Local tabularity via filtrations

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A logic L is *locally tabular* if, for any finite n, there exist only finitely many pairwise nonequivalent formulas in L built from the variables $p_1, \ldots, p_n$.

Equivalently, a logic L is locally tabular if the variety of its algebras is *locally finite*, i.e., every finitely generated L-algebra is finite.

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Segerberg-Maksimova criterion for extensions of $\mathrm{K4}$

Formulas of finite height

$$B_1 = p_1 \to \Box\Diamond p_1, \quad B_{i+1} = p_{i+1} \to \Box(\Diamond p_{i+1} \vee B_i)$$

Theorem (Segerberg, Maksimova)

A logic $\mathrm{L} \supseteq \mathrm{K4}$ *is locally tabular iff* L *contains* B_h *for some* $h > 0$.

New results on local tabularity of normal unimodal logics

- A necessary syntactic condition:

  a logic is locally tabular, only if it is *pretransitive* and is of *finite height*.

- A semantic criterion:

  $Log(\mathcal{F})$ is locally tabular iff $\mathcal{F}$ is of uniformly finite height and has the *ripe cluster property*.

- Segerberg – Maksimova syntactic criterion for extensions of logics much weaker than $K4$:

  if $m \geq 1$, $\Diamond^{m+1}p \to \Diamond p \vee p \in \mathrm{L}$, then L is locally tabular iff it is of finite height.

Frames of finite height

A poset $\mathbb{F}$ is of *finite height* $\leq n$ if every its chain contains at most n elements.

Skeleton

R^* is the transitive reflexive closure of R.
Clusters are maximal subsets where R^* is universal:
$\sim_R$ is the equivalence relation $R^* \cap R^{*-1}$, an equivalence class modulo $\sim_R$ is a *cluster* in (W, R).
The *skeleton of* (W, R) is the poset $(W/{\sim_R}, \leq_R)$, where for clusters C, D,

$$C \leq_R D \text{ iff } xR^*y \text{ for some (for all) } x \in C, y \in D.$$

Height of a frame is the height of its skeleton.

Transitive logics of finite height

For any transitive F,

$$\mathrm{F} \models B_h \iff ht(\mathrm{F}) \leq h,$$

where

$$B_1 = p_1 \to \Box\Diamond p_1, \quad B_{i+1} = p_{i+1} \to \Box(\Diamond p_{i+1} \vee B_i).$$

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Pretransitive relations and logics

$R^{\leq m} = \bigcup_{0 \leq i \leq m} R^i$.

R is *m-transitive*, if $R^{\leq m} = R^*$, or equivalently, $R^{m+1} \subseteq R^{\leq m}$.

R is pretransitive, if it is m-transitive for some $m \geq 0$.

$\Diamond^0 \varphi := \varphi$, $\Diamond^{i+1}\varphi := \Diamond\Diamond^i\varphi$,

$\Diamond^{\leq m}\varphi := \bigvee_{i=0}^{m} \Diamond^i \varphi$, $\Box^{\leq m}\varphi := \neg\Diamond^{\leq m}\neg\varphi$.

Proposition

R is m-transitive iff $(W, R) \vDash \Diamond^{m+1} p \to \Diamond^{\leq m} p$.

A logic L is *m-transitive*, if $(\Diamond^{m+1} p \to \Diamond^{\leq m} p) \in L$;

L is *pretransitive*, if it is m-transitive for some $m \geq 0$.

Pretransitive logics of finite height

$\varphi^{[m]}$ is obtained from φ by replacing $\Diamond$ with $\Diamond^{\leq m}$ and $\Box$ with $\Box^{\leq m}$.

Proposition

For an m-transitive frame F, $\mathrm{F} \vDash B_h^{[m]} \iff ht(\mathrm{F}) \leq h$.

A pretransitive L *is of finite height*, if L contains $B_h^{[m]}$, where m is the least such that L is m-transitive.

Theorem

Every locally tabular logic is pretransitive of finite height.

The converse is not true in general.

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Pretransitive logics of finite height

All m-transitive logics of finite height

$$\mathrm{K} + (\Diamond^{m+1} p \to \Diamond^{\leq m} p) + B_h^{[m]}$$

have the FMP [Kudinov and Sh, 2015].

However, for $m > 1$, none of them are locally tabular: the 2-transitive logic of height 1

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Pretransitive logics are much more complex than $\mathrm{K4}$.
E.g., the FMP (and even the decidability) of the logics $\mathrm{K} + (\Diamond^{m+1} p \to \Diamond^{\leq m} p)$ is unknown for $m \geq 2$.

Semantic criterion

Partitions, the finite model property, and local tabularity

In modal logic, the FMP is often proved via constructing special partitions of Kripke frames and models (*filtrations*).

Local tabularity in terms of partitions:

If F is an L-frame and $\mathcal{A}$ is a finite partition of F, then there exists a finite refinement of $\mathcal{A}$ with a special properties.

As usual, a partition $\mathcal{A}$ of a non-empty set W is a set of non-empty pairwise disjoint sets such that $W = \cup\mathcal{A}$. The corresponding equivalence relation is denoted by $\sim_{\mathcal{A}}$, so $\mathcal{A} = W/{\sim_{\mathcal{A}}}$.
A partition $\mathcal{B}$ refines $\mathcal{A}$, if each element of $\mathcal{A}$ is the union of some elements of $\mathcal{B}$, or equivalently, $\sim_{\mathcal{B}} \subseteq \sim_{\mathcal{A}}$.

