The Riesz representation theorem for valuations on distributive lattices

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Motivation

Definition

Let D be a bounded distributive lattice. A valuation on D is a function $v\colon D \to \mathbb{R}$ satisfying $v(\bot) = 0$ and

$$v(x \vee y) + v(x \wedge y) = v(x) + v(y) \qquad x, y \in D.$$

Valuations on distributive lattices appear in

- Geometry and measure theory
- Probability theory
- Functional analysis

Geometry

Example

Let $\mathscr{B}(X)$ be the Borel σ-algebra over a bounded measurable set $X \subset \mathbb{R}^n$. These functions are valuations on $\mathscr{B}(X)$:

- Lebesgue measure on $\mathscr{B}(X)$
- $p(A) := \#(A \cap \mathbb{Z}^n), \quad A \in \mathscr{B}(X)$

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Theorem (Hadwiger, 1955)

Let $\mathcal{C}$ be the family of compact convex sets in $\mathbb{R}^n$ and $\mathcal{U}$ be the lattice of polyconvex sets. Then there is a unique valuation χ on $\mathcal{U}$ such that $\chi(C) = 1$ for each nonempty $C \in \mathcal{C}$.

Probability

Every probability on a Boolean algebra is a valuation. Probability functions are often extended to non-classical setting:

- **Heyting algebras** and **MV-algebras** have distributive lattice reducts.
- Virtually any probability-like functional studied on those algebras thus becomes a valution.

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Example (Mundici)

Let M be an MV-algebra and $s\colon M \to [0,1]$ be a state: $s(\top) = 1$ and

$$s(a \oplus b) = s(a) + s(b), \quad a, b, \in M \text{ with } a \odot b = \bot.$$

Then s is a valuation.

Riesz theorem

Let X be a compact Hausdorff space and $C(X)$ be the Banach space of continuous functions $X \to \mathbb{R}$. There is a bijection between

1. bounded linear functionals L on $C(X)$ and
2. signed Baire measures μ on X

such that

$$L(f) = \int_X f \, \mathrm{d}\mu \qquad f \in C(X).$$

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Note

Here we can think of X as the dual topological space to $C(X)$ on which the linear functional L is represented by μ.

Our goal

- We will look at valuations from the perspective of **Stone duality** for distributive lattices.
- The mirror image of a valuation will be a measure over a certain family of subsets of the **spectral space**.
- Our main result is a representation theorem for valuations by measures named **spectral Baire measures**.

Outline

From valuations to charges

From charges to measures

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Terminology

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Let v be a valuation on a bounded distributive lattice D. We call v

- monotone if $x \leq y$ implies $v(x) \leq v(y)$
- normalised if $v(\top) = 1$

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A real function c on a Boolean algebra B is a charge if $c(\bot) = 0$ and $c(x \vee y) = c(x) + c(y)$ whenever $x \wedge y = \bot$. A charge c is positive if $c(x) \geq 0$ for any $x \in B$.

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Lemma

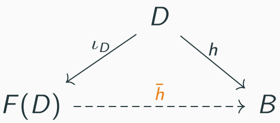
The valuations on a Boolean algebra B are exactly the charges on B.
*The **monotone** valuations on B are exactly the **positive** charges on B.*

Extending valuations to charges (1)

We will pass from D to the Boolean algebra $F(D)$ freely generated by D.

Definition

The free Boolean extension of a distributive lattice D is a pair $(F(D), \iota_D)$ where $\iota_D \colon D \to F(D)$ is a homomorphism such that the following diagram commutes for any Boolean algebra B and a homomorphism $h \colon D \to B$:



Extending valuations to charges (2)

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Any valuation v on D extends uniquely to a valuation v' on $F(D)$. Moreover, v' is monotone normalised iff v is monotone normalised.

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Hint

Let w be a valuation on $F(D)$ extending v. Then

$$v(x) = w(x) = w(x \wedge y) + w(x \wedge \neg y) = v(x \wedge y) + w(x \wedge \neg y).$$

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Then

$$w(x \wedge \neg y) = v(x) - v(x \wedge y).$$

The general formula for v' is derived using DNF of elements in $F(D)$.

Finite case

If D is a finite distributive lattice, then valuations are determined by their values on the join-irreducible elements $\mathcal{JI}(D)$ of D (Rota).

Lemma

Let D be a finite distributive lattice. Then there is a bijection between

- *valuations on D and*
- *functions $p\colon \mathcal{JI}(D) \to \mathbb{R}$.*

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In the case of infinite D we need to work with valuations over lattices of sets in the **Stone space** of D.

Dual space

Let D be a bounded distributive lattice:

- $X := \mathrm{Spec}\, D$ is its Stone space
- $\mathrm{Kn}(X)$ is the lattice of compact open sets
- Stone map $x \in D \mapsto \hat{x} \in \mathrm{Kn}(X)$

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We construct the free Boolean extension of the lattice $\mathrm{Kn}(X) \cong D$:

- Let π be the patch topology on X, which is generated by

$$\{A \mid A \in \mathrm{Kn}(X)\} \cup \{A^C \mid A \in \mathrm{Kn}(X)\}.$$

- Then $\mathrm{Clop}(X, \pi) =$ the free Boolean extension of $\mathrm{Kn}(X)$.

Representation of valuations by charges

Lemma

Let μ be a charge on $\mathrm{Clop}(X, \pi)$. Define a function $v_\mu \colon D \to \mathbb{R}$ by setting

$$v_\mu(x) := \mu(\hat{x}) \qquad x \in D.$$

Then

$$\mu \mapsto v_\mu$$

is a **bijection** *between charges on $\mathrm{Clop}(X, \pi)$ and valuations on D.*

$$\begin{array}{ccc} D & \xrightarrow{\cong} & \mathrm{Kn}(X) \\ {\scriptstyle \iota_D}\big\downarrow & & \big\downarrow{\scriptstyle \iota_{\mathrm{Kn}(X)}} \\ F(D) & \xrightarrow{\cong} & \mathrm{Clop}(X, \pi) \end{array}$$

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We will further extend the charges to measures.

