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ON SOME HOMOMORPHISMS OF DIRECT SUMS OF MODULES

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Abstract. This work gives a series of descriptions of homomorphisms of direct sums of R-modules, which will be used in the second part of the paper in order to characterize some direct sums of R-modules with the direct summand intersection property (for short D.S.I.P.).

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0. Short history and introduction

The history of the modules with the direct summand intersection property begins with Kaplansky, which in [12, ex. 51, p. 49] asks to be proved that if A is a free module over a principal ideal ring, then the intersection of any finite number of direct summands of A is again a direct summand of A. This fact suggested to Fuchs in [9] to ask, in his turn, for the solution of the same exercise for free abelian groups (ex. 14.4), countable free abelian groups or free abelian groups of the power of the continuum (ex. 19.5). Moreover, Fuchs proposes the following open problem (problem 9): "Characterize the groups in which the intersection of two direct summands is again a direct summand". The groups (or the modules) for which the intersection of two direct summands is a direct summand as well will be called groups (modules) with the direct summand intersection property, for short D.S.I.P. We will also say that the abelian group (or the R-module) A has the strong direct summand intersection property (in short S.D.S.I.P.) if the intersection of any number of direct summands of A is a direct summand too. Obviously, if a group (an R-module) has S.D.S.I.P., it also has D.S.I.P. and the converse is generally false (see [22, p. 32]).

Solutions to this Fuchs's problem can be found in [1], [2], [4], [10], [11], [16], [17], [18], [19] and in [22].

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More recently (February 1997) Fuchs proposes the following problem: "Find necessary and/or sufficient conditions over an R-module A (specifically an abelian group) so that, being given any index set I (finite or infinite), $\oplus_I A$ to have D.S.I.P.". Specific solutions to this new Fuchs's problem can be found in [10] and in [19].

In this work we will consider: R an associative, commutative ring with unity, $\{A_j\}_{j\in J}$ and $\{B_i\}_{i\in I}$ two families of R-modules and $f : \oplus_{j\in J}A_j \to \oplus_{i\in I}B_i$ any homomorphism of R-modules, where I and J are any two index sets. We will describe the images and the kernels of such homomorphisms, as well as necessary and sufficient conditions for which [$\oplus_{j\in J}A_j$ has D.S.I.P. if and only if End($\oplus_{j\in J}A_j$)-the endomorphism ring of R-module $\oplus_{j\in J}A_j$ is regular]. We will use the following terms:

– End(A), for the endomorphism ring of R-module A,
– $E(A)$, for the injective hull (envelope) of R-module A,
– $\oplus_I A$, for the direct sum (indexed by the set I) of copies of A.

The paper is structured in two section: the first one refers exclusively, to homomorphisms of direct sums of R-modules. A series of elementary results referring to such homomorphisms are presented and the kernels and their images are described. The second section contains descriptions of the images and the kernels of homomorphisms of direct sums of R-modules with D.S.I.P., as well as a few classes of families of R-modules $\{A_j\}_{j\in J}$ for which: $\oplus_{j\in J}A_j$ having D.S.I.P. involves End($\oplus_{j\in J}A_j$)-the endomorphism ring of $\oplus_{j\in J}A_j$ is regular (the converse of this implication being always true, according to [19, 2.1, 2.5]).

All through this paper we will mark with R an associative, commutative ring with unity, the considered modules will be over R and by group we mean abelian group in additive notation. Other conditions about the ring R or R-modules will be imposed whenever the case.

1. Homomorphisms of direct sums of modules

Let R be an associative, commutative ring with unity, $\{A_j\}_{j\in J}$ and $\{B_i\}_{i\in I}$ two families of R-modules. We consider $f : \oplus_{j\in J}A_j \to \oplus_{i\in I}B_i$ an R-homomorphism of R-modules. Then f induces a two times indexed family $\{f_j^i : A_j \to B_i\}_{(i,j)\in I\times J}$, where $f_j^i = p^i \circ f \circ u_j$, $u_j : A_j \to \oplus_{j\in J}A_j$ is the j^{th} canonical injection, and $p^i : \oplus_{i\in I}B_i \to B_i$ is the i^{th} canonical projection, such that the following diagram is commutative:

$$\begin{array}{ccc} \oplus_{j\in J}A_j & \xrightarrow{f} & \oplus_{i\in I}B_i \\ {\scriptstyle u_j}\uparrow & & \downarrow{\scriptstyle p^i} \\ A_j & \xrightarrow{f_j^i} & B_i \end{array} \qquad (1)$$

We define $f_j : A_j \to \oplus_{i\in I}B_i$, $f_j = f \circ u_j$ and $f^i : \oplus_{j\in I}A_j \to B_i$, $f^i = p^i \circ f$.

It can be easily proved that for every $j \in J$ and for every $a_j \in A_j$, the set $\{f_j^i(a_j) \neq 0\}_{i\in I}$ is finite, that is for every $j \in J$, the family $\{f_j^i\}_{i\in I}$ is almost everywhere trivial.

If f, $g : \oplus_{j\in J}A_j \to \oplus_{i\in I}B_i$ are two homomorphisms of R-modules which induce the two times indexed families $\{f_j^i : A_j \to B_i\}_{(i,j)\in I\times J}$ and $\{g_j^i : A_j \to B_i\}_{(i,j)\in I\times J}$ of homomorphisms of R-modules such that for every $j \in J$ and for every $i \in I$, $f_j^i = g_j^i$, then $f = g$.

