On the completion of pointfree function rings

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Let $\mathfrak{L}(\mathbb{R})$ denote the *frame of reals* [1], presented by generators and relations, that is, the frame generated by all ordered pairs (p, q) of rationals, subject to the relations

(R1) $(p,q) \wedge (r,s) = (p \vee r, q \wedge s)$,
(R2) $(p,q) \vee (r,s) = (p,s)$ whenever $p \le r < q \le s$,
(R3) $(p,q) = \bigvee\{(r,s) \mid p < r < s < q\}$,
(R4) $\bigvee\{(p,q) \mid p,q \in \mathbb{Q}\} = 1$.

For any frame L, a *continuous real function* on L is a frame homomorphism $\mathfrak{L}(\mathbb{R}) \to L$. Let $\mathrm{C}(L)$, resp. $\mathrm{C}^*(L)$ denote the lattice-ordered ring of continuous, resp. bounded continuous real functions on L. It is well known that $\mathrm{C}(L)$ and $\mathrm{C}^*(L)$ are distributive lattices. In general, however, due to axiom (R2) above, they are not Dedekind (order) complete: arbitrary non-void sets of continuous real functions in $\mathrm{C}(L)$ and $\mathrm{C}^*(L)$ bounded from above need not have a least upper bound in the lattices $\mathrm{C}(L)$ and $\mathrm{C}^*(L)$.

In this talk, we will present the Dedekind completions of $\mathrm{C}(L)$ and $\mathrm{C}^*(L)$, in two different ways, in terms of:

(1) partial continuous real functions on L ([3]), and

(2) normal semicontinuous real functions on L ([4]).

The first approach evokes the classical description of the completion of $\mathrm{C}^*(X)$ due to Dilworth [2], and simplified and extended to $\mathrm{C}(X)$ by Horn [5]. Our results extend Dilworth's construction to the pointfree setting, but the pointfree situation is not merely a mimic of the classical one, what makes the whole picture more interesting. Our main device will be the frame $\mathfrak{L}(\mathbb{IR})$ of *partially defined real numbers*, obtained from $\mathfrak{L}(\mathbb{R})$ just by dropping relation (R2).

The second approach allows us to show that for any completely regular frame L, the completion of $\mathrm{C}^*(L)$ is isomorphic to some $\mathrm{C}^*(M)$, namely $\mathrm{C}^*(\mathfrak{B}(L))$, where $\mathfrak{B}(L)$ denotes the Booleanization of L. In the general non-bounded case, the Gleason cover $\mathfrak{G}(L)$ of L takes the role of the Booleanization but an assumption on the frame L is required, namely, the *weak cb property*.

References

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