Localic Krull Dimension

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Motivation I

Definition

R: commutative ring

$\mathrm{Spec}(R)$: set of prime ideals of R

Krull dimension of R: supremum of lengths of chains in $\mathrm{Spec}(R)$ ordered by $\subseteq$

Extends to:

- Spectral spaces via specialization order
- (Bounded) distributive lattices via Stone duality

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Motivation II

Isbell 1985

- [Krull dimension is] "spectacularly wrong for the most popular spaces, vanishing for all non-empty Hausdorff spaces; but it seems to be the only dimension of interest for the Zariski spaces of algebraic geometry."
- Remedy: graduated dimension

Goal:

Modify Krull dimension motivated by applications in modal logic

Point free approach

1. Locale of open subsets $\Rightarrow$ Heyting algebras and intuitionistic logic
2. Power set closure algebra $\Rightarrow$ modal logics above **S4**

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Closure algebras

Definition

Closure Algebra $\mathfrak{A} = (A, \mathbf{C})$: Boolean algebra A with a closure operator $\mathbf{C} : A \to A$ satisfying the Kuratowski axioms

$$\mathbf{C}(a \vee b) = \mathbf{C}a \vee \mathbf{C}b \qquad \mathbf{CC}a \leq \mathbf{C}a$$
$$\mathbf{C}0 = 0 \qquad a \leq \mathbf{C}a$$

Interior operator: $\mathbf{I} : A \to A$ is dual to $\mathbf{C}$; i.e. $\mathbf{I}a = -\mathbf{C}(-a)$

Natural examples:

- $(\wp X, \mathbf{C})$ where $\wp X$ is the power set of X, a topological space with closure operator $\mathbf{C}$
- $(\wp W, R^{-1})$ where (W, R) is a quasi-ordered set and $R^{-1}(A) := \{w \in W \mid \exists v \in A,\ wRv\}$

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Heyting Algebra $\mathfrak{H}$: bounded distributive lattice such that $\wedge$ has residual $\to$ satisfying $a \leq b \to c$ iff $a \wedge b \leq c$

Natural examples:

- Open subsets of a topological space X; a.k.a. the locale $\Omega(X)$
- Upsets of a partially ordered set

Connecting closure and Heyting algebras

Heyting algebra of open elements of $\mathfrak{A} = (A, \mathbf{C})$:
$\mathfrak{H}(\mathfrak{A}) = \{\mathbf{I}a \mid a \in A\}$
Closure algebra associated with $\mathfrak{H}$: $\mathfrak{A}(\mathfrak{H})$ free Boolean extension of $\mathfrak{H}$ with 'appropriate' closure operator
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Krull dimension of a closure algebra

Recall

For $\mathfrak{A} = (A, \mathbf{C})$, let $\mathfrak{A}_*$ be the set of ultrafilters of A

- Quasi-order $\mathfrak{A}_*$: xRy iff $\forall a \in A$, $a \in y \Rightarrow \mathbf{C}a \in x$
- *R-chain*: finite sequence $\{x_i \in \mathfrak{A}_* \mid i < n\}$ such that $x_i R x_{i+1}$ and $x_{i+1} \not R x_i$ for all i
- *length* of R-chain $\{x_i \mid i < n\}$ is $n-1$
  allow the empty R-chain which has length -1

Definition

The *Krull dimension* $\mathrm{kdim}(\mathfrak{A})$ of a closure algebra $\mathfrak{A}$ is the supremum of the lengths of R-chains in $\mathfrak{A}_*$. If the supremum is not finite, then we write $\mathrm{kdim}(\mathfrak{A}) = \infty$.

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Some easy examples

Let $\mathfrak{A} = (A, \mathbf{C})$ be a closure algebra:

Example 1

If $\mathfrak{A}$ is trivial then $\mathrm{kdim}(\mathfrak{A}) = -1$ (since A has no ultrafilters, $\mathfrak{A}_* = \varnothing$)

Example 2

If $\mathbf{C} = \mathbf{id}_A$ then $\mathrm{kdim}(\mathfrak{A}) = 0$ (since the relation for $\mathfrak{A}_*$ is equality)

Example 3: $A = \{0, a, b, 1\}$

Let $\mathbf{C}a = \mathbf{C}b = 1$. Then $\mathrm{kdim}(\mathfrak{A}) = 0$ (since $\mathfrak{A}_*$ is two element cluster)

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Some easy examples cont.

