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GEOMETRIC AND ALGEBRAIC REPRESENTATIONS OF COMMUTATIVE CANCELLATIVE MONOIDS

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Abstract. Part I is devoted to the structural description of normal and seminormal monoids and relevent ring-theoretic properties of the corresponding monoid rings and their completions with respect to the natural augmentation ideals.

Part II concerns with divisor class groups of normal monoids. Here the ranks and torsion parts of these groups are characterized in terms of the combinatorical data of these monoids.

Part III deals with K-theoretical properties of monoid rings. In particular:

(a) various examples of K-regular monoid rings which are not filtered unions of regular rings are derived;

(b) by establishing excision property in monoid rings for p-divisible monoids ($p > 1$), higher K-analogs of Vorst's results on Serre's problem for discrete Hodge algebras are obtained;

(c) a passing procedure (for Grothendieck groups) from the monoid rings corresponding to p-divisible monoids ($p > 1$) to arbitrary ones is descibed in a categorial framework; as a result "almost categorial" proof of the triviality of the groups $SK_0(R[M])$ for wide class of R-s and M-s is derived; in an attempt to develop the analogous approach to the study of the elementary action on unimodular rows over monoid rings an intermediate result is obtained for subintegral ring extensions which seems to be of independent interest;

(d) the same topics are considered in the concluding section for commutative, cancellative, but not necessarily torsion free monoids.

I

II

III K

K

p $(p > 1)$

K

p

$(p > 1)$

$SK_0(R[M])$

" R M

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Introduction

In this paper we continue our study of commutative cancellative torsion free monoids and their representations started back in [G1-G5]. The present work is organized as follows:

In Part I (Φ-correspondence and normal monoids) for the readers convenience we describe once more the Φ-correspondence between the (commutative, cancellative, torsion free) monoids and convex sets. Thereafter from the established new structural theorems we derive the well-known necessary and sufficient conditions for a monoid domain to be normal or seminormal [AA-Gi]. The developed approach simultaneously enables us to receive analogous necessary and sufficient conditions for a completion of a monoid domain (with respect to the natural augmentation ideal) to be completely integrally closed or seminormal. This result seems to be new.

Part II (Divisor class groups of normal monoids) is devoted to the study of arithmetic properties of normal monoids. Namely, we research the structure of their divisor class groups (as defined in [C1]). More precisely, the basis of $\mathbb{Q}$-spaces $\mathbb{Q} \otimes Cl(M)$ for finitely generated normal monoids M without nontrivial units are explicitly constructed. As a result we obtain the following relation (M as above) $\operatorname{rank}(Cl(M)) + \operatorname{rank}(M) = F(M)$, where $F(M)$ denotes the number of codimension 1 faces of the polyhedron $\Phi(M)$. We note that the same relation could be obtained by identifying $\operatorname{Div}(M)$ with the free abelian group spanned by the set of height 1 prime ideals of M (as one has for rings in [Bo])[1] which in their turn are in one-to-one correspondence with the mentioned set of faces of $\Phi(M)$. There is also given some partial pure geometrical characterization of the torsion part $Cl(M)_{tor}$.

Part III (K-theory of monoid rings) consists of some relatively independent sections, devoted to the study of K-theoretical properties of monoid rings.

In §3.I we give examples of normal monoids M which are neither divisible by some natural number $c > 1$ (and, hence, do not fall into the class of monoids treated in [G3]) nor filtered unions of free monoids, but the homomorphisms $K_i(R) \to K_i(R[M])$ are isomorphisms for all regular rings R and all indices i. It should be noted that the positive answer to the Vorst-Weibel's problem [LLPS, Sect. 2.10] would imply the absence of such monoids in the class of finitely generated monoids save the case of free monoids.

