq, Agda,...

See:

- Per Martin-Löf – *Intuitionistic type theory*, Bibliopolis 1984.
- Bengt Nordstroem, Kent Petersson and Jan M. Smith – *Programming in Martin-Löf's type theory*. Oxford University Press, 1990.

Categorical correspondences

Similar categorical structures appear in homotopy theory and type theory:

- ∞-groupoids
- Quillen model structures

Aim today

Add another example to the list: *path categories*.

Section 1

Path categories

Setting

Path category: setting for axiomatic homotopy theory (like Quillen model structures).

A path category is a category $\mathcal{C}$ equipped with two classes of maps:

- *fibrations*
- *weak equivalences*

Terminology:

- A map which is both a fibration and a weak equivalence will be called an *acyclic fibration*.
- If we can factor the diagonal $B \to B \times B$ as a weak equivalence $r : B \to PB$ followed by a fibration $(s, t) : PB \to B \times B$, then PB is a *path object* for B.

Examples

1. Groupoids.
2. Topological spaces.
3. Simplicial sets (Kan complexes).
4. The fibrant objects in any Quillen model structure in which every object is cofibrant.
5. Cubical sets with uniform Kan fibrations (Coquand et al).

Category with path objects, or path category

Axioms

1. $\mathcal{C}$ has a terminal object 1 and $X \to 1$ is always a fibration.
2. Fibrations are closed under composition.
3. The pullback of a fibration along any other map exists and is again a fibration.
4. The pullback of an acyclic fibration along any other map is again an acyclic fibration.
5. Weak equivalences satisfy 2-out-of-6.
6. Isomorphisms are acyclic fibrations and every acyclic fibration has a section.
7. Every object B has at least one path object.

(This strengthens Brown's notion of a *category of fibrant objects* in two ways: we have 2-out-of-6 for weak equivalences instead of 2-out-of-3 and we demand that acyclic fibrations have sections.)

First basic facts about path categories

- Every map $f : Y \to X$ factors as a weak equivalence followed by a fibration:

$$\begin{array}{ccccccc} Y & \xrightarrow{(1,rf)} & P_f & \xrightarrow{p_2} & PX & \xrightarrow{t} & X \\ & \searrow^{1} & \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle s} & & \\ & & Y & \xrightarrow[f]{} & X & & \end{array}$$

- This means that if $f : Y \to X$ is a fibration, then we can factor $Y \to Y \times_X Y$ as

$$Y \longrightarrow P_X(Y) \longrightarrow Y \times_X Y,$$

  where the first is a weak equivalence and the second a fibration.
- Corollary: Let $\mathcal{C}(X)$ be the full subcategory of $\mathcal{C}/X$ whose objects are fibrations. Then $\mathcal{C}(X)$ is again a path category.

Homotopy in a path category

If $f, g : Y \to X$ are two parallel maps, then we say that f and g are *homotopic* and write $f \simeq g$ if there is a map $h : Y \to PX$ making

$$\begin{array}{ccc} & & PX \\ & \overset{h}{\nearrow} & \downarrow{\scriptstyle (s,t)} \\ Y & \xrightarrow[(f,g)]{} & X \times X \end{array}$$

commute.

Theorem

The homotopy relation $\simeq$ is a congruence on $\mathcal{C}$.

The quotient is the *homotopy category* of $\mathcal{C}$. A map which becomes an isomorphism in the homotopy category is called a *homotopy equivalence*.

Theorem

The weak equivalences and homotopy equivalences coincide in a path category.

Two hard results

Theorem (Moerdijk-BvdB)

If w is a weak equivalence and p is a fibration fitting into a commutative square

$$\begin{array}{ccc} D & \xrightarrow{l} & B \\ {\scriptstyle w}\downarrow & \overset{d}{\nearrow} & \downarrow{\scriptstyle p} \\ C & \xrightarrow[k]{} & A, \end{array}$$

then there is a filler $d : C \to B$ such that $pd = k$ and $dw \simeq_A l$. Moreover, such fillers d are unique up to fibrewise homotopy over A.

Here *fibrewise homotopy* and $\simeq_A$ refer to the path object $P_A(B)$ in $\mathcal{C}(A)$.

Theorem (Brown)

Weak equivalences are stable under pullback along fibrations.