Conversely, let $\{f_j^i : A_j \to B_i\}_{(i,j)\in I\times J}$ be a two times indexed family of homomorphisms of R-modules, with property that for every $j \in J$, the family $\{f_j^i\}_{i\in I}$ is almost everywhere trivial. For every $a_j \in A_j$, we mark with $S(a_j) = \{i \in I | f_j^i(a_j) \neq 0\}$. From hypothesis it follows that for every $j \in J$ and for every $a_j \in A_j$, the set $S(a_j)$ is finite. Now, for every $j \in J$, we define $f_j : A_j \to \oplus_{i\in I}B_i$ by: if $a_j \in A_j$, $f_j(a_j) = \sum\limits_{i\in S(a_j)} f_j^i(a_j) = \sum\limits_{i\in I} f_j^i(a_j)$. It can be easily proved that f_j is a homomorphism of R-modules. Now, we can define $f : \oplus_{j\in J}A_j \to \oplus_{i\in I}B_i$, by: for every $a_{j_1} + a_{j_2} + \cdots + a_{j_n} \in A_{j_1} \oplus A_{j_2} \oplus \cdots \oplus A_{j_n}$,

$$f(a_{j_1} + a_{j_2} + \cdots + a_{j_n}) = f_{j_1}(a_{j_1}) + f_{j_2}(a_{j_2}) + \cdots + f_{j_n}(a_{j_n}) \in \oplus_{i\in I}B_i.$$

It is certain that the map defined in such a way is a homomorphism of R-modules.

Remark 1.1. If for every $j \in J$ and for every $i \in I$, f_j^i, f_j and f are as above, then the two triangles and the rectangle below are commutative:

$$\begin{array}{ccc} A_j & \xrightarrow{f_j^i} & B_i \\ {\scriptstyle u_j}\downarrow & \searrow^{f_j} & \uparrow{\scriptstyle p^i} \\ \oplus_{j\in J}A_j & \xrightarrow{f} & \oplus_{i\in I}B_i \end{array} \tag{2}$$

We can also define the following homomorphism of R-modules: $f^i : \oplus_{j\in J}A_j \to B_i$ by: if $a = a_{j_1} + a_{j_2} + \cdots + a_{j_n} \in \oplus_{j\in J}A_j$, then $f^i(a) = f_{j_1}^i(a_{j_1}) + f_{j_2}^i(a_{j_2}) + \cdots + f_{j_n}^i(a_{j_n})$.

Remark 1.2. If for every $j \in J$ and for every $i \in I$, f_j^i, f^i, f are as above and $\overline{f} : \oplus_{j\in J}A_j \to \oplus_{i\in I}B_i$ is defined by: for every $a \in \oplus_{j\in J}A_j$, $\overline{f}(a) = \sum\limits_{i\in I} f^i(a)$, then $\overline{f} = f$ and the two triangles and the rectangle below are commutative:

$$\begin{array}{ccc} A_j & \xrightarrow{f_j^i} & B_i \\ {\scriptstyle u_j}\downarrow & \nearrow^{f^i} & \uparrow{\scriptstyle p^i} \\ \oplus_{j\in J}A_j & \xrightarrow{\overline{f}=f} & \oplus_{i\in I}B_i \end{array} \tag{3}$$

We notice that any system which is two times indexed by homomorphisms of R-modules $\{f_j^i : A_j \to B_i\}_{(i,j)\in I\times J}$ can be represented as a matrice of order $I \times J$.

If we mark with $M^*_{I\times J}$ the set of all matrices of order $I\times J$, whose entries are homomorphisms of R-modules $f^i_j : A_j \to B_i$ with property that for every $j \in J$, the family $\{f^i_j\}_{i\in I}$ is almost everywhere trivial and we mark with $M(f) = M(f^i_j) \in M^*_{I\times J}$-the matrice determined by (or which determines) the map f, then we have the following result, now very easy to prove:

Theorem 1.3. *There is an isomorphism of R-modules $\Psi : \mathrm{Hom}(\oplus_{j\in J}A_j, \oplus_{i\in I}B_i) \to M^*_{I\times J}$, where for every $f \in \mathrm{Hom}(\oplus_{j\in J}A_j, \oplus_{i\in I}B_i)$, $\Psi(f) = M(f) \in M^*_{I\times J}$.*

Corollary 1.4. *If $\{A_i\}_{i\in I}$ is a family of R-modules, then the endomorphism algebra of the R-module $\oplus_{i\in I}A_i$ (regarded to the addition and the usual composition of functions as well as their multiplication by a scalar from R) is isomorphic to the algebra of all matrices of order $I \times I$, M^*_I, and whose entries are homomorphisms of R-modules $f^i_j : A_j \to A_i$, with property that for every $j \in J$, the family $\{f^i_j\}_{i\in I}$ is almost everywhere trivial, an algebra evaluated considering the usual addition and multiplication of matrices as well as their multiplication by a scalar from R.*