In §3.2, basing on Suslin-Wodzicki's recent important work on excision in algebraic K-theory, we prove the higher K-analogue in the class of p divisible monoids (for some

[1] This is actually done in [Gi, Sects. 15 and 16].

prime p) of Vorst's theorem concerning Serre's problem for discrete Hodge algebras [V]. More precisely, the validity of the following implication is shown ($i \in \mathbb{N}$)

$$K_i(R) \xrightarrow{\sim} K_i(R[M]) \Rightarrow K_i(R) \xrightarrow{\sim} K_i(R[M]/RI),$$

where M is p divisible for some prime p and I is a proper radical ideal ($\sqrt{I} = I$) of M. We note that if the divisibility condition on M is omitted, then the mentioned implication will not hold (according to [Sw2]). Thereafter the analogous approach is used to obtain a partial positive answer to the question raised in [G3]. Namely, the natural isomorphisms $K_i(R) \xrightarrow{\sim} K_i(R[M])$ are established, where i is any natural index (we always mean Quillen's K-groups), R is a (not necessarily commutative) regular ring and M is any intermediate p divisible monoid

$$\mathbb{Z}_+^r \subset M \subset \mathbb{Q}_+^r$$

for some prime p and natural r. Previously the above mentioned isomorphisms for any p divisible monoid M without nontrivial units were established in the case $i = 1$ [G3] and $i = 2$ [Mu]. We remark that nontriviality of p divisible intermediate monoids $\mathbb{Z}_+^r \subset M \subset \mathbb{Q}_+^r$ follows from the observation that any finite abelian group can be realized as $Cl_0(M)$ for M normal of the mentioned type, where $Cl_0(M)$ is some canonical subgroup of $Cl(M)$ (see Example 2.5.10 below). It should be also noted that the mentioned monoids, as usual, are not filtered unions of free monoids [G3, Remark 1.12].

In §3.3 for certain class of functors from (commutative) rings to abelian groups it is proved that (denoting such a functor by G) $G(R[\mathbb{Z}_+^r]) = 0 \Rightarrow G(R[M]) = 0$, where R is a commutative ring, $r \in \mathbb{N}$ and M is an intermediate monoid $\mathbb{Z}_+^r \subset M \subset \mathbb{Q}_+^r$. This class of functors contains SK_0 and, thus, we obtain "almost categorial" proof of the implication $K_0(R[\mathbb{Z}_+^r]) = \mathbb{Z} \Rightarrow K_0(R[M]) = \mathbb{Z}$ for M as above which, in addition, is normal. Therefore, using [G4] we get an essentially new proof of the freeness of projective $R[M]$-modules for any PID R and any intermediate seminormal monoid $\mathbb{Z}_+^r \subset M \subset \mathbb{Q}_+^r$. This proof avoids Horrocks monic inversion technique [L] that was involved in the original (and independent) proofs of the same theorems as presented in [G1] and [C2]. We recall that in [G2] (which used Horrock's mentioned technique as a starting point) the freeness of all projective $R[M]$-modules, where R is a PID and M is any seminormal monoid (as conjectured by Anderson [A1,A2]) was proved.

Herein we develop the analogous approach to the question raised in [G4]: whether the action of $E_n(R[M])$ on the set $Um_n(R[M])$ is transitive, where R is a commutative Noetherian ring of Krull $\dim R = d < \infty$, M is any (commutative, cancellative, torsion free) monoid and $n \geq \max(d+2, 3)$. The "classical" situation is the case of polynomial rings (i.e. $M = \mathbb{Z}_+^r$) treated in [Su]. The mentioned transitivity for a rather wide class of monoids including, for example, all monoids with the rank of the corresponding groups of quotients ≤ 3 and all intermediate monoids $\mathbb{Z}_+^r \subset M \subset \mathbb{Q}_+^r$ ($r \in \mathbb{N}$) was established in [G4, G5]. First we prove here some technical results on unimodular rows over subintegral ring extensions which seems to be of independent interest, and then we propose a hypothetical approach (involving results from [Sc]) to the aforementioned question on unimodular rows.

Commutative, cancellative, but not necessarily torsion free monoids, appear in the concluding §3.4. We investigate the analogous transitivity property of the action on unimodular rows and the triviality of SK_0-groups for the corresponding monoid rings. Positive results are obtained in the special cases when $K(M)_{tor}$ is either cyclic or p-group for some prime p (if in addition $p \cdot 1 = 0$ in a coefficient ring).

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Remark. After this paper has been written, Wilberd van der Kallen sent me his topological example showing that Conjecture 3.3.6 fails in general. However, we still may hope that our conjecture is true in the special case we are interested in, i.e. in the case of monoid rings.