Path lifting and transport

As a first consequence of these results we have:

Corollary (Transport)

If $f : Y \to X$ is a fibration, then there is a map $\nabla : Y \times_X PX \to Y$ such that $f\nabla = tp_2$ and $\nabla(1, rf) \simeq_X 1$. Such "connections" are unique up to fibrewise homotopy.

Corollary (Lifting paths)

If $f : Y \to X$ is a fibration and ∇ is a connection on it, then there is map $L : Y \times_X PX \to PY$ such that $sL = p_1$ and $tL = \nabla$.

Groupoid structure

Corollary

If X is a path object structure on X, then PX gives X the structure of a groupoid up to homotopy.

In fact, we have:

Theorem (Garner-BvdB)

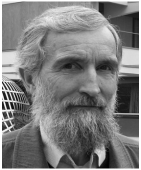
In a path object category every object carries the structure of an ∞-groupoid in the sense of Batanin-Leinster.

Section 2

Type theory

Martin-Löf's type theory

Type theory as an alternative foundation for constructive mathematics:



- There are *terms* and *types* and every term has a specific type ($t : A$).
- One may write $s = t$, but only if s and t have the same type ($s = t : A$).
- One can have parametrised (dependent) types $B(a)$, with $a : A$.
- Statements are always made in context.
- Every type in inductively generated.

General shape of a judgement

$$\begin{aligned}
&\Gamma \vdash B(x_0, \ldots, x_n)\text{Type}\\
&\Gamma \vdash B(x_0, \ldots, x_n) = C(x_0, \ldots, x_n)\\
&\Gamma \vdash a(x_0, \ldots, x_n) : B(x_0, \ldots, x_n)\\
&\Gamma \vdash a(x_0, \ldots, x_n) = b(x_0, \ldots, x_n) : B(x_0, \ldots, x_n)
\end{aligned}$$

where

$$\Gamma = [\, x_0 : A_0, x_1 : A_1(x_0), \ldots, x_n : A_n(x_0, \ldots, x_{n-1})\,],$$

is a context.

An example

Formation

$$\frac{}{\vdash \mathbb{N}\ \mathsf{Type}}$$

Introduction

$$\frac{}{\vdash 0 : \mathbb{N}} \qquad \frac{\vdash n : \mathbb{N}}{\vdash s(n) : \mathbb{N}}$$

Elimination

$$\frac{\begin{array}{c} n : \mathbb{N} \vdash P(n)\ \mathsf{Type} \\ \vdash c : P(0) \\ n : \mathbb{N}, x : P(n) \vdash g(x, n) : P(s(n)) \end{array}}{n : \mathbb{N} \vdash \mathsf{rec}(c, g, n) : P(n)}$$

Computation

$$\mathsf{rec}(c, g, 0) = c : P(0)$$

$$\mathsf{rec}(c, g, s(n)) = g(\mathsf{rec}(c, g, n), n) : P(s(n))$$

Another example

Formation $\dfrac{A \text{ Type} \quad B \text{ Type}}{A \times B \text{ Type}}$

Introduction $\dfrac{\vdash a : A \quad \vdash b : B}{\vdash p(a, b) : A \times B}$

Elimination $\dfrac{\begin{array}{c} x : A \times B \vdash P(x) \text{ Type} \\ a : A, b : B \vdash f(a, b) : P(p(a, b)) \end{array}}{x : A \times B \vdash \text{prodrec}(f, x) : P(x)}$

Computation $\text{prodrec}(f, p(a, b)) = f(a, b) : P(p(a, b))$

Towards an identity type

- There should also be a propositional equality, that is, a type $\mathsf{Id}_A(a, b)$ of proofs of the equality of a and b.
- Of course, it has to be inductively generated and the rules for it should conform to the general pattern.
- Idea: *equality is the least reflexive relation*.

Rules for the identity type

Formation $\dfrac{\vdash x : A, y : A}{\vdash \mathrm{Id}_A(x, y)\ \mathrm{Type}}$

Introduction $\dfrac{\vdash a : A}{\vdash r(a) : \mathrm{Id}_A(a, a)}$

Elimination $\dfrac{\begin{array}{l} x : A, y : A, z : \mathrm{Id}_A(x, y) \vdash C(x, y, z)\ \mathrm{Type} \\ x : A \vdash d(x) : C(x, x, r(x)) \end{array}}{x : A, y : A, z : \mathrm{Id}_A(x, y) \vdash J(x, y, z, d) : C(x, y, z)}$

Computation $J(x, x, r(x), d) = d(x)$

You should realise that identity types can be nested:

$$\alpha : Id_{Id_A(x,y)}(f, g)$$

So there are proofs of the equality of certain equality proofs, and proofs of the equality of those, et cetera!