Proof. Let be f, $g \in \mathrm{End}(\oplus_{i\in I}A_i)$ and let Ψ be the map from Theorem 1.3. If the family (of homomorphisms) $\{f^i_j : A_j \to A_i\}_{(i,j)\in I\times I}$ is determined by f and the family $\{g^k_i : A_i \to A_k\}_{(k,i)\in I\times I}$ is determined by g, then $g \circ f$ determines the family $\big\{ \sum\limits_{i\in I} g^k_i \circ f^i_j :\ A_j \to A_k\big\}_{(k,j)\in I\times I}$. If for every $j \in I$, the family $\{f^i_j\}_{i\in I}$ is almost everywhere trivial and for every $i \in I$, the family $\{g^k_i\}_{k\in I}$ is almost everywhere trivial, it can be easily proved that for every $j \in I$, the family $\big\{ \sum\limits_{i\in I} g^k_i \circ f^i_j\big\}_{(k,j)\in I\times I}$ is almost everywhere trivial too. Since $M(g \circ f) = M\big(\sum\limits_{i\in I} g^k_i \circ f^i_j\big) = M(g^k_i)M(f^i_j) = M(g)M(f)$, it follows that Ψ is an isomorphism of algebras. $\square$

Corollary 1.5. *If A is an R-module, then the endomorphism ring of R-module $\oplus_I A$ is isomorphic to the ring of all matrices of order $I \times I$, M^*_I, whose entries are endomorphisms of A and whose columns have a finite number of non-null entries, a ring evaluated considering the usual addition and multiplication of matrices.*

Further, for every homomorphism $f : \oplus_{j\in J}A_j \to \oplus_{i\in I}B_i$, we will present a list with some elementary results concerning $\ker f$ and $\mathrm{Im}\, f$, results which will be used in the second part of the paper in order to characterize some direct sums of R-modules with D.S.I.P.. The proof of these results being elementary (we use only the definitions of f, f_j and f^i as well as (1.1)–(1.2)), we will omit it.

Proposition 1.6. *Let R be an associative, commutative ring with unity, $\{A_j\}_{j\in J}$ and $\{B_i\}_{i\in I}$ two families of R-modules. If $f : \oplus_{j\in J}A_j \to \oplus_{i\in I}B_i$ is a homomorphism of R-modules, then the following relationships occur:*

(1) $\ker f = \bigcap_{i \in I} \ker f^i,$ (4)

(2) $\operatorname{Im} f = \oplus_{i \in I} \operatorname{Im} f^i = \sum_{j \in J} \operatorname{Im} f^i_j,$ (5)

(3) $\oplus_{j \in J} \ker f_j \leq \ker f,$ (6)

(4) *for every* $j \in J$, $\ker f_j = \bigcap_{i \in I} \ker f^i_j,$ (7)

and

$$\operatorname{Im} f_j = \oplus_{i \in I} \operatorname{Im} f^i_j, \tag{8}$$

(5) *for every* $i \in I$, $\oplus_{j \in J} \ker f^i_j \leq \ker f^i,$ (9)

and

$$\operatorname{Im} f^i = \sum_{j \in J} \operatorname{Im} f^i_j. \tag{10}$$

The immediate remarks impose themselves here.

Remarks 1.7.

(a) Generally the equality does not occur in relationship (6).

(b) Generally the equality does not occur in relationship (9).

(c) In (9) the equality takes place if and only if in (10) the sum is direct. Therefore, for every $i \in I$:

$$\ker f^i = \oplus_{j \in J} \ker f^i_j \tag{11}$$

if and only if

$$\operatorname{Im} f^i = \oplus_{j \in J} \operatorname{Im} f^i_j. \tag{12}$$

(d) In (10) the sum is direct if and only if the following equality occurs:

$$\ker f = \bigcap_{i \in I} \big(\oplus_{j \in J} \ker f^i_j \big). \tag{13}$$

(e) If the index set J is finite and $\oplus_{j \in J} A_j$ has the lattice of its all submodules distributive, then in (7) the equality takes place. In this case also the following equality occurs:

$$\ker f = \oplus_{j \in J} \Big(\bigcap_{i \in I} \ker f^i_j \Big). \tag{14}$$

(f) If the index set J is finite, in (10) the sum is direct and $\oplus_{j \in J} A_j$ has the lattice of its all submodules distributive, then the following equality occurs:

$$\bigcap_{i \in I} \big(\oplus_{j \in J} \ker f^i_j \big) = \oplus_{j \in J} \Big(\bigcap_{i \in I} \ker f^i_j \Big). \tag{15}$$

At the end of this section we will present a case in which in relationship (6) the equality takes place. It is known that if the R-module A has the property that $\operatorname{End}(A)$ is a regular ring, then for every B, direct summand of A, the endomorphism ring of B is regular too, but the conversely is generally false, that is if $A = \oplus_{k \in K} A_k$ and for every $k \in K$, $E_k = \operatorname{End}(A_k)$ is a regular ring, it does not result that $E = \operatorname{End}(A)$ is a regular ring too (see [20, Lemma 3.3, Example 3.4]. The following result proves that this is true if $\{A_k\}_{k\in K}$ is a family of fully invariant direct summands of A (in which case in relationship (6) the equality takes place).