PART I

Φ-Correspondence and Normal Monoids

§ 1.1. Conversations. Let $\mathbb{N}$ denote the natural numbers and $r \in \mathbb{N}$. By S^{r-1} will be denoted the standard unit sphere in $\underset{r}{\oplus}\mathbb{R}$, where $\mathbb{R}$ denotes reals. A subset $X \subset S^{r-1}$ will be called convex if for any points $x, y \in X$, which are not opposite on S^{r-1}, the uniquely determined shortest line connecting x and y lies entirely in X. A convex polyhedron $P \subset S^{r-1}$ is determined as an intersection of finite system of closed hemispheres of S^{r-1}. Since we shall not deal with other types of polyhedra, the word "convex" will be omitted. A polyhedron P will be called finite if it is contained in the interior of some hemisphere. A point $x \in S^{r-1}$ will be called rational if it determines a rational radial direction in $\underset{r}{\oplus}\mathbb{R}$. A convex hull of a subset $X \subset S^{r-1}$ is determined as an intersection of all convex sets containing X. We shall also say that X spans this convex hull. Similarly to the classical "flat" situation [Br] any polyhedron can be obtained as a convex hull of some finite set. A convex subset $X \subset S^{r-1}$ will be called rational if it is spanned by some rational points. Of course, rational points determine a dense subset of S^{r-1}. A polyhedron $P \subset S^{r-1}$ will be called simplex if its geometrical dimension $+1$ is equal to the quantity of codimension 1 faces. For a convex set $X \subset S^{r-1}$ $\operatorname{int}(X)$ will denote the relative interior of it. Finally, a subsphere of S^{r-1} always will mean the one with the same centre.

§ 1.2. Φ-Correspondence. All the considered monoids are assumed to be commutative, cancellative and torsion free (in the sense that the corresponding groups of quotients have no torsion elements). For a monoid M its group of quotients will be denoted by $K(M)$. $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ will denote the additive monoids of integer, rational and real numbers, while $\mathbb{Z}_+$, $\mathbb{Q}_+$, $\mathbb{R}_+$ will denote those of the corresponding non-negetive numbers. Because of our conditions on a monoid M one has the natural injective homomorphisms

$$M \longrightarrow K(M) \longrightarrow \mathbb{Q} \otimes K(M) \xrightarrow{\sim} \underset{r}{\oplus}\mathbb{Q} \longrightarrow \underset{r}{\oplus}\mathbb{R}$$

The mentioned cardinal r (may be infinite) will be denoted by $\operatorname{rank}(M)$. If the contrary will not be explicitly stated, the $\operatorname{rank}(M)$ will be supposed to be finite. Put $M^+ = M \backslash \{0\}$, where 0 is the neutral element of M (in particular, $(\mathbb{Z}_+)^+ = \mathbb{N}$). We shall identify M with its image in $\underset{r}{\oplus}\mathbb{R}$ (for some fixed isomorphism $\mathbb{Q} \otimes K(M) \xrightarrow{\sim} \underset{r}{\oplus}\mathbb{Q}$). For $m \in M^+$ by $\Phi(m)$ will be denoted the (rational) point of intersection of the radial direction of m with S^{r-1}. $\Phi(M)$ is defined as a convex hull of $\{\Phi(m) \mid m \in M^+\}$ which in its turn coincides with the set of all rational points of $\Phi(M)$.

Remark. For $\operatorname{rank}(M) = 0$ (or equivalently, $M = 0$) we put $\Phi(M) = \varnothing$.

The group of units of a monoid M (i.e. the largest subgroup of M) will be denoted by $U(M)$. For a finitely generated monoid M the set $\Phi(M)$ actually is a polyhedron and in the case $U(M) = 0$ it can be shown that $\Phi(M)$ is a finite one [G2].

For a monoid M and a convex subset $X \subset \Phi(M)$ put $M(X) = \{m \in M^+ \mid \Phi(m) \in X\} \cup \{0\}$. Of course, $M(X)$ is a submonoid of M. For brevity $M(\operatorname{int}(\Phi(M)))$ will be denoted by M_*.

Proposition 1.2.1 ([G3,G4]). *For a monoid M and a convex subset $X \subset \Phi(M)$ such that $\dim X = \dim \Phi(M)$ we have $K(M(X)) = K(M)$.*

Gordan's Lemma 1.2.2 [G3,G4]). *A monoid M is finitely generated if and only if $K(M)$ is finitely generated and $\Phi(M)$ is a polyhedron.*

§ 1.3. Normal monoids. We shal use additive notations (unless otherwise stated) for a monoid operation.