Example 4: $A = \{0, a, b, 1\}$

Let $\mathbf{C}a = a$ and $\mathbf{C}b = 1$. Then $\mathrm{kdim}(\mathfrak{A}) = 1$ (since $\mathfrak{A}_*$ is two element chain)

Observe: $\mathbf{IC}a = \mathbf{I}a = -\mathbf{C} - a = -\mathbf{C}b = -1 = 0$

Definitions

Let $\mathfrak{A} = (A, \mathbf{C})$ be a closure algebra and $a \in A$:

- *a is nowhere dense in $\mathfrak{A}$*: provided $\mathbf{IC}a = 0$
- *Relativization $\mathfrak{A}_a$ of $\mathfrak{A}$ to a*: the interval $[0, a]$ with operations $\wedge, \vee$ as in $\mathfrak{A}$, the complement of $b \in \mathfrak{A}_a$ is $a - b$, and closure of $b \in \mathfrak{A}_a$ is $a \wedge \mathbf{C}b$

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Internal definition

kdim is not defined point free–requires $\mathfrak{A}_*$!

Internal definition

For a closure algebra $\mathfrak{A} = (A, \mathbf{C})$,

$$
\begin{array}{lll}
\mathrm{kdim}(\mathfrak{A}) = -1 & \text{if} & \mathfrak{A} \text{ is the trivial algebra,} \\
\mathrm{kdim}(\mathfrak{A}) \leq n & \text{if} & \forall d \text{ nowhere dense in } \mathfrak{A},\ \mathrm{kdim}(\mathfrak{A}_d) \leq n-1, \\
\mathrm{kdim}(\mathfrak{A}) = n & \text{if} & \mathrm{kdim}(\mathfrak{A}) \leq n \text{ and } \mathrm{kdim}(\mathfrak{A}) \not\leq n-1, \\
\mathrm{kdim}(\mathfrak{A}) = \infty & \text{if} & \mathrm{kdim}(\mathfrak{A}) \not\leq n \text{ for any } n = -1, 0, 1, 2, \ldots.
\end{array}
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Observation

Both definitions for $\mathrm{kdim}(\mathfrak{A})$ are equivalent

Finite kdim is expressible by a modal formula

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kdim is not defined point free–requires $\mathfrak{A}_*$!

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Interpreting the modal language in CA

Interpretations in $\mathfrak{A} = (A, \mathbf{C})$

letters	elements of $\mathfrak{A}$
Classical connectives	Boolean operations of A
diamond	$\mathbf{C}$
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Formula φ *is valid in* $\mathfrak{A}$: φ evaluates to 1 for all interpretations; written $\mathfrak{A} \models \varphi$

The bd formulas

Let $n \geq 1$:

$$\mathrm{bd}_1 := \Diamond\Box p_1 \to p_1,$$

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Characterizing finite kdim for CA

Theorem

Let $\mathfrak{A}$ be a nontrivial closure algebra and $n \geq 1$. TFAE:

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Krull dimension of a Heyting algebra

Recall

For a Heyting algebra $\mathfrak{H}$, let $\mathfrak{H}_*$ be the set of prime filters

$\mathfrak{H}_*$ can be partially ordered by $\subseteq$ (closely related to R for $\mathfrak{A}(\mathfrak{H})_*$)

Definition

The *Krull dimension* $\mathrm{kdim}(\mathfrak{H})$ of a Heyting algebra $\mathfrak{H}$ is the supremum of the lengths of chains in $\mathfrak{H}_*$. If the supremum is not finite, then we write $\mathrm{kdim}(\mathfrak{H}) = \infty$.

Lemma

- If $\mathfrak{A}$ is a closure algebra, then $\mathrm{kdim}(\mathfrak{A}) = \mathrm{kdim}(\mathfrak{H}(\mathfrak{A}))$.
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An example

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Let $\mathfrak{H}$ be a Heyting algebra and $a \in \mathfrak{H}$:
a is dense in $\mathfrak{H}$: provided $\neg a := a \to 0 = 0$

Example 4 revisited

$\mathfrak{H} = \{0, b, 1\}$ open elements from previous Example 4
b is dense in $\mathfrak{A}$ ... also in $\mathfrak{H}$

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Let $\mathfrak{H}$ be a Heyting algebra and $a \in \mathfrak{H}$: *Relativization $\mathfrak{H}_a$ of $\mathfrak{H}$ to a*: the interval $[a, 1]$ with operations $\wedge, \vee, \to$ as in $\mathfrak{H}$ and bottom a

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Interpreting the intuitionistic lang. in HA

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3. There does not exist a sequence $1 > b_1 > \cdots > b_n > 0$ in $\mathfrak{H}$ such that b_{i-1} is dense in $\mathfrak{H}_{b_i}$ for each $i \in \{1, \ldots, n\}$.
4. $\mathrm{depth}(\mathfrak{H}_*) \leq n$.