Proposition 1.8. *Let $A = \oplus_{k\in K} A_k$ be an R-module such that:*
(1) *for every $k \in K$, $End(A_k)$ is a regular ring,*
(2) *for every $k \in K$, A_k is fully invariant in A.*
Then the following statements occur:
(a) *in the relationship* (6) *the equality takes place,*
(b) $\mathrm{End}(A)$ *is a regular ring,*
(c) $\mathrm{End}(A)$ *is a direct product of regular rings.*

Proof. (a) Let f be an endomorphism of A. According to the hypothesis, it follows that for every i, $j \in K$, $i \neq j$, $\mathrm{Hom}(A_i, A_j) = \mathrm{Hom}(A_j, A_i) = 0$. So for every i, $j \in K$, $i \neq j$ and for every homomorphism $f_j^i : A_j \to A_i$, $\ker f_j^i = A_j$ and $\mathrm{Im}\, f_j^i = 0$. If $f^i : A \to A_i$, then $\mathrm{Im}\, f^i = \mathrm{Im}\, f_i^i$. Since $\mathrm{End}(A_i)$ is a regular ring, it follows that $\mathrm{Im}\, f_i^i$ is a direct summand in A_i. In this case, from (1.6) it follows that $\mathrm{Im}\, f$ is a direct summand in A. On the other side, $\ker f_j = \ker f_j^j$ is a direct summand in A_j. If $a = a_{j_1} + a_{j_2} + \cdots + a_{j_n} \in \ker f$, where for every $k = \overline{1,n}$, $a_{j_k} \in A_{j_k}$, then $f(a_{j_1} + a_{j_2} + \cdots + a_{j_n}) = f_{j_1}(a_{j_1}) + f_{j_2}(a_{j_2}) + \cdots + f_{j_n}(a_{j_n}) = f_{j_1}^{j_1}(a_{j_1}) + f_{j_2}^{j_2}(a_{j_2}) + \cdots + f_{j_n}^{j_n}(a_{j_n}) = 0$, that is for every $k = \overline{1,n}$, $a_{j_k} \in \ker f_{j_k}^{j_k}$. Therefore in relationship (6) the equality takes place.

(b) The statement follows from what has been proved at point (a) and [20, Corollary 3.2].

(c) The statement follows from [17, 2.3]. □

2. Applications to R-modules with D.S.I.P.

In this section, we will start the presentation of the results with the proof of the following results:

Proposition 2.1. *Let $\{A_k\}_{k\in K}$ be a family of R-modules with property that $A = \oplus_{k\in K} A_k$ has S.D.S.I.P. Then for any I and J two subsets of K with $I \cap J = \varnothing$ and for every homomorphism of R-modules $f : \oplus_{j\in J} A_j \to \oplus_{i\in I} A_i$, the following equality occurs:*

$$\ker f = D \oplus \big(\oplus_{j\in J} \ker f_j \big), \tag{16}$$

where, for every $j \in J$, $f_j : A_j \to \oplus_{i\in I} A_i$ has the significance from section 1, and D is a direct summand in A.

Proof. We suppose that $\{A_k\}_{k\in K}$, $A = \oplus_{k\in K} A_k$, I, J and f are the same as in the statement. Then A has D.S.I.P. and according to [10, 1.4] $\ker f$ is a direct summand in A. From [10, 1.2] it follows that for every $i \in I$ and for every $j \in J$, $A_j \oplus A_i$ has D.S.I.P. and so $\ker f_j^i$ is a direct summand in A_j. Since A_j, $j \in J$, has S.D.S.I.P. too, it follows that $\ker f_j = \bigcap\limits_{i\in I} \ker f_j^i$

is a direct summand in A_j. Then $\oplus_{j\in J}\ker f_j$ is a direct summand in A. It follows that $\ker f = D \oplus (\oplus_{j\in J}\ker f_j)$, where D is a direct summand in A. $\square$

The following remarks improse themselves here too:

Remarks 2.2.

(a) Since $D \cap A_j \leq \ker f \cap A_j = \ker f_j$, it follows that $D \cap A_j = 0$, for every $j \in J$.

(b) If $a = a_{j_1} + a_{j_2} + \cdots + a_{j_n} \in D$, with $n \in \mathbf{N}^*$ and for every $k = \overline{1,n}$, $a_{j_k} \in A_{j_k}$, then for every $j_l \in \{1,\ldots,n\}$ there is $j_m \in \{1,\ldots,n\}$, $j_l \neq j_m$, with property that $\mathrm{Ann}(a_{j_l}) = \mathrm{Ann}(a_{j_m})$; particularly, if there is a $j_l \in \{1,\ldots,n\}$ such that a_{j_l} is torsion-free, then there is $j_m \in \{1,\ldots,n\}$, $j_l \neq j_m$, with property that a_{j_m} is torsion-free too ($\mathrm{Ann}(a)$ is the annihilator of a).

(c) There are R-modules for which the summand D from the relationship (16) is 0; for example: let R be a principal ideal domain, $\{p_k\}_{k\in K}$ a family of unassociated prime elements from R, and let $\{A_k\}_{k\in K}$ be a family of elementary $p_k - R$-modules. Then, from [22, Proposition 9, Lemma 5] and (2.1), it follows that for every I and J two subsets of K with $I \cap J = \varnothing$ and for any homomorphism of R-modules $f : \oplus_{j\in J}A_j \to \oplus_{i\in I}A_i$, the following equality occurs:

$$\ker f = \oplus_{j\in J}\ker f_j. \tag{17}$$

Corollary 2.3. *Let $\{A_k\}_{k\in K}$ be a family of R-modules with property that $A = \oplus_{k\in K}A_k$ has the regular endomorphism ring. Then for any I and J two finite subsets of K with $I \cap J = \varnothing$ and for every homomorphism of R-modules $f : \oplus_{j\in J}A_j \to \oplus_{i\in I}A_i$, the equality* (16) *occurs.*