Minimal filtrations

The *minimal filtration of* (W, R) *through* $\mathcal{A}$ is the frame $(\mathcal{A}, R_\mathcal{A})$, where for $U, V \in \mathcal{A}$

$$U \, R_\mathcal{A} \, V \iff \exists u \in U \, \exists v \in V \; uRv.$$

Let $\mathrm{M} = (W, R, \theta)$ be a model, Γ a set of formulas. A partition $\mathcal{A}$ of M *respects* Γ, if for all $x, y \in W$

$$x \sim_A y \quad \Rightarrow \quad \forall \varphi \in \Gamma(\mathrm{M}, x \vDash \varphi \iff \mathrm{M}, y \vDash \varphi).$$

Filtration lemma (late 1960s)

Let Γ be a set of formulas closed under tanking subformulas, $\mathcal{A}$ respect Γ. Then, for all $x \in W$ and all formulas $\varphi \in \Gamma$,

$$\mathrm{M}, x \vDash \varphi \iff (\mathcal{A}, R_\mathcal{A}, \theta_\mathcal{A}), [x]_\mathcal{A} \vDash \varphi.$$

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Fact

Consider a Kripke complete logic $\mathrm{L} = Log(W, R)$. If for every finite partition $\mathcal{A}$ of W there exists a finite $\mathcal{B}$ such that $\mathcal{B}$ refines $\mathcal{A}$ and $(\mathcal{B}, R_{\mathcal{B}}) \models \mathrm{L}$, then L has the FMP.

Special minimal filtrations: tuned partitions

Definition

A partition $\mathcal{A}$ of $\mathrm{F} = (W, R)$ is R-*tuned*, if for any $U, V \in \mathcal{A}$

$$U\, R_{\mathcal{A}}\, V \quad \Rightarrow \quad \forall u \in U\ \exists v \in V\ \ uRv,$$

that is,

$$\exists u \in U\ \exists v \in V\ \ uRv \quad \Longleftrightarrow \quad \forall u \in U\ \exists v \in V\ \ uRv.$$

Fact (Franzen, early 1970s)

If $\mathcal{A}$ is R-tuned, then $Log(W, R) \subseteq Log(\mathcal{A}, R_{\mathcal{A}})$.

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If for every finite partition $\mathcal{A}$ of W there exists a finite R-tuned refinement $\mathcal{B}$ of $\mathcal{A}$, then $Log(W, R)$ has the FMP.

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Fact

If for every finite partition $\mathcal{A}$ of W there exists a finite R-tuned refinement $\mathcal{B}$ of $\mathcal{A}$, then $Log(W, R)$ has the FMP.

Example

$Log(\mathbb{N}, <)$ and $Log(\mathbb{N}, \leq)$ have the FMP.

Semantic criterion

Definition

A frame F is *ripe*, if there exists a monotonic $f : \mathbb{N} \to \mathbb{N}$, such that for every finite partition $\mathcal{A}$ of W there exists an R-tuned refinement $\mathcal{B}$ of $\mathcal{A}$ such that $|\mathcal{B}| \leq f(|\mathcal{A}|)$.

A class of frames $\mathcal{F}$ is ripe if all frames $\mathrm{F} \in \mathcal{F}$ are ripe for a fixed f.

Theorem (Intermediate criterion)

$Log(\mathcal{F})$ is locally tabular iff $\mathcal{F}$ is ripe.

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Example

Local tabularity of $\mathrm{S5}$ is an immediate consequence of the theorem: in a frame with the universal relation, any partition is tuned. But even for preorders of finite height > 1, to construct tuned refinements is an exercise.

However, it is enough to construct partitions only for clusters.

Semantic criterion. Main result

Definition

A class $\mathcal{F}$ of frames has the *ripe cluster property*, if the class of clusters in its frames $\{\mathrm{C} \mid \exists \mathrm{F} \in \mathcal{F} \text{ s.t. } \mathrm{C} \text{ is a cluster in } \mathrm{F}\}$ is ripe. A logic has the ripe cluster property, if the class of its frames has.

Theorem

A logic $Log(\mathcal{F})$ is locally tabular iff $\mathcal{F}$ is of uniformly finite height and has the ripe cluster property.

Example

Some logics with the ripe cluster property:

- $\mathrm{S4}$, $\mathrm{K4}$ (any partition of a cluster is tuned, so $f(n) = n$);
- $\mathrm{wK4} = \mathrm{K} + \Diamond\Diamond p \to \Diamond p \vee p$ (again, $f(n) = n$);
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Corollaries

Theorem

Suppose L_0 *is a canonical pretransitive logic with the ripe cluster property. Then for any logic* $\mathrm{L} \supseteq \mathrm{L}_0$*:*

$$\mathrm{L} \textit{ is locally tabular iff it is of finite height.}$$

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Intuitionistic case

$Log(\mathbb{N}, \leq)$ is not locally tabular: it is of infinite height.
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For every partition $\mathcal{A}$ of $\mathbb{N}$ there exists a finite $\leq$-tuned refinement $\mathcal{B}$ of $\mathcal{A}$. So $Log(\mathbb{N}, \leq)$ have the fmp.

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But $(\mathbb{N}, \leq)$ is not ripe enough: for any natural n there exists a two-element partition of $\mathbb{N}$ such that for every $\leq$-tuned refinement $\mathcal{B}$ of $\mathcal{A}$ we have $|\mathcal{B}| > n$. So $Log(\mathbb{N}, \leq)$ is not locally tabular.

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If $\mathcal{A}$ is induced by upward-closed sets, then $\mathcal{A}$ consists of intervals, so it is $\leq$-tuned already.

Two problems

Problem

A syntactic criterion for local tabularity over K*.*

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A syntactic criterion for local tabularity of intermediate logics.

Thank you!