From charges to measures

Measures and pre-measures

Definition

Let Σ be a σ-**complete** Boolean algebra. A function $\mu\colon \Sigma \to \mathbb{R}^+$ is a measure if $\mu(\bot) = 0$ and

$$\mu\left(\bigvee_{i\in\mathbb{N}} x_i\right) = \sum_{i\in\mathbb{N}} \mu(x_i)$$

for any countable subset $\{x_i \mid i \in \mathbb{N}\} \subseteq \Sigma$ such that $x_i \wedge x_j = \bot$, $i \neq j$.

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Definition

Let B be a Boolean algebra. A function $\mu\colon B \to \mathbb{R}^+$ is a pre-measure if $\mu(\bot) = 0$ and

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for any countable subset $\{x_i \mid i \in \mathbb{N}\} \subseteq B$ such that $x_i \wedge x_j = \bot$, $i \neq j$, whenever $\bigvee_{i\in\mathbb{N}} x_i \in B$.

Hahn-Kolmogorov theorem

Let B be a Boolean algebra of subsets of a set X and Σ be the Boolean σ-algebra generated by B in 2^X. Then any **pre-measure** on B uniquely extends to a **measure** on Σ.

Borel and Baire

Definition

Let X be a topological space.

- The Borel algebra $\mathscr{B}(X)$ is the σ-algebra generated by the open sets.
- The Baire algebra $\mathscr{B}_c(X)$ is the smallest σ-algebra making all continuous functions $X \to \mathbb{R}$ measurable.

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For every compact Hausdorff space X,

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However, we are working with spectral spaces.

Spectral spaces

Definition

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- X is compact, T_0, sober and
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- $\mathrm{Kn}(X)$ is a lattice and a basis for the topology.

A continuous map $f: X \to Y$ between spectral spaces X and Y is called spectral when

$$f^{-1}(A) \in \mathrm{Kn}(X) \qquad \text{for every } A \in \mathrm{Kn}(Y).$$

Indicator functions into Sierpiński space

We will present the Borel algebra of X in analogy with the Baire algebra.

Definition

The Sierpiński space is a space $\mathbf{2} := \{0, 1\}$ with the collection of open sets $\{\emptyset, \{0\}, \{0, 1\}\}$.

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Definition

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Lemma

For any space X, the Borel algebra $\mathscr{B}(X)$ coincides with the smallest σ-algebra making all the continuous functions $X \to \mathbf{2}$ measurable.

Spectral Baire algebra

Definition

Let X be a spectral space. The spectral Baire algebra $\mathscr{S}(X)$ is the smallest σ-algebra making all the **spectral maps** $X \to \mathbf{2}$ measurable.

Lemma

Let X be a spectral space. Then $\mathscr{S}(X)$ is generated by the compact open sets $\mathsf{Kn}(X)$.

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Now we have almost all ingredients.

From charges to measures

Recall what we have already proved:

Lemma

Let μ *be a charge on* $\mathrm{Clop}(X,\pi)$*. Define a function* $v_\mu \colon D \to \mathbb{R}$ *by setting*

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We want to extend μ onto the spectral Baire algebra $\mathscr{S}(X)$ by applying Hahn-Kolmogorov theorem:

Is μ a pre-measure on $\mathrm{Clop}(X,\pi)$?

Diagram

$$\begin{array}{ccc} D & \xrightarrow{\cong} & \mathrm{Kn}(X) \\ \iota_D \big\downarrow & & \big\downarrow \iota_{\mathrm{Kn}(X)} \\ F(D) & \xrightarrow{\cong} & \mathrm{Clop}(X, \pi) \\ & & \big\downarrow \\ & & \mathscr{S}(X) \end{array}$$

Infinite joins never exist

μ is indeed a pre-measure on $\mathrm{Clop}(X, \pi)$ as a consequence of the following observation.

Lemma

Let X be a spectral space and $\mathrm{Kn}(X)$ be the lattice of compact open sets. Suppose that $\{A_i \mid i \in \mathcal{I}\} \subseteq \mathrm{Kn}(X)$ is a family such that the following conditions hold:

- $A_i \neq \emptyset$ *for each* $i \in \mathcal{I}$.
- $A_i \cap A_j = \emptyset$ *for each* $i \neq j$.
- $\bigcup_{i \in \mathcal{I}} \in \mathrm{Kn}(X)$.

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Then $\mathcal{I}$ is finite.

Representation theorem

Theorem

Let D be a bounded distributive lattice, $X := \operatorname{Spec} D$ be its spectral space and $\mu\colon \mathscr{S}(X) \to \mathbb{R}^+$ be a spectral measure. Define a function $v_\mu\colon D \to \mathbb{R}^+$ by setting

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We can extend the theorem to the bijection between signed spectral measures and **bounded** valuations.

Applications

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Riesz theorem for spectral spaces

Let X be a **spectral space** and $C(X)$ be the Banach space of continuous functions $X \to \mathbb{R}$. There is a bijection between

1. bounded linear functionals L on $C(X)$ and
2. signed Baire measures μ

such that $L(f) = \int_X f \, \mathrm{d}\mu$ for every $f \in C(X)$.

Items for future research

- Reproving the classical Riesz theorem by our results for spectral spaces.
- Development of measure theory for spectral spaces.

Questions?