A Vietoris functor for d-frames

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2. Lawson topology $(D; \bigsqcup^{\uparrow}) \mapsto (D; \sigma_D \vee \lambda_D)$
3. Priestley duality

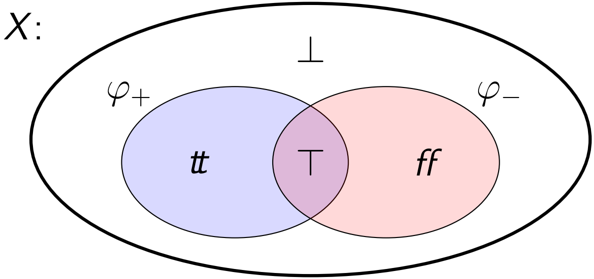
   spec: $(D; \wedge, \vee, 1, 0) \mapsto ([D \to \mathbf{2}];\ \tau_+ \vee \tau_-,\ \leq_{\tau_+})$

$$\Phi_+(x) = \{h \colon D \to \mathbf{2} \mid h(x) = 1\}$$
$$\Phi_-(x) = \{h \colon D \to \mathbf{2} \mid h(x) = 0\}$$

4. ...

Bitopological spaces

$(X;\, \tau_+, \tau_-)$ is a *bitopological space* (or *bispace* for short) if $(X;\, \tau_+)$ and $(X;\, \tau_-)$ are topological spaces.



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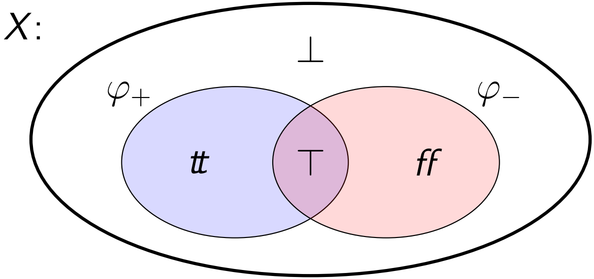
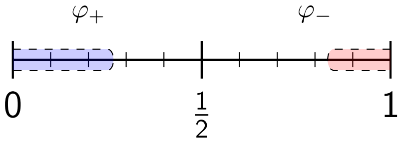


Example

$\varphi(x) \stackrel{\text{def}}{\equiv} x \leq \frac{1}{2}$



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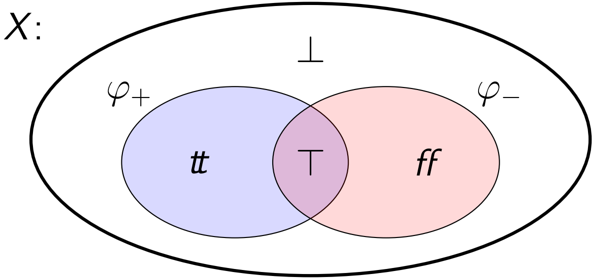

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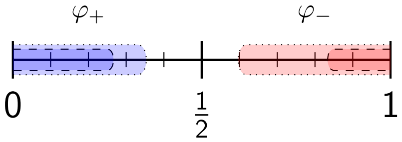

Two orders of $\tau_+ \times \tau_-$

Let $\varphi = (\varphi_+, \varphi_-)$, $\psi = (\psi_+, \psi_-) \in \tau_+ \times \tau_-$

Inf. order: $\varphi \sqsubseteq \psi$ iff $\varphi_+ \subseteq \psi_+$ and $\varphi_- \subseteq \psi_-$:

$$
\begin{aligned}
\varphi \sqcup \psi &\stackrel{\text{def}}{\equiv} (\varphi_+ \cup \psi_+, \varphi_- \cup \psi_-) \quad \bot \stackrel{\text{def}}{\equiv} (\emptyset, \emptyset) \\
\varphi \sqcap \psi &\stackrel{\text{def}}{\equiv} (\varphi_+ \cap \psi_+, \varphi_- \cap \psi_-) \quad \top \stackrel{\text{def}}{\equiv} (X, X)
\end{aligned}
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Logic order: $\varphi \leq \psi$ iff $\varphi_+ \subseteq \psi_+$ and $\varphi_- \supseteq \psi_-$:

$$\varphi \vee \psi \stackrel{\text{def}}{\equiv} (\varphi_+ \cup \psi_+, \varphi_- \cap \psi_-) \quad \mathit{ff} \stackrel{\text{def}}{\equiv} (\emptyset, X)$$
$$\varphi \wedge \psi \stackrel{\text{def}}{\equiv} (\varphi_+ \cap \psi_+, \varphi_- \cup \psi_-) \quad \mathit{tt} \stackrel{\text{def}}{\equiv} (X, \emptyset)$$