Proof. If $\mathrm{End}(A)$ is a regular ring, then $\oplus_{k\in I\cup J}A_k$ has the regular endomorphism ring too, according to [20, Lemma 3.3]. From [20, Corollary 3.2], it follows that for every $\varphi \in \mathrm{End}(\oplus_{k\in I\cup J}A_k)$, $\ker\varphi$ is a direct summand in $\oplus_{k\in I\cup J}A_k$. According to [19, 2.1], it follows that $\oplus_{k\in I\cup J}A_k$ has D.S.I.P. Then, for every homomorphism of R-modules $f : \oplus_{j\in J}A_j \to \oplus_{i\in I}A_i$, $\ker f$ is a direct summand in $\oplus_{j\in J}A_j$. Since $I \cap J = \varnothing$, it follows that for every $i \in I$ and for every $j \in J$, $A_j \oplus A_i$ has D.S.I.P. and so $\ker f_j^i$ is a direct summand in A_j. Since A_j, $j \in J$, has D.S.I.P. too, it follows that $\ker f_j = \bigcap_{i\in I}\ker f_j^i$ is a direct summand in A_j. Further on we reason as in (2.1). $\square$

In [19] we have proved that the set of R-modules A with property that $\mathrm{End}(A)$ is a regular ring is strictly contained in the set of R-modules with D.S.I.P. Further on we will present some cases in which D.S.I.P. for an R-module A involves $\mathrm{End}(A)$-regular ring. We will begin by describing the kernels and the images of homomorphisms, as well as of endomorphisms of semi-simple R-modules which have S.D.S.I.P. and whose endomorphism rings are regular.

Theorem 2.4. *If $\{A_k\}_{k\in K}$ is a family of simple R-modules and $A = \oplus_{k\in K}A_k$, then for any I and J two subsets of K and for every homomorphism of R-modules $f : \oplus_{j\in J}A_j \to \oplus_{i\in I}A_i$, there is a subset L of K such that* $\ker f = \oplus_{k\in L}A_k$, *and* $\operatorname{Im} f \cong \oplus_{k\in J\setminus L}A_k$.

Proof. Let $f : \oplus_{j\in J}A_j \to \oplus_{i\in I}A_i$ be a homomorphism of R-modules and $\{f_j^i : A_j \to A_i\}_{(i,j)\in I\times J}$ the two times indexed family of homomorphisms of R-modules determined by f, with property that for every $j \in J$, the family $\{f_j^i\}_{i\in I}$ is almost everywhere trivial (see the section 1). Since $\ker f_j^i$ is a submodule of A_j, according to the hypothesis, then either $\ker f_j^i = 0$ or $\ker f_j^i = A_j$. If $f_j : A_j \to \oplus_{i\in I}A_i$, $f_j = f \circ u_j$ has the significance from Section 1, then, according to the relationship (4), either $\ker f_j = 0$ or $\ker f_j = A_j$. Therefore $\ker f_j$ is a trivial direct summand in A_j, in which case there is a subset L of J such that $\oplus_{j\in J}\ker f_j = \oplus_{k\in L}A_k$, which is such a direct summand in $\oplus_{j\in J}A_j$. Since $\oplus_{j\in J}A_j$ is a semi-simple R-module, $\ker f$ is a direct summand in $\oplus_{j\in J}A_j$, which contains $\oplus_{j\in J}\ker f_j$. Hence, $\ker f = B \oplus (\oplus_{j\in J}\ker f_j)$, where B is a direct summand of $\oplus_{j\in J}A_j$. If $B \neq 0$, then, for every $j \in J$, $B \cap A_j$ is either 0 or A_j, which is impossible (see [3, 9.4.] or [15, Lemma 3.14, Corollary]) It follows that $\ker f = \oplus_{l\in L}A_l$ and $\operatorname{Im} f \cong \oplus_{k\in J\setminus L}A_k$. □

Corollary 2.5. *If $\{A_k\}_{k\in K}$ is a family of simple R-modules and $A = \oplus_{k\in K}A_k$, then for every $f \in \operatorname{End}(A)$, there is a subset L of K such that* $\ker f = \oplus_{k\in L}A_k$ *and* $\operatorname{Im} f \cong \oplus_{k\in K\setminus L}A_k$.

As a particular case of (2.5) we have:

Corollary 2.6. *If A is a simple R-module, then for any index set I and for every $f \in \operatorname{End}(\oplus_I A)$ there is a subset L of I such that* $\ker f = \oplus_L A$ *and* $\operatorname{Im} f \cong \oplus_{I\setminus L}A$.