Localic Krull dimension of a space

Definition

The *localic Krull dimension* of a topological space X is

$$\mathrm{ldim}(X) = \mathrm{kdim}(\Omega(X)) = \mathrm{kdim}(\wp X, \mathbf{C})$$

Corollary: recursive definition of ldim

$$\begin{array}{lll} \mathrm{ldim}(X) = -1 & \text{if} & X = \varnothing, \\ \mathrm{ldim}(X) \leq n & \text{if} & \forall D \text{ nowhere dense in } X,\ \mathrm{ldim}(D) \leq n-1, \\ \mathrm{ldim}(X) = n & \text{if} & \mathrm{ldim}(X) \leq n \text{ and } \mathrm{ldim}(X) \not\leq n-1, \\ \mathrm{ldim}(X) = \infty & \text{if} & \mathrm{ldim}(X) \not\leq n \text{ for any } n = -1, 0, 1, 2, \ldots. \end{array}$$

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Characterizing finite localic Krull dimension

Interpretation in X

Via the closure algebra $(\wp X, \mathbf{C})$

Thus $\Diamond$ and $\Box$ are $\mathbf{C}$ and $\mathbf{I}$ resp.

φ *is valid in* X: φ evaluates to X under all interpretations; written $X \models \varphi$

Theorem

Let $X \neq \varnothing$, $n \geq 1$, and $\mathfrak{F}_{n+1}$ be the $(n+1)$-element chain. TFAE:

1. $\mathrm{ldim}(X) \leq n-1$.
2. $X \models \mathrm{bd}_n$.
3. There does not exist a sequence $E_0, \dots, E_n$ of nonempty closed subsets of X such that $E_0 = X$ and E_{i+1} is nowhere dense in E_i for each $i \in \{0, \dots, n-1\}$.
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Comparing ldim to other dimension functions

Lemma

Let X be a space

- If X is a spectral space, then $\mathrm{kdim}(X) \leq \mathrm{ldim}(X)$.
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Drawbacks and benefits via some examples

- $\mathrm{ldim}(\mathbb{R}^n) = \mathrm{ldim}(\mathbb{Q}) = \mathrm{ldim}(\mathcal{C}) = \infty$
- $\mathrm{ldim}(\omega^n) = n - 1$; and so $\mathrm{ldim}(\omega^n + 1) = n$
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ldim AND T_1 SPACES: I

Lemma

Let X be a T_1 space:

- $\text{ldim}(X) \leq 0$ iff X is discrete
- $\text{ldim}(X) \leq 1$ iff X is nodec (every nowhere dense set is closed)

Definition

The Zeman formula is zem $:= \Box\Diamond\Box p \to (p \to \Box p)$

Esakia et al. 2005

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The n-Zeman formula is
$\mathsf{zem}_n := \Box(\Box(\Box p_{n+1} \to \mathsf{bd}_n) \to p_{n+1}) \to (p_{n+1} \to \Box p_{n+1})$

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Some logics and properties

Definition

For $n \geq 1$, put $\mathbf{S4}_n := \mathbf{S4} + \mathrm{bd}_n$ and $\mathbf{S4.Z}_n := \mathbf{S4} + \mathrm{zem}_n$

Lemma

- $\mathbf{S4}_{n+1} \subsetneq \mathbf{S4.Z}_n$
- $\mathbf{S4.Z}_n$ has the finite model property
- $\mathbf{S4.Z}_n$ is the logic of uniquely rooted finite frames of depth $n+1$

Incompleteness

No logic in $[\mathbf{S4}_{n+1}, \mathbf{S4.Z}_n)$ is complete with respect to a class of T_1 spaces

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The logics $\mathbf{S4.Z}_n$

Goal:

Construct a countable crowded Tychonoff space Z_n with $\mathrm{ldim}(Z_n) = n$ whose logic is $\mathbf{S4.Z}_n$

Ingredients

1. Single frame $\mathfrak{B}_n$ determining $\mathbf{S4.Z}_n$
2. Adjunction spaces (gluing); e.g. wedge sum
3. Čech-Stone compactification and Gleason cover
4. Key ingredient: building block Y, a countable crowded ω-resolvable Tychonoff nodec space such that there is a subspace of $\beta Y \setminus Y$ that is homeomorphic to $\beta\omega$ and for any nowhere dense $D \subseteq Y$, $\mathbf{C}D$ and $\beta\omega$ are disjoint.