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Lattices: $(\tau_+ \times \tau_-; \wedge, \vee, \mathit{tt}, \mathit{ff})$, $(\tau_+ \times \tau_-; \sqcap, \sqcup, \top, \bot)$

D-frames (Jung & Moshier)

D-frame is a structure $\mathcal{L} = (L_+ \times L_-; \mathrm{con}, \mathrm{tot})$ where

- L_+ and L_- are frames (i.e. $\bigvee_i (a_i \wedge b) = (\bigvee_i a_i) \wedge b$)
- con $\subseteq L_+ \times L_-$
- tot $\subseteq L_+ \times L_-$
  (+ axioms, e.g. $\alpha \sqsubseteq \beta \in$ con implies $\alpha \in con$)

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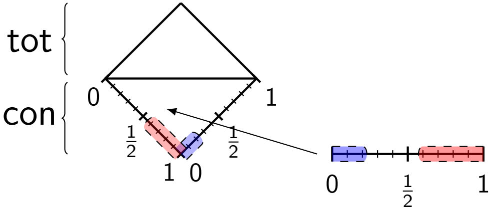

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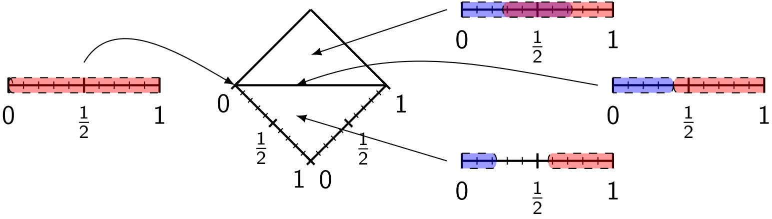

Stone duality for d-frames

- D-frame homomorphisms $\mathcal{L} \to \mathcal{M}$ are $(h_+, h_-)\colon L_+ \times L_- \to M_+ \times M_-$

$$h[\mathrm{con}_{\mathcal{L}}] \subseteq \mathrm{con}_{\mathcal{M}}, \quad h[\mathrm{tot}_{\mathcal{L}}] \subseteq \mathrm{tot}_{\mathcal{M}}$$

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- $\Sigma_d(\mathcal{L}) = ([\mathcal{L} \to \mathbf{2}\times\mathbf{2}];\ \Phi_+[L_+], \Phi_-[L_-])$ where

$$\Phi_+(x) = \{h\colon \mathcal{L} \to \mathbf{2}\times\mathbf{2} \mid h_+(x) = 1\}$$
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- $\Omega_d(X;\ \tau_+, \tau_-) = (\tau_+ \times \tau_-; \mathrm{con}_X, \mathrm{tot}_X)$ where
$$(U, V) \in \mathrm{con}_X \quad \text{iff} \quad U \cap V = \emptyset$$
$$(U, V) \in \mathrm{tot}_X \quad \text{iff} \quad U \cup V = X$$

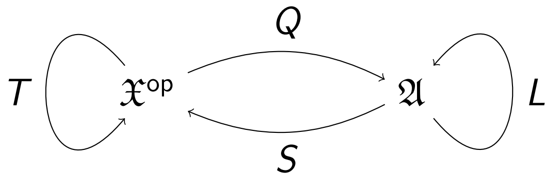
Coalgebras

$$\xi \colon X \to T(X)$$

Syntax of a type functor:

$$T ::= \mathsf{Id} \mid A \mid T_1 + T_2 \mid T_1 \times T_2 \mid T^B \mid \mathcal{P}\, T$$

Coalgebraic logics (Kurz's framework)