These results can be generalized as it follows:

Theorem 2.7. *Let $\{A_k\}_{k\in K}$ be a family of indecomposable R-modules, with property that $A = \oplus_{k\in K}A_k$ has D.S.I.P. and for every $k \in K$,* $\operatorname{End}(A_k)$ *is a regular ring. Then the following statements occur:*

(a) *for every $f \in \operatorname{End}(A)$, there are L and L' two subsets of K such that $\oplus_{k\in L}A_k \leq \ker f$ and $\oplus_{k\in L'}A_k \leq \operatorname{Im} f$,*

(b) *for every $i, j \in K$, $i \neq j$, either*

(i) $\operatorname{Hom}(A_i, A_j) = 0$ *or*

(ii) *A_i is isomorphic to a submodule of A_j.*

Proof. Let f be an endomorphism of A. If $\{f_j^i : A_j \to A_i\}_{(i,j)\in K\times K}$ is the two times indexed family of homomorphisms determined by f, then for every i, $j \in K$, $\ker f_j^i$ is a direct summand in A_j. According to the hypothesis, it follows that for every i, $j \in K$, either $\ker f_j^i = 0$, in which case $A_j \cong \operatorname{Im} f_j^i \leq A_i$, or $\ker f_j^i = A_j$, in which case $\operatorname{Im} f_j^i = 0$. Analogously it is shown that either $A_i \cong \operatorname{Im} f_j^i \leq A_j$ or $\operatorname{Im} f_j^i = 0$. If $j \in K$ and

$f_j : A_j \to \oplus_{k\in K} A_k$, $f_j = f \circ u_j$ has the same meaning as in the first section, then, according to the relationship (4), either $\ker f_j = 0$ or $\ker f_j = A_j$ and there is $L \subseteq K$ such that $\oplus_{k\in K} \ker f_k = \oplus_{k\in L} A_k \leq \ker f$ (see the relationship (6)). On the other side, from hypothesis and relationships (10) and (5) it follows that, if $i \in K$ and $f^i : A \to A_i$ has the same meaning as in first section, then there is $L' \subseteq K$ such that $\oplus_{k\in L'} \operatorname{Im} f^k = \oplus_{k\in L'} A_k \leq \operatorname{Im} f$. Now, we show that for every $i, j \in K$, $i \neq j$, [$\operatorname{Hom}(A_i, A_j) = 0$ if and only if $\operatorname{Hom}(A_j, A_i) = 0$]. We suppose that $\operatorname{Hom}(A_i, A_j) = 0$ and there is $0 \neq g : A_j \to A_i$ a homomorphism. Then g is monomorphism and since $End(A_i \oplus A_j)$ is regular, there is a homomorphism $h : A_i \to A_j$ such that $h \circ g = 1_{A_j}$, which is a contradiction with the presupposition made. It follows that the statement from point (b) of theorem occurs. □

Corollary 2.8. *Let $\{A_k\}_{k\in K}$ be a family of indecomposable R-modules and $A = \oplus_{k\in K} A_k$, with property that $A \oplus A$ has D.S.I.P. If for every $k \in K$, $\operatorname{End}(A_k)$ is a local ring, then $\operatorname{End}(A)$ is a regular ring.*

Proof. If $A \oplus A$ has D.S.I.P., then the kernel of each endomorphism of A is a direct summand in A. If $f \in \operatorname{End}(A)$, then, according to (2.7) (a), $\ker f = B \oplus (\oplus_{k\in L} A_k)$, where B is a direct summand in A and L is a subset of K. Then for every $k \in K \setminus L$, $B \cap A_k$ is a direct summand in A_k. Since for every $k \in K \setminus L$, $B \cap A_k \neq A_k$, it follows that for every $k \in K \setminus L$, $B \cap A_k = 0$. Now, from hypothesis and [3, 12.6, 12.3] it follows that $B = 0$. Hence $\ker f = \oplus_{k\in L} A_k$ and $\operatorname{Im} f = \oplus_{k\in K\setminus L} A_k$. □

Corollary 2.9. *If A is an indecomposable R-module such that for any index set I, $A^* = \oplus_I A$ has D.S.I.P., and $\operatorname{End}(A)$ is a local ring, then $\operatorname{End}(A^*)$ is a regular ring.*

Definition. A family $\{A_k\}_{k\in K}$ of R-modules is said to be rigid system (of R-modules) if for every $i, j \in K$, $i \neq j$, $\operatorname{Hom}(A_i, A_j) = 0$.

Definition. Let A and B be two torsion-free R-modules. A quasi-homomorphism from A to B is a homomorphism $\varphi : E(A) \to E(B)$ with property that there is a $r \in R$ such that $r\varphi E(A) \subseteq E(B)$. A quasi-homomorphism φ is a quasi-isomorphism if it is an isomorphism of $E(A)$ with $E(B)$ and φ^{-1} is a quasi-homomorphism.

Now from (2.7.) and [9, §92] we obtain:

Corollary 2.10. *If $A = \oplus_{k\in K} A_k$ is an R-module with D.S.I.P. and for every $k \in K$, A_k is indecomposable, torsion-free and of finite rank, then $\{A_k\}_{k\in K}$ contains a rigid subsystem or pairwise quasi-isomorphic R-modules. Particularly, if for every $k \in K$, A_k is indecomposable, torsion-free and of rank one, then $\{A_k\}_{k\in K}$ contains a rigid subsystem or pairwise isomorphic R-modules.*

For injective modules we obtain the following result:

Proposition 2.11. *Let $\{A_k\}_{k\in K}$ be a family of indecomposable injective R-modules, with property that $A = \oplus_{k\in K} A_k$ has D.S.I.P. Then the following statements occur:*

(a) *for every f-endomorphism of A,* $\ker f$ *and* $\operatorname{Im} f$ *are direct summands in A, so* $\operatorname{End}(A)$ *is a regular ring.*