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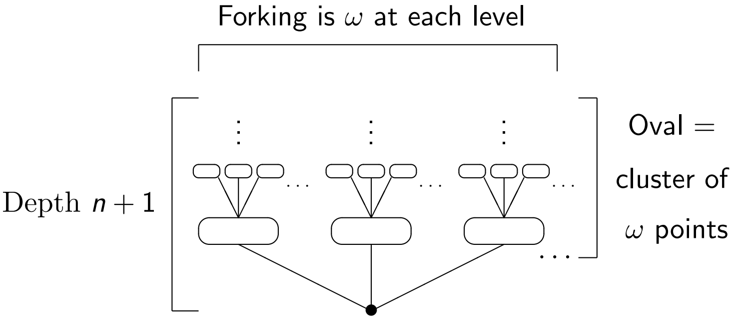
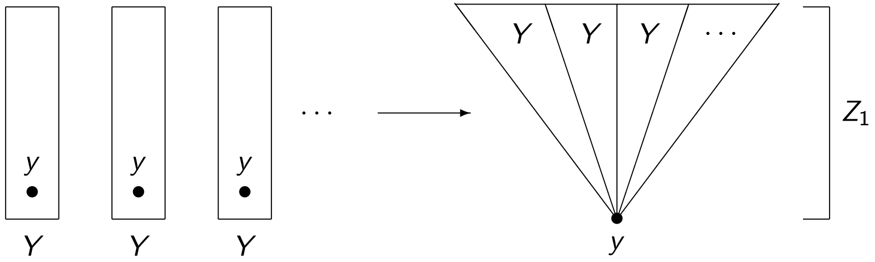
STEP 1: IDENTIFY $\mathfrak{B}_n$

$\mathfrak{B}_n$ is the following frame and determines $\mathbf{S4.Z}_n$
$\mathfrak{B}_n$ is obtained using a refined unraveling technique



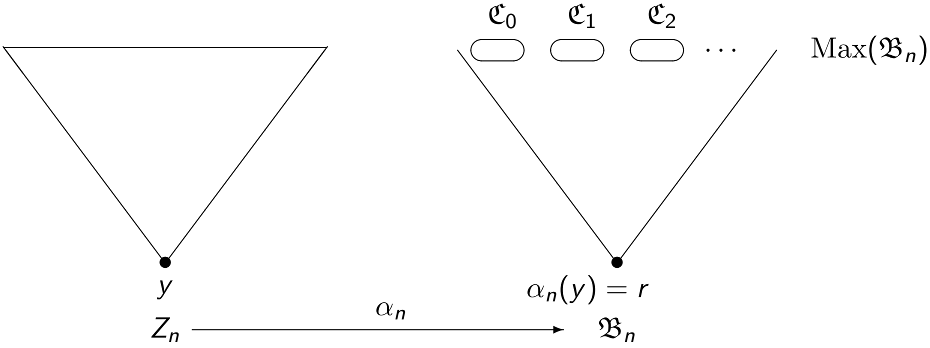
Step 2: base step–build Z_1

Choose and fix a point $y \in Y$
Take topological sum of ω copies of Y
Identify each copy of the point y to get Z_1
Note $\mathfrak{B}_1$ is an interior image of Z_1