Definition 1.3.1. (a) For an extension of monoids $M \subset N$ an element $x \in N$ will be called integral over M if $nx \in M$ for some $n \in \mathbb{N}$;

(b) For an extension of monoids $M \subset N$ the monoid M will be called normal in N if there does not exist an element $x \in N \backslash M$ which is integral over M;

(c) A monoid M will be called normal if it is normal in $K(M)$.

Obviously, an extension $M \subset N$ is integral iff $\Phi(M) = \Phi(N)$, and M is normal in N if $x \in M$ for all $x \in N^+$ such that $\Phi(x) \in \Phi(M)$.

Now we give another characterization of finitely generated normal monoids without nontrivial units.

Theorem 1.3.2. *A finitely generated monoid M with $U(M) = 0$ is normal iff there exists an isomorphic to M submonoid $N \subset \mathbb{Z}_+^r$ which is normal in $\mathbb{Z}_+^r$, where $r = \operatorname{rank}(M)$.*

To prove this theorem we shall use the following

Lemma 1.3.3. *For any natural r and any rational hemisphere $h \subset S^{r-1}$ the monoid $\mathbb{Z}_+^r(h)_* (= \mathbb{Z}^r(\operatorname{int}(h)))$ is a filtered union of monoids isomorphic to $(\mathbb{Z}_+^r)_*$ which, in addition, are normal submonoids in $\mathbb{Z}^r(h)_*$.*

Proof. Let dh denote the boundary of h. Hence dh is a rational subsphere of S^{r-1} having dimension $r-2$ (without loss of generality here we assume $r \geq 2$, because $\mathbb{Z}(h) = \mathbb{Z}_+$ and $(\mathbb{Z}_+)_* = \mathbb{Z}_+$ otherwise). Hence $\mathbb{Z}^r(dh) \approx \mathbb{Z}^{r-1}$. Choose arbitrarily a basis $\{x_1, \dots, x_{r-1}\}$ of $\mathbb{Z}^r(dh)$. Denote by G the free monoid $\mathbb{Z}_+x_1 + \cdots + \mathbb{Z}_+x_{r-1}$. Consider now the quotient group $\mathbb{Z}^r/\mathbb{Z}^r(dh)$. Since $\mathbb{Z}^r(dh)$ is obviously normal in $\mathbb{Z}^r$, this quotient group is torsion free and, hence, isomorphic to $\mathbb{Z}$. This just means that there exists $y \in \mathbb{Z}^r$ such that $\{x_1, \dots, x_{r-1}, y\}$ is a basis of $\mathbb{Z}^r$. Of course, $\Phi(y) \notin dh$. Considering $-y$, if necessary, we can assume that $\Phi(y) \in \operatorname{int}(h)$. Choose arbitrarily $z \in G_*^+$ and consider the system of free monoids $F_c = \mathbb{Z}_+x_1 + \cdots + \mathbb{Z}_+x_{r-1} + \mathbb{Z}_+(y - cz)$, where c varies over $\mathbb{N}$. Obviously, $K(F_c) = \mathbb{Z}^r$ for all c. Since free monoids are normal, we obtain that F_c are normal in $\mathbb{Z}^r$ for all c. The condition $\Phi(y) \in \operatorname{int}(h)$ implies $(F_c)_* \subset \mathbb{Z}^r(h)_*$. We also have $F_c \subset F_{c'}$ for $c < c'$. Therefore, $(F_c)_*$ is a filtered system of monoids ($c \in \mathbb{N}$). All these observations imply that we will be done whenever the inclusion $\operatorname{int}(h) \subset \bigcup_{\mathbb{N}} \Phi(F_c)$ is shown. Put $p_i = \Phi(x_i)$, $i \in [1, r-1]$, $q_c = \Phi(y - cz)$ and $q_\infty = \Phi(-z)$. Let us introduce the following notation: for $u, v \in S^{r-1}$, which are not opposite, by $[u, v]$ will be denoted the uniquely determined shortest line $l \subset S^{r-1}$ connecting u with v. From the elementary geometrical observation we can see that $q_c \to q_\infty$ when $c \to \infty$. Consequently, since q_∞ for each i is not opposite to p_i, we get $[p_i, q_c] \to [p_i, q_\infty]$ when $c \to \infty$ (the convergence here is meant in an obvious sense). Denote by H_{ic} the convex hull of $\{p_1, \dots, p_{i-1}, q_c, p_{i+1}, \dots, p_{r-1}\}$, where $c \in \mathbb{N} \cup \{\infty\}$, and by D that of $\{p_1, \dots, p_{r-1}\}$. Since $\Phi(z) \in \Phi(G_*)$, we come to the conclusion (by the immediate geometrical observation) that