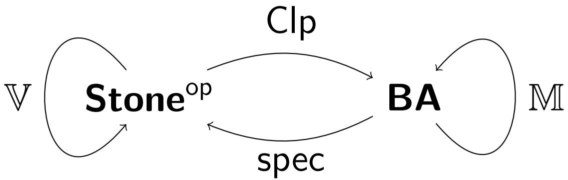
and

$$\delta \colon LQ \Longrightarrow QT$$

If $S \circ Q \cong \mathrm{Id}$, $Q \circ S \cong \mathrm{Id}$ and δ is a natural iso, it lifts to

$$\mathrm{Coalg}(T)^{\mathrm{op}} \cong \mathrm{Alg}(L)$$

Jónsson-Tarski duality



- Coalg($\mathbb{V}$) $\cong$ descriptive (general) frames
- Alg($\mathbb{M}$) $\cong$ modal Boolean algebras

$\mathbb{M}\colon A \;\mapsto\; \mathbb{BA}\langle\, \Box a, \Diamond a : a \in A\,\rangle/_{\approx}$ where $\approx$ is generated by

$$\Box a \wedge \Box b \approx \Box(a \wedge b) \quad \Box 1 \approx 1$$

$$\Diamond a \vee \Diamond b \approx \Diamond(a \vee b) \quad \Diamond 0 \approx 0$$

$$\Box a \wedge \Diamond b \preceq \Diamond(a \wedge b) \quad \Box(a \vee b) \preceq \Box a \vee \Diamond b$$

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$\mathbb{V}\colon (X,\tau) \;\mapsto\; (\mathcal{K}X, \mathbb{V}\tau)$ where

- $\mathcal{K}X =$ closed subsets of X
- $\mathbb{V}\tau$ is generated by $\boxtimes U$ and $\diamondplus U$, $\forall U \in \tau$:

$$K \in \boxtimes U \;\stackrel{\text{def}}{\equiv}\; K \subseteq U \quad \text{and} \quad K \in \diamondplus U \;\stackrel{\text{def}}{\equiv}\; K \cap U \neq \emptyset$$

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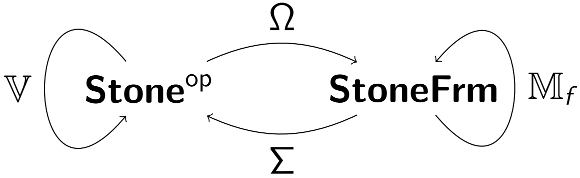
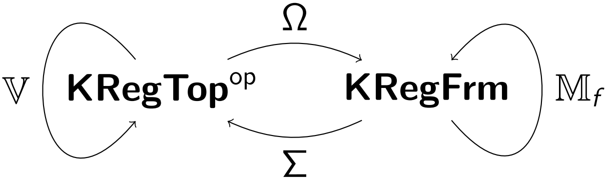
$$\delta_X\colon \mathbb{M} \circ \mathrm{Clp}(X) \to \mathrm{Clp} \circ \mathbb{V}(X), \quad \begin{aligned} \Box U &\mapsto \{K \in \mathcal{K}X \mid K \subseteq U\} \\ \Diamond U &\mapsto \{K \in \mathcal{K}X \mid K \cap U \neq \emptyset\} \end{aligned}$$


Ω
𝕍
Stone^op
StoneFrm
𝕄_f
Σ


Ω
𝕍
KRegTop^op
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Johnstone's powerlocale

$$\mathbb{M}_f : L \longmapsto \mathbb{F}\mathrm{r}\langle\, \Box a, \Diamond a : a \in L \,\rangle/_{\approx}$$

where $\approx$ is generated by

$$\begin{aligned} \Box a \wedge \Box b &\approx \Box(a \wedge b) & \Box 1 &\approx 1 \\ \Diamond a \vee \Diamond b &\approx \Diamond(a \vee b) & \Diamond 0 &\approx 0 \end{aligned}$$

$$\Box a \wedge \Diamond b \preceq \Diamond(a \wedge b) \quad \Box(a \vee b) \preceq \Box a \vee \Diamond b$$

$$\bigvee^{\uparrow} \Box a_i \approx \Box(\bigvee^{\uparrow} a_i) \quad \bigvee^{\uparrow} \Diamond a_i \approx \Diamond(\bigvee^{\uparrow} a_i)$$

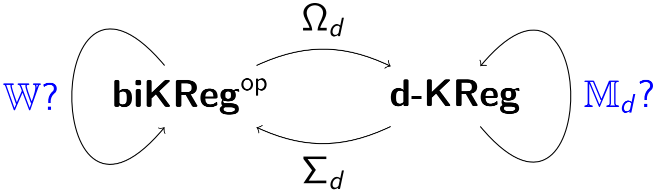
4-valued coalgebraic logic?