Section 3

The connection

Classifying category

To understand matters better, we should organise the syntax into a category!

Suppose Δ and Γ are contexts and

$$\Gamma = [\, x_1 : A_1, x_2 : A_1(x_1), \ldots, x_n : A_n(x_1, \ldots, x_{n-1})\,].$$

A *context morphism* $f : \Delta \to \Gamma$ is an n-tuple of terms $t_1, \ldots, t_n$ such that the following statements are derivable in type theory:

$$\begin{array}{l} \Delta \vdash t_1 : A_1 \\ \Delta \vdash t_2 : A_2(t_1) \\ \ldots \\ \Delta \vdash t_n : A_n(t_1, \ldots, t_{n-1}) \end{array}$$

The contexts together with the context morphisms form a category: the *syntactic* or *classifying category*.

Classifying category is a path category

Within the classifying category there is a special class of morphisms: those isomorphic to maps of the form

$$(x_1, \ldots, x_i) : [\, x_1 : A_1, x_2 : A_2(x_1), \ldots, x_n : A_n(x_0, \ldots, x_{n-1}) \,] \to$$
$$[\, x_1 : A_1, x_2 : A_2(x_1), \ldots, x_i : A_I(x_1, \ldots, x_{i-1}) \,],$$

where $i \leq n$. These are called *dependent projections*.

Theorem (Gambino-Garner, Avigad-Kapulkin-Lumsdaine)

Let $\mathcal{C}$ be the syntactic category associated to a dependent type theory with identity types. Then $\mathcal{C}$ carries the structure of a path category in which the dependent projections are the fibrations.

In this structure the identity types are the path objects: that is, the factorisation of the diagonal $[x : A] \to [x : A] \times [x : A] \cong [x : A, y : A]$ is precisely

$$[x : A] \to [x : A, y : A, p : \mathrm{Id}_A(x, y)] \to [x : A, y : A].$$

Classifying category is a path category, part 2

Theorem (Gambino-Garner, Avigad-Kapulkin-Lumsdaine)

Let $\mathcal{C}$ be the syntactic category associated to a dependent type theory with identity types. Then $\mathcal{C}$ carries the structure of a path category in which the dependent projections are the fibrations.

Corollary (Lumsdaine, Garner-BvdB)

In type theory every type carries the structure of an ∞-groupoid in the sense of Batanin-Leinster.

Theorem (BvdB)

The two results above still hold if we weaken the computation rule from $J(x, x, r(x), d) = d(x)$ to requiring the existence of a proof term $h(d, x)$ of type $\mathrm{Id}_{C(x,x,rx)}(J(x, x, r(x), d), d(x))$.

Soundness and completeness

Theorem (BvdB)

The results from the previous page still hold if we weaken the computation rule from $J(x, x, r(x), d) = d(x)$ to requiring the existence of a proof term $h(d, x)$ of type $\mathrm{Id}_{C(x,x,rx)}(J(x, x, r(x), d), d(x))$.

Theorem (Moerdijk-BvdB)

Let $\mathcal{C}$ be a path category. Modulo coherence problems related to substitution $\mathcal{C}$ is a model of a basic type theory with identity types for which the computation rule holds only in a weak ("propositional") form.

These two results can be summarised as follows:

To summarise

Morally, path categories are a sound and complete semantics for type theory with propositional identity types.

References

1. J. Avigad, K. Kapulkin and P.L. Lumsdaine, *Homotopy limits in type theory*. Mathematical Structures in Computer Science 25, 2015.
2. B. van den Berg. *Path categories and propositional identity types*. arXiv:1604.06001, April 2016.
3. B. van den Berg and R. Garner. *Types are weak ω-groupoids*. Proceedings of the London Mathematical Society 102, 2011.
4. B. van den Berg and I. Moerdijk. *Exact completion of path categories and algebraic set theory*. arXiv:1603.02456, March 2016.
5. N. Gambino and R. Garner. *The identity type weak factorisation system*, Theoretical Computer Science 409, 2008.
6. P.L. Lumsdaine. *Weak ∞-categories from intensional type theory*. Logical Methods in Computer Science 6, 2010.