$$\Big(\bigcup_{i=1}^{r-1} H_{i\infty}\Big) \cup D = dh.$$

Since for each $c \in \mathbb{N}$

$$\Big(\bigcup_{i=1}^{r-1} H_{ic}\Big) \cup D = d\big(\Phi(F_c)\big),$$

where $d(\Phi(F_c))$ denotes the boundary of $\Phi(F_c)$, we obtain $d(\Phi(F_c)) \to dh$ when $c \to \infty$. Now the conditions $D \subset dh$ and $q_c \in \operatorname{int}(h)$ imply that the convex hulls (in S^{r-1}) of $d(\Phi(F_c))$ (coinciding obviously with $\Phi(F_c)$)) almost cover $\operatorname{int}(h)$ whenever c is sufficiently large. Hence, $\operatorname{int}(h) \subset \bigcup_{\mathbb{N}} \Phi(F_c)$. ■

Proof of Theorem 1.3.2. As it was mentioned above, $\Phi(M)$ is a finite polyhedron. Of course, the hemisphere $h \subset S^{r-1}$ ($r = \operatorname{rank}(M)$), for which $\Phi(M) \subset \operatorname{int}(h)$, can be chosen to be rational. By Lemma 1.3.3 there exists a free monoid $F \subset K(M) (\approx \mathbb{Z}^r)$ such that F (or, equivalently, F_*) is normal in $K(M)$ and $M \subset F_*$. Since $K(M) = K(F)$ (Proposition 1.2.1) and M is normal, we see that M is normal in F. Fixing arbitrarily an isomorphism $F \approx \mathbb{Z}_+^r$ and denoting by N the image of M under this isomorphism, we obtain the desired extension $N \subset \mathbb{Z}_+^r$. ■

Here is one more characterization of finitely generated normal monoids without nontrivial units.

Proposition 1.3.4. *A finitely generated monoid M with $U(M) = 0$ is normal iff there exists an extension of monoids $N \subset \mathbb{Z}_+^s$ for some $s \in \mathbb{N}$ such that $N \approx M$ and $N = K(N) \cap \mathbb{Z}_+^s$, where the intersection is considered in $\mathbb{Z}^s$.*

Proof can be found in [H].

Corollary 1.3.5. *A finitely generated monoid M with $U(M) = 0$ is normal if and only if there exists a pullback diagram (of monoids) of the type*

$$\begin{array}{ccc} M & \longrightarrow & \mathbb{Z}_+^r \\ \big\downarrow & & \big\downarrow \\ \mathbb{Z}_+^s & \longrightarrow & L \end{array}$$

consisting of embeddings $(r, s \in \mathbb{N})$ such that $U(L) = 0$.

Proof. Let $f : M \to \mathbb{Z}_+^r$ be the embedding mentioned in Theorem 1.3.2 (thus $r = \operatorname{rank}(M)$) and $g : M \to \mathbb{Z}_+^s$ that mentioned in Proposition 1.3.4. Then $K(f(M)) = \mathbb{Z}^r$ and $K(g(M)) \subset \mathbb{Z}^s$. Consider the diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & \mathbb{Z}_+^r \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle h} \\ \mathbb{Z}_+^s & \longrightarrow & \mathbb{Z}^s \end{array},$$

where the bottom row is the identity embedding and h coincides with the composite map $\mathbb{Z}_+^r \hookrightarrow \mathbb{Z}^r = K(f(M)) \xrightarrow[\sim]{gf^{-1}} K(g(M)) \hookrightarrow \mathbb{Z}^s$. This diagram is Cartesian due to the determining properties of f and g (the proof is straightforward). Put $L = \operatorname{im}(h) + \mathbb{Z}_+^s$. It is also straightforward (due to the same properties of f and g) that $U(L) = 0$. Hence the diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & \mathbb{Z}_+^r \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle h} \\ \mathbb{Z}_+^s & \longrightarrow & L \end{array}$$

is the desired one.