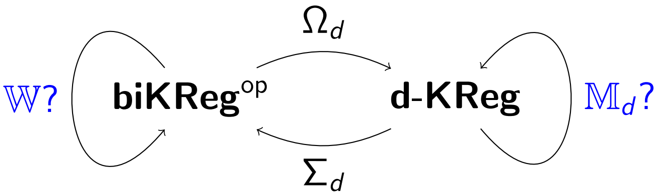
We need:

- $\mathcal{X} \subseteq$ **biSpace**
- $\mathcal{A} \subseteq$ **d-Frame**
  such that $\Omega_d \colon \mathcal{X}^{\mathrm{op}} \cong \mathcal{A}$

- $\mathbb{W} \colon \mathcal{X} \to \mathcal{X}$
- $\mathbb{M}_d \colon \mathcal{A} \to \mathcal{A}$
  such that there is a $\delta \colon \mathbb{M}_d \circ \Omega_d \cong \Omega_d \circ \mathbb{W}$

$$\mathbf{biKReg}^{\mathsf{op}} \underset{\Sigma_d}{\overset{\Omega_d}{\rightleftarrows}} \mathbf{d\text{-}KReg}$$


Ω_d
$\mathbb{W}$?
$\mathbf{biKReg}^{\mathsf{op}}$
$\mathbf{d\text{-}KReg}$
$\mathbb{M}_d$?
Σ_d



$\mathbb{W}\colon (X;\, \tau_+, \tau_-) \mapsto (\mathcal{K}X;\, \mathbb{V}\tau_+, \mathbb{V}\tau_-)$ where

1. $\mathcal{K}X =$ compact convex subsets of X

   (Note: $(\leq_{\tau_+}) = (\geq_{\tau_-})$)

2. $\mathbb{V}\tau_+$ is generated by $\boxtimes U_+$ and $\diamondplus U_+$, $\forall U_+ \in \tau_+$
3. $\mathbb{V}\tau_-$ is generated by $\boxtimes U_-$ and $\diamondplus U_-$, $\forall U_- \in \tau_-$

Free d-frame construction

For $(B_+ \times B_-; \text{con}_1, \text{tot}_1)$ where

1. $B_+ \subseteq L_+$ and $\langle B_+ \rangle = L_+$
2. similarly for B_-
3. $\text{con}_1, \text{tot}_1 \subseteq B_+ \times B_-$

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set

$$\mathtt{TOT}\,\langle \mathsf{tot}_1 \rangle \overset{\text{def}}{\equiv} \uparrow\!\mathbb{DL}_{\vee,\wedge}\,\langle \mathsf{tot}_1 \rangle$$

$$\mathtt{CON}\,\langle \mathsf{con}_1 \rangle \overset{\text{def}}{\equiv} \mathbb{DCPO}_{\bigsqcup^{\uparrow}}\,\langle \downarrow\!\mathbb{DL}_{\vee,\wedge}\,\langle \mathsf{con}_1 \rangle \rangle$$

Then, $(L_+ \times L_-;\, \mathtt{CON}\,\langle \mathsf{con}_1 \rangle\,, \mathtt{TOT}\,\langle \mathsf{tot}_1 \rangle)$ is a *pre*-d-frame.