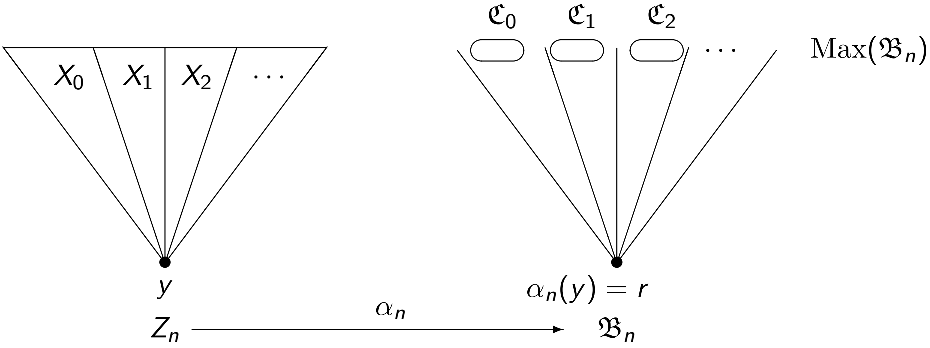
Step 2: Recursive case–Chopping Z_n

Start with $\alpha_n : Z_n \to \mathfrak{B}_n$



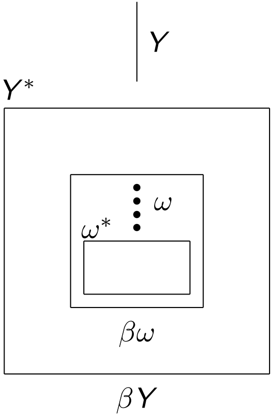
Step 2: Recursive case–Chopping Z_n

'Chop' Z_n into $X_i = \alpha_n^{-1}(R^{-1}\mathfrak{C}_i)$



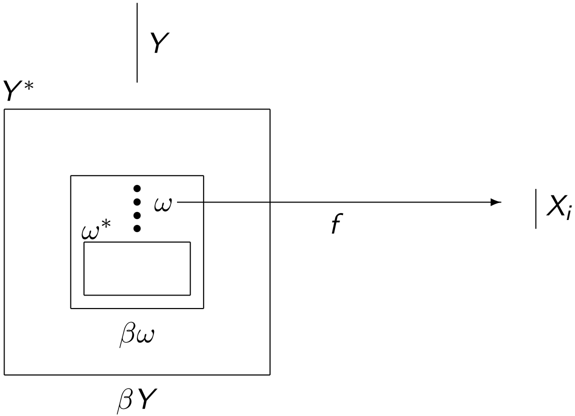
Step 2: Recursive case–general use of Y

For each countable (Tychonoff) space X_i



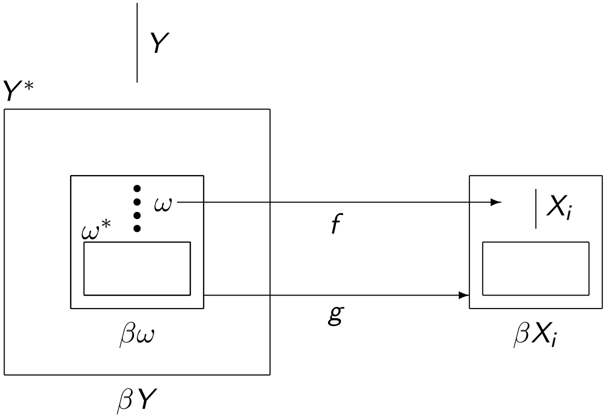
Step 2: Recursive case–general use of Y

There is continuous bijection $f : \omega \to X_i$



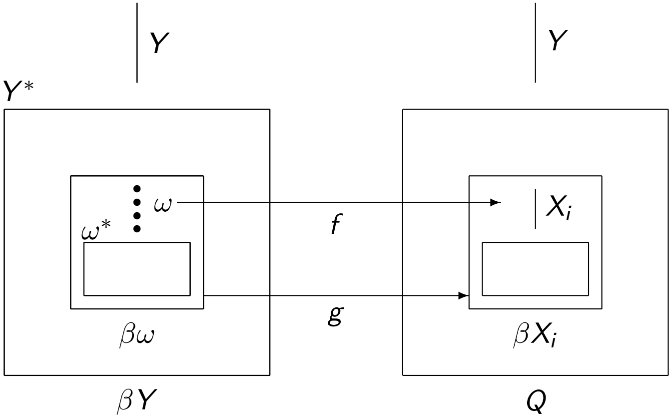
Step 2: Recursive case–general use of Y

f continuously extends to $g : \beta\omega \to \beta X_i$



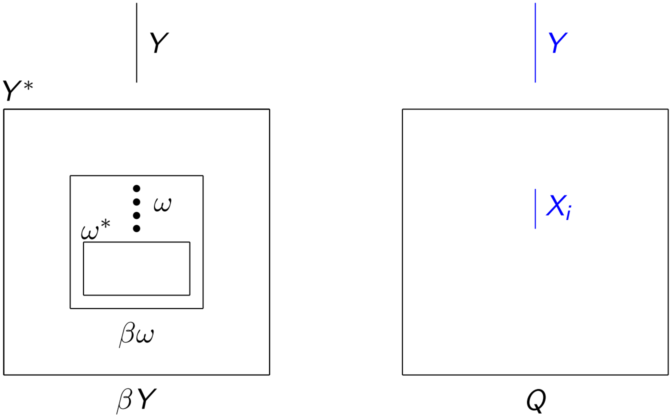
Step 2: Recursive case–general use of Y

Form quotient Q via fibers of g ...