It only remains to note that the normality of M at the existence of the mentioned diagram is obvious. ■

We state here without proof the following

Proposition 1.3.6. *A finitely generated monoid M with $U(M) = 0$ is normal iff there exists a limit diagram of the type*

$$\begin{tikzcd}
& & M \arrow[dll] \arrow[dl] \arrow[drr] & & \\
\mathbb{Z}^r_+ \arrow[drr, "f_1"'] & \mathbb{Z}^r_+ \arrow[dr, "f_2"] & & & \mathbb{Z}^r_+ \arrow[dll, "f_n"] \\
& & \mathbb{Z}^r & &
\end{tikzcd}$$

consisting of embeddings, where $r = \operatorname{rank}(M)$ *and* n *is the quantity of codimension 1 faces of* $\Phi(M)$*.*

The proof is based on the fact that a convex polyhedron is an intersection of half spaces bounded by the supporting hyperplanes.

§ 1.4. Localization of monoids. For an extension of monoids $L \subset M$ put $L^{-1}M = \{m - l \mid m \in M,\, l \in L\} \subset K(M)$.

Definition 1.4.1. A monoid M is called seminormal if $x \in K(M)$, $2x \in M$, $3x \in M$ imply $x \in M$.

The following proposition will be used in the next section.

Proposition 1.4.2. *Let* M *be a finitely generated normal (seminormal) monoid. Then there exists a finitely generated normal (seminormal) monoid* N *with* $U(N) = 0$ *and its submonoid* L *such that* $L^{-1}N = M$*.*

We shall need

Lemma 1.4.3. *For a seminormal monoid* M *the monoid* M_* *is normal.*

Proof can be found in [G2, Proposition 1.6].

Proof of Proposition 1.4.2. Let M be a finitely generated normal or seminormal monoid. Assume $U(M) \neq 0$. Choose arbitrarily $m \in U(M)^+$ and a rational finite polyhedron $P \subset S^{r-1}$ ($r = \operatorname{rank}(M)$) of dimension $r-1$, for which $\Phi(m)$ is contained in its interior. Denote by Q the rational finite polyhedron $\Phi(M) \cap P$. We claim that $L^{-1}N = M$, where $N = M(Q)$ and $L = \mathbb{Z}_+ m$. First observe that $U(N) = 0$ (since the polyhedron Q is finite) and N is finitely generated (by Gordan's Lemma). It is also obvious that the normality of M implies that of N and similarly for the seminormality. Since $\dim Q = \dim P$, by Proposition 1.2.1 $K(N) = K(M)$. Hence $K(L^{-1}N) = K(M)$. Let us show that $\Phi(L^{-1}N) = \Phi(M)$. Since $K(L) \subset L^{-1}N$, it suffices to show that $\Phi(x) \in \Phi(L^{-1}N)$ for any $x \in M \backslash K(L)$. Indeed, assume $x \in M \backslash K(L)$. Then $\Phi(x)$ cannot be the point opposite (on S^{r-1}) to $\Phi(m)$. Let $[\Phi(x), \Phi(m)]$ be the uniquely determined shortest line on S^{r-1} connecting $\Phi(x)$ with $\Phi(m)$. By the definition of Q, the intersection $Q \cap [\Phi(x), \Phi(m)]$ does not degenerate into $\Phi(m)$, i.e. there exists $y \in N^+$ such that $\Phi(y) \neq \Phi(m)$ and $[\Phi(y), \Phi(m)] \subset [\Phi(x), \Phi(m)]$. From the elementary geometrical observations we obtain $\Phi(x) \in [\Phi(y), \Phi(m)] \cup [\Phi(y), \Phi(-m)] \subset \Phi(L^{-1}N)$.

Now assume M is normal. Since localizations preserve normality, $L^{-1}N$ will be normal (the normality of M obviously implies that of N). It follows from the equality

$K(L^{-1}N) = K(M)$ that $L^{-1}N$ is normal in M. We also have the integrality of the extension $L^{-1}N \subset M$ ($\Leftrightarrow \Phi(L^{-1}N) = \Phi(M)$). The last two conditions imply $L^{-1}N = M$, as claimed above.

It remains to consider the case of seminormal monoids. First of all observe that localizations in monoids preserve seminormality. Indeed, if $