(con-tot) $\alpha \in \text{con}, \beta \in \text{tot}$ and

$$\alpha_+ = \beta_+ \text{ or } \alpha_- = \beta_- \implies \alpha \sqsubseteq \beta$$

(con-tot) $\alpha \in \mathsf{con}, \beta \in \mathsf{tot}$ and
$\alpha_+ = \beta_+$ or $\alpha_- = \beta_- \implies \alpha \sqsubseteq \beta$

Lemma

If B_+ and B_- are meet-semilattices and

$$\frac{\gamma \in \mathsf{tot}_\wedge,\ (\alpha_i, \beta) \in {\downarrow}\mathbb{DL}_{\vee,\wedge}\langle \mathsf{con}_1 \rangle,\ \beta \in B_-}{\gamma_+ \leq \bigvee^{\uparrow}{}_i\, \alpha_i \implies \beta \leq \gamma_i}$$

(+ symmetrical rule),

then $(L_+ \times L_-; \mathtt{CON}\langle \mathsf{con}_1 \rangle, \mathtt{TOT}\langle \mathsf{tot}_1 \rangle)$ *is a d-frame, i.e. satisfies* (con-tot)*.*

Vietoris functor for d-frames

$$\mathbb{M}_d : (L_+ \times L_-; \mathsf{con}, \mathsf{tot}) \mapsto (\mathbb{M}_f L_+ \times \mathbb{M}_f L_-;\ \mathtt{CON}\left\langle \mathsf{con}_1 \right\rangle, \mathtt{TOT}\left\langle \mathsf{tot}_1 \right\rangle)$$

where

$$\mathsf{tot}_1 = \{(\Box a, \Diamond b), (\Diamond a, \Box b) : (a, b) \in \mathsf{tot}\}$$
$$\mathsf{con}_1 = \{(\Box a, \Diamond b), (\Diamond a, \Box b) : (a, b) \in \mathsf{con}\}$$

(from: $\Box a \wedge \Diamond b \leq \Diamond(a \wedge b) \quad \Box(a \vee b) \leq \Box a \vee \Diamond b$)

Theorem

Let $\mathcal{L}$ be a compact regular d-frame. Then $\mathbb{M}_d\mathcal{L}$ is also compact and regular.

Moreover, if $\mathcal{L}$ was zero-dimensional then $\mathbb{M}_d\mathcal{L}$ also is.

Theorem

$\mathcal{L}$ can be embedded in $\mathbb{M}_d\mathcal{L}$ ($\mathbb{M}_d\mathcal{L} \twoheadrightarrow \mathcal{L}$).

(Proofs are similar to those by Johnstone.)

Points of d-Vietoris construction

$P_{\mathcal{L}} \subseteq L_+ \times L_-$ be such that $\alpha \in P_{\mathcal{L}}$ iff

$$\text{(P+)} \quad (\alpha_+ \vee u_+, \alpha_-) \in \text{tot} \implies (u_+, \alpha_-) \in \text{tot}$$

$$\text{(P−)} \quad (\alpha_+, \alpha_- \vee u_-) \in \text{tot} \implies (\alpha_+, u_-) \in \text{tot}$$

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Lemma

The map $(U_+, U_-) \mapsto X \setminus (U_+ \cup U_-)$ is a bijection between $P_{\Omega_d(X)}$ and $\mathcal{K}X$.

Lemma ($\star$)

The map $p \mapsto (\bigvee\{x \in L_+ \mid p_+(\Diamond x) = 0\}, \dots)$ is a bijection between $[\mathbb{M}_d\mathcal{L} \to \mathbf{2}\times\mathbf{2}]$ and $P_{\mathcal{L}}$.

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It gives a natural iso: $\Sigma_d \circ \mathbb{M}_d \cong \mathbb{W} \circ \Sigma_d$

Proof of Lemma ($\star$)

$$p\colon \mathbb{M}_d\mathcal{L} \to \mathbf{2}\times\mathbf{2}, \quad \alpha_\pm = \bigvee\{x \in L_\pm \mid p_\pm(\Diamond x) = 0\}$$

1. Let $(\alpha_+ \vee u, \alpha_-) \in \mathrm{tot}_\mathcal{L}$

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2. $(\Box(\alpha_+ \vee u), \Diamond\alpha_-) \in \mathtt{TOT}\,\langle \mathsf{tot}_1 \rangle$
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3. $p(\Box(\alpha_+ \vee u), \Diamond\alpha_-) \in \mathsf{tot}_{\mathbf{2}\times\mathbf{2}}$
4. $p(\Diamond\alpha_+ \vee \Box u, \Diamond\alpha_-) \in \mathsf{tot}_{\mathbf{2}\times\mathbf{2}}$
   (from $\Box(\alpha_+ \vee u) \leq \Diamond\alpha_+ \vee \Box u$)