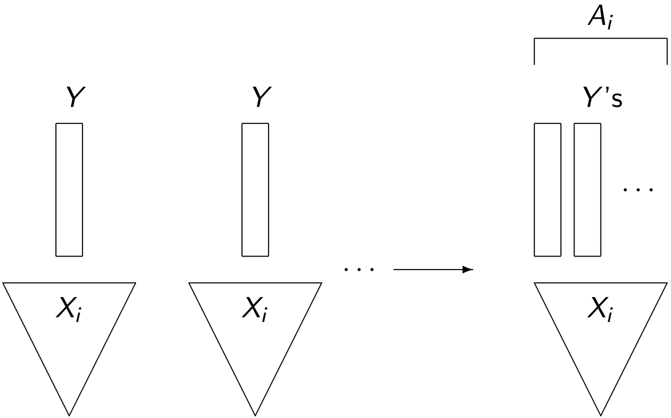
Step 2: Recursive case–general use of Y

Form quotient Q via fibers of g ... **take subspace** $Y \cup X_i$



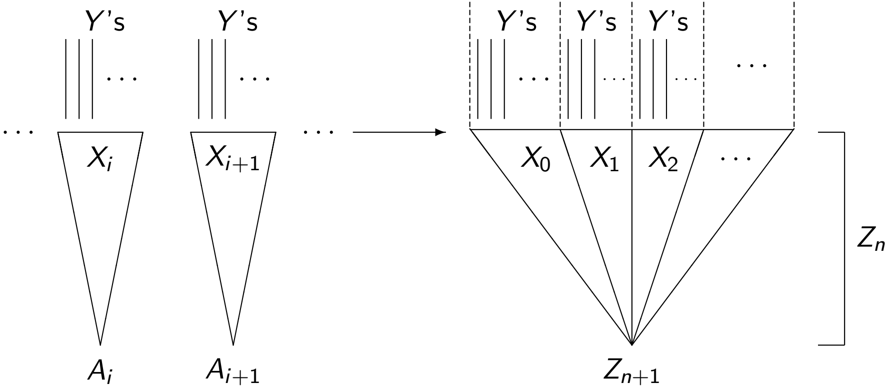
Step 2: Recursive case–gluing $Y \cup X_i$'s

Take topological sum of ω copies of $Y \cup X_i$
Identify through each copy of X_i to get A_i



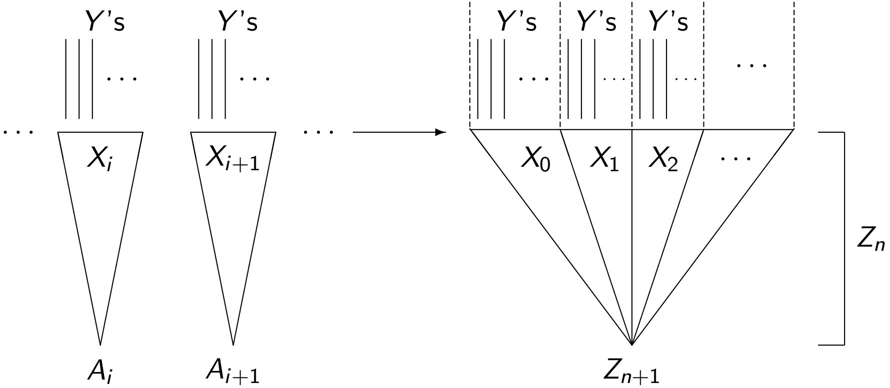
Step 2: Recursive step–gluing A_i's to get Z_{n+1}

Each X_i is a subset of Z_n
Identify the copies of points from Z_n



Step 2: Recursive step–gluing A_i's to get Z_{n+1}

Each X_i is a subset of Z_n
Identify the copies of points from Z_n
Send Y's 'above' X_i to cluster's above $\mathfrak{C}_i$ in $\mathfrak{B}_{n+1}$



Conclusions

Theorem

The logic of Z_n is $\mathbf{S4.Z}_n$

Complete details available at:
http://www.illc.uva.nl/Research/Publications/Reports/PP-2016-19.text.pdf

Thank You... Organizers and Audience

Questions ...

Conclusions

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