On Equivalence of Hilbert Lattices and Quantum Dynamic Algebras

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MOTIVATION

- Finding the right structures for Quantum Theory:

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  - Enough structure

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  - Avoiding unnecessary complexity
- Hilbert Lattices are well-studied example of such structure

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  - A "testable property": a closed linear subspaces of $\mathcal{H}$, say A and the cor. test by the projectors P_A on A, (idempotent, self-adjoint linear maps on $\mathcal{H}$)
  - Unitaries: Invertible linear maps on $\mathcal{H}$.

PRELIMINARIES

Hilbert Lattice:

- **Definition:** A Hilbert lattice is the lattice of closed linear subspaces of a Hilbert space, $\mathcal{H}$, ordered by inclusion (a lattice for testable properties)
- It is well stablished that the unitaries operators on $\mathcal{H}$ are the lattice automorphisms of its Hilbert lattice.

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- In line with the genral trend of the dynamification of logic: many non-classical propositional logics can be viewed as being about actions, rather than about propositions.
- Study of information fellows, Action Logic, Game Logic, Belief Revision
- The information systems are essentially dynamic systems: state of the system are essentially identified with the set of actions that can be preformed at that state; actions are primary

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- Quantum actions as information updates
  - They are similar to dynamic opertors in dynamic logics
  - They are also different: they are not just about the epistemics but have also ontic aspects: preforming an action changes the system

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- We can extended this view to other physically meaningful actions: unitaries and the combination of unitaries and tests
  - composition of actions: $\alpha.\beta$
  - non-deterministic choice of actions $[\bigsqcup_{i \in I} \alpha_i]$

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  - traditional orthomodular quantum logic does not give a complete axiomatisation
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- This setting seems more promising in this regard since for example it allows encoding "higher order properties" in the first order language using the dynamic modalities.

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  - Q is a set (of quantum actions)
  - $\bigsqcup : \mathcal{P}(Q) \to Q$ an infinitary operation on Q
  - $\cdot : Q \times Q \to Q$ a binary operation on Q
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Definition: A Quantum Dynamic Algebra is a generalised dynamic algebra satisfying a set of conditions.

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- $x[y] := \sim\sim (x.y)$
- $At(\mathcal{L}_{\mathcal{Q}}) := \{p \in \mathcal{L}_{\mathcal{Q}} \,|\, \forall q \in \mathcal{L}_{\mathcal{Q}}, (0 \neq q \leq p \implies q = p)\}$

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Proposition: For a QDA, $\mathcal{Q}$, $\mathcal{L}_\mathcal{Q}$ is a Hilbert lattice.

Hilbert Space Realisation

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Let $\mathcal{H}$ be a finite dimensional Hilbert space over $\mathbb{C}$ Let H be the underlying set of vectors. For $R \subseteq H \times H$

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Every liner map on $\mathcal{H}$ has an adjoint (Hermitian conjugate)

HILBERT SPACE REALISATION

Define $\mathcal{Q}(\mathcal{H}) = (Q, \bigsqcup, ., \sim, \dagger)$ by

- $Q = \mathcal{P}(LM(\mathcal{H}, \mathcal{H}))$
- $\bigsqcup$ is the set theoretic union
- $A, B \subset LM(\mathcal{H}, \mathcal{H})$, $A \cdot B = \{a \cdot b \,|\, a \in A, b \in B\}$
- $A \subset LM(\mathcal{H}, \mathcal{H})$, $\sim A = \{P_{B^\perp}\}$ where $B = Im(\bigcup_{a \in A} a)$
- $A^\dagger = \{a^\dagger \,|\, a \in A\}$

Note that

- **Proposition:** For a Hilbert lattice $\mathcal{H}$, $\mathcal{Q}(\mathcal{H})$ is a QDA.

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the other direction is trivial: Every QDA, $\mathcal{Q}$, has a Hilbert lattice, $\mathcal{L}_{\mathcal{Q}}$, embedded in it

Equivalence of Hilbert Lattices and QDA

Translation from Hilbert lattices to QDAlgebras:

Let $(\mathcal{L}, \leq, \sim)$ be a HL We will define a QDA as follows:

- Let U be the set of ortholattice automorphisms on $\mathcal{L}$
- for each $p \in \mathcal{L}$, let $f_p : \mathcal{L} \to \mathcal{L}$ by $f_p(a) = p \wedge (p' \vee a)$
- Let $F_{\mathcal{L}} = \{f_p \,|\, p \in \mathcal{L}\}$
- $F_{\mathcal{U}} = \mathcal{U}$
- $F_{\mathcal{J}}$ the smallest set of functions that contains $F_{\mathcal{L}}$ and $F_{\mathcal{U}}$ and is closed under composition.
- for each $a \in F_{\mathcal{L}}$ let $a^{\dagger} = a$ and for every $a \in F_{\mathcal{U}}$, $a^{\dagger} = a^{-1}$.
- We extend this to compositions by $(a \cdot b)^{\dagger} = b^{\dagger} \cdot a^{\dagger}$

EQUIVALENCE OF HILBERT LATTICES AND QDA

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$$\mathcal{Q}(\mathcal{L}) = (Q(\mathcal{L}), \bigsqcup, ; \sim, \dagger)$$

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- $A.B = \{f.g \,|\, f \in A, g \in B\}$
- $\sim A = \{f_{q^\perp}\}$ where $q = \bigvee_{f \in A} f(\top)$
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- **Weak morphisms:** preserving meets, complementation
- **Strong morphisms:** bijective, preserving, meets, joins and complementation

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We define two notions of morphisms for QDA:

- **Weak morphisms:** Partial functions $\psi : \mathcal{Q}_1 \to \mathcal{Q}_2$ such that
  - $\mathcal{L}_{\mathcal{Q}_1} = \{\sim a \,|\, a \in Q_1\}$ is in the domain of ψ
  - restriction of ψ to $\mathcal{L}_{\mathcal{Q}_1}$, preserves $\sqcup$ and $\sim$.
- **Strong morphisms:** Total functions $\psi : \mathcal{Q}_1 \to \mathcal{Q}_2$ such that
  - restriction of ψ to $\mathcal{L}_{\mathcal{Q}_1}$ preserves $\cdot$, $\sqcup$ and $\sim$.

Equivalence of Hilbert Lattices and QDA

Taking a QDA morphism $\psi : \mathcal{Q}_1 \to \mathcal{Q}_2$, restriction of ψ maps $\mathcal{L}_{\mathcal{Q}_1}$ to $\mathcal{L}_{\mathcal{Q}_2}$ and preserves meets.

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For **weak** Hilbert lattice morphism $\psi : \mathcal{L}_1 \to \mathcal{L}_2$

$$\Psi : \mathcal{Q}(\mathcal{L}_1) \to \mathcal{Q}(\mathcal{L}_2)$$

$$\Psi(\{f_p\}) = \{f_{\psi(p)}\}$$

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Proposition: Ψ is a weak QDA morphism.

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For **strong** Hilbert lattice morphism $\psi : \mathcal{L}_1 \to \mathcal{L}_2$

$$\Psi : \mathcal{Q}(\mathcal{L}_1) \to \mathcal{Q}(\mathcal{L}_2)$$

Let $\ell_\psi(q) = \bigwedge_{\psi(s) \leq q} a$

$$\Psi(f)(q) = \psi \cdot f \cdot \ell_\psi(q)$$

EQUIVALENCE OF HILBERT LATTICES AND QDA

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- **Proposition:** $\mathcal{U}_{\mathcal{Q}_1}$ is mapped to unitaries $\mathcal{U}_{\mathcal{Q}_2}$
- $\Psi(u)$ preserves meets, joins, and complementation and thus $\Psi(u)$ is a Hilbert lattice automorphism. Thus $\Psi(u) \in U_{\mathcal{Q}_2}$
- for some $\{f \cdot g\} \in Q$, $\Psi(f \cdot g) = \psi \cdot f\dot{g} \cdot \ell_\psi = \psi \cdot f\dot{\ell}_\psi \cdot \psi \cdot g \cdot \ell_\psi = \Psi(f) \cdot \Psi(g)$

Equivalence of Hilbert Lattices and QDA

- Thus taking two QDAlgebras $\mathcal{Q}_1, \mathcal{Q}_2$ and a morphism Ψ between them we get Hilbert lattices $\mathcal{L}_{\mathcal{Q}_1}$ and $\mathcal{L}_{\mathcal{Q}_2}$ and the restriction of Ψ to $\mathcal{L}_{\mathcal{Q}_1}$ given a Hilbert lattice morphism.
- Thus taking two Hilbert lattices $\mathcal{L}_1, \mathcal{L}_2$ and a weak/strong morphism ψ between them. We define QDAlgebras $\mathcal{Q}(\mathcal{L}_1), \mathcal{Q}(\mathcal{L}_2)$ and the morphism

$$\Psi(A) = \{\psi \cdot a \cdot \ell_\psi \,|\, a \in A\}$$

gives a weak/strong morphism of QDAlgebras.