Proof of Lemma ($\star$)

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1. Let $(\alpha_+ \vee u, \alpha_-) \in \mathsf{tot}_{\mathcal{L}}$
2. $(\Box(\alpha_+ \vee u), \Diamond\alpha_-) \in \mathtt{TOT}\,\langle \mathsf{tot}_1 \rangle$
3. $p(\Box(\alpha_+ \vee u), \Diamond\alpha_-) \in \mathsf{tot}_{\mathbf{2}\times\mathbf{2}}$
4. $p(\Diamond\alpha_+ \vee \Box u, \Diamond\alpha_-) \in \mathsf{tot}_{\mathbf{2}\times\mathbf{2}}$
   (from $\Box(\alpha_+ \vee u) \leq \Diamond\alpha_+ \vee \Box u$)
5. $p_+(\Box u) = 1$

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7. $(x, x^*) \in \mathsf{con}$ and $(\Box x, \Diamond(x^*)) \in \mathsf{con}_{\mathbb{M}_d\mathcal{L}}$

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7. $(x, x^*) \in \mathsf{con}$ and $(\Box x, \Diamond(x^*)) \in \mathsf{con}_{\mathbb{M}_d\mathcal{L}}$
8. $p_-(\Diamond(x^*)) = 0$ therefore $x^* \leq \alpha_-$

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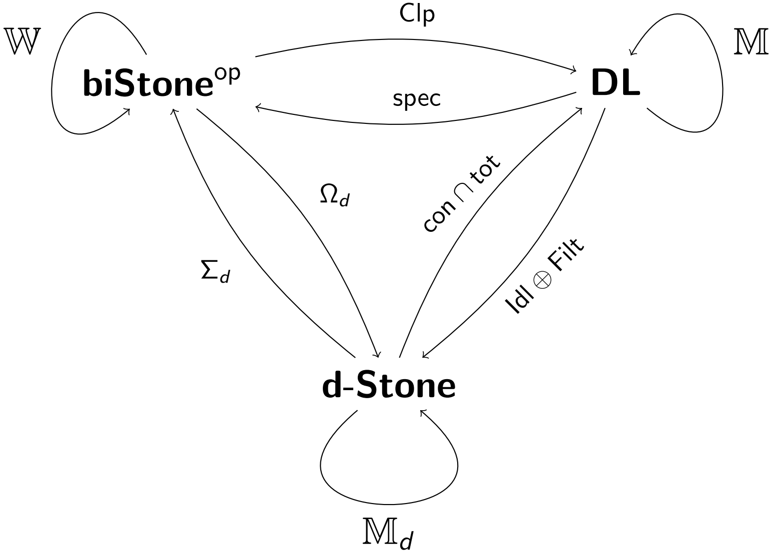
8. $p_-(\Diamond(x^*)) = 0$ therefore $x^* \leq \alpha_-$
9. $(u, \alpha_-) \in \mathsf{tot}$ $\Box$



Clp
$\mathbb{W}$
$\mathbb{M}$
biStone^{op}
DL
spec
Ω_d
con ∩ tot
Σ_d
Idl ⊗ Filt
d-Stone
$\mathbb{M}_d$

Thank you!

Topological properties

	frames	d-frames
a^* :	$\bigvee\{x \mid x \wedge a = 0\}$	$\bigvee\{x \in L_- \mid (a, x) \in \mathrm{con}\}$
$a \lhd b$:	$b \vee a^* = 1$	$(b, a^*) \in \mathrm{tot}$

$$\text{Regularity:} \quad a = \bigvee\{x \mid x \lhd a\}$$

$$\text{Zero-dimensionality:} \quad a = \bigvee\{x \mid x \lhd x \leq a\}$$

Compactness:

For all $U \subseteq L$:

$$\bigvee U = 1 \implies \exists F \subseteq_{\text{fin}} U \text{ s.t. } \bigvee F = 1$$

For all $\mathcal{U} \subseteq L_+ \times L_-$:

$$\bigsqcup \mathcal{U} \in \mathrm{tot} \implies \exists \mathcal{F} \subseteq_{\text{fin}} \mathcal{U} \text{ s.t. } \bigsqcup \mathcal{F} \in \mathrm{tot}$$