(b) $\operatorname{End}(A)$ *is a direct product of regular rings.*

Proof. If $\{A_k\}_{k\in K}$ and A are the same as in the statement, then, according to [19, 3.1], for every i, $j \in I$, either $\operatorname{Hom}(A_j, A_i) = 0$ or $A_j \cong A_i$. So, reasoning analogously with (2.7), we obtain that there is a subset L of K such that $\operatorname{Im} f = \oplus_{k\in L} A_k$ and so $\operatorname{Im} f$ and $\ker f$ are direct summands in A and $\operatorname{End}(A)$ is a regular ring. According to [19, 4.1], over K we can define the following equivalence relationship, marked by "$\approx$": for every $k_1, k_2 \in K$, $k_1 \approx k_2$ if and only if $A_{k_1} \cong A_{k_2}$. If for $k \in K$, we mark with $k^* = \{i \in K | A_i \cong A_k\}$, then $\{k^*\}_{k\in K}$ is the partition corresponding to "$\approx$" over K. It follows that $A = \oplus_{k^*\in K/\approx} A_{k^*}$, where $A_{k^*} = \oplus_{i\in k^*} A_i$ and as we have already stated in [19, 4.1] for every $k_1, k_2 \in K$ which are not equivalent, $A_{k_1^*}$ and $A_{k_2^*}$ are fully invariant. Then, according to (1.8), $\operatorname{End}(A) = \prod\limits_{k^*\in K/\approx} \operatorname{End}(A_{k^*})$, and from [15, Proposition 3.12] and (2.9) it follows that for every $k^* \in K/\approx$, the ring $\operatorname{End}(A_{k^*})$ is regular. □

We obtain the results from (2.11) in the following case too:

Theorem 2.12. *Let $A = \oplus_{k\in K} A_k$ be an R-module such that:*

(1) *for every $k \in K$,* $\operatorname{End}(A_k)$ *is a principal ideal domain,*

(2) *for every i, $j \in K$, either* $\operatorname{Hom}(A_i, A_j) = 0$ *or* $A_i \cong A_j$.

Then the following statements occur:

(a) $\operatorname{End}(A)$ *is a regular ring.*

(b) $\operatorname{End}(A)$ *is a direct product of regular rings.*

(c) *for every $f \in \operatorname{End}(A)$, there are L and L' two subsets of K such that* $\oplus_{k\in L} A_k = \ker f$ *and* $\oplus_{k\in L'} A_k = \operatorname{Im} f$.

Proof. If $\operatorname{End}(A_k)$ is a principal ideal domain, then A_k is indecomposable, according to [10, Lemma 3.1]. Considering over K the equivalent relationship "$\approx$", stated in the proof of (2.11) and $\{k^*\}_{k\in K}$-the partition corresponding to "$\approx$" over K, we obtain again that $A = \oplus_{k^*\in K/\approx} A_{k^*}$, where $A_{k^*} = \oplus_{i\in k^*} A_i$ and for every k_1 and $k_2 \in K$ which are not equivalent, $A_{k_1^*}$ and $A_{k_2^*}$ are fully invariant. From [10, Proposition 3.2] it follows that $A_{k^*} = \oplus_{i\in k^*} A_i$ has D.S.I.P. and the kernel of each endomorphism of A_{k^*} is a direct sum of copies of A_i, $i \in k^*$. So, for every $k^* \in K/\approx$, the ring $\operatorname{End}(A_{k^*})$ is regular. Now we can apply (1.8). Since A has D.S.I.P., according to [19, 4.1], from (2.7) and from the above proved facts, it follows that for every $f \in \operatorname{End}(A)$, $\ker f = D \oplus (\oplus_{k^*\in K/\approx} \ker f_{k^*}^{k^*})$, where $f_{k^*}^{k^*} \in \operatorname{End}(A_{k^*})$, and D is a direct summand in A. If $a = a_{k_1^*} + a_{k_2^*} + \cdots + a_{k_n^*} \in D$, where for every $i = \overline{1,n}$, $n \in \mathbf{N}^*$, $a_{k_i^*} \in A_{k_i^*}$, then $0 = f(a) = f_{k_1^*}(a_{k_1^*}) + \cdots + f_{k_n^*}(a_{k_n^*})$.

It follows that for every $i = \overline{1,n}$, $a_{k_i^*} \in \ker f_{k_i^*} = \ker f_{k_i^*}^{k_i^*}$ – which is impossible. Therefore the statement from point (b) of the theorem occurs. $\square$

The result form (2.12) can be improved:

Theorem 2.13. *Let $A = \oplus_{k\in K} A_k$ be an R-module such that:*

(1) *for every $k \in K$, $E_k = \mathrm{End}(A_k)$ is a hereditary ring,*

(2) *for every $k \in K$, A_k is indecomposable, E_k-flat (left) and self small,*

(3) *for every $i, j \in K$, either $\mathrm{Hom}(A_i, A_j) = 0$ or $A_i \cong A_j$.*

Then, for any index set I, the following statements occur:

(a) *the R-module $\oplus_I A$ has D.S.I.P.,*

(b) $\mathrm{End}\,(\oplus_I A)$ *is a regular ring,*

(c) $\mathrm{End}\,(\oplus_I A)$ *is a direct product of regular rings.*

(d) *for every $f \in \mathrm{End}\,(\oplus_I A)$, there are L and L' two subsets of K such that $\oplus_{k\in L}(\oplus_I A_k) = \ker f$ and $\oplus_{k\in L'}(\oplus_I A_k) = \mathrm{Im}\, f$.*

Proof. If for every $k \in K$, A_k is indecomposable, E_k-flat (left) and self small and $E_k = \mathrm{End}(A_k)$ is a hereditary ring, according to [10, Theorem 2.1], for any index set I, $\oplus_I A_k$ has D.S.I.P. On the other side, $A = \oplus_{k^*\in K/\approx} A_{k^*}$, where $A_{k^*} = \oplus_{i\in k^*} A_i$ and for every k_1 and $k_2 \in K$ which are not equivalent, $A_{k_1^*}$ and $A_{k_2^*}$ are fully invariant. Again from [10, Theorem 2.1] and (1.6), it follows that $A_{k^*} = \oplus_{i\in k^*} A_i$ has D.S.I.P., the kernel of each endomorphism of A_{k^*} is a direct sumand of A_{k^*}, and $\mathrm{End}(A_{k^*})$ is a regular ring. It follows that for any index set I, $\oplus_I A = \oplus_I(\oplus_{k^*\in K/\approx} A_{k^*}) = \oplus_{k^*\in K/\approx}(\oplus_I A_{k^*})$, where for every $k^* \in K/\approx$, $\oplus_I A_{k^*}$ is a fully invariant submodule (of $\oplus_I A$) with D.S.I.P. and $\mathrm{End}(\oplus_I A_{k^*})$ is regular. Now we can do as in the proof of (2.12), replacing A_{k^*} with $\oplus_I A_{k^*}$. $\square$

References

1. U. Albrecht and J. Hausen, Mixed Abelian groups with the summand intersection property. *In: Lectures Notes in Pure and Appl. Math.* **182**, 123–132, *Marcel Dekker, New-York*, 1996.

2. U. Albrecht and J. Hausen, Modules with the quasi-summand intersection property. *Bull. Austral. Math. Soc.* **44**(1991), 189–201.

3. F. W. Anderson and K. R. Fuller, Rings and categories of modules. *Springer-Verlag, Berlin–Heidelberg–New York*, 1974.

4. D. M. Arnold and J. Hausen, A characterization of modules with the summand intersection property. *Comm. Algebra* **18(2)**(1990), 519–528.

5. J. D. Botha and P. J. Gräbe, On torsion-free Abelian groups whose endomorphism rings are principal ideal domains. *Comm. Algebra* **11, 12**(1983), 1343–1354.

6. G. Călugăreanu, Introduction to Abelian groups theory. (Romanian) *Editura Expert, Cluj-Napoca*, 1994.

7. C. Faith, Lectures on injective modules and quotient rings. *Lectures Notes in Math.* **49**, *Springer-Verlag, Berlin–Heidelberg–New York,* 1967.

8. C. Faith, Algebra II, Ring Theory. *Springer-Verlag, Berlin–Heidelberg–New York*, 1976.

9. L. Fuchs, Infinite Abelian Groups, **1**–**2**. *Academic Press, New York and London*, **36**, *Lecture Notes in Pure and Applied Mathematics*, 1970–1973,

10. J. Hausen, Modules with the summand intersection property. *Comm. Algebra* **17**(1989), 135–148.

11. F. F. Kamalov, Intersections of direct summand of abelian groups. (Russian) *Izv. Vyssh. Uchebn. Zaved. Mat.* **5**(1977), 45–56.

12. I. Kaplansky, Infinite Abelian Groups. *Univ. of Michigan Press, Ann Arbor, Michigan*, 1954, 1969.

13. E. Matlis, Injective modules over Noetherian rings. *Pacific J. Math.*, 1958, No. 8, 511–528.

14. I. Purdea, Treatise of modern algebra, **1**–**2**. (Romanian) *Editura Academiei R.S.R., Bucureşti*, 1977, 1982.

15. D. W. Sharpe and P. Vámos, Injective modules. *Cambridge University Press*, 1972.

16. D. Vălcan, Abelian groups with the direct summand intersection property. (Romanian) *The Annual Conference of the Romanian Society of Mathematical Sciences,* 1997*, May* 29*–June* 1*, Bucharest, Romania*, **1**–**2**, 111–120.

17. D. Vălcan, Other characterization of abelian groups with the direct summand intersection property. *Studia Univ. Babes-Bolyai Math.* **43**(1998), No. 1, 95–122.

18. D. Vălcan, Subgroups and quotient groups of abelian groups with D.S.I.P. *Sci. Bull. Nord. Univ. Baia-Mare, Seria B, Fascicol Mathematica-Intromatica,* **13**(1997), No. 1–2, 17–32.

19. D. Vălcan, Injective modules with the direct summand intersection property. *Sci. Bull. Moldovian Acad. Sci. Seria Mathematica,* **31**(1999), No. 3, 39–50.

20. R. Ware, Endomorphism rings of projective modules, *Trans. Amer. Math. Soc.* **155**(1971), No. 1, 233–256.

21. Jr. R. B. Warfield, Decompositions of injective modules. *Pacific J. Math.* **31**(1969), No. 1, 263–276.

22. G. V. Wilson, Modules with the summand intersection property. *Comm. Algebra* **14**(1986), 21–38.

23. X. H. Zheng, Characterizations of Noetherian and Hereditary Rings. *Proc. Amer. Math. Soc.* **93**(1985), No. 3, 414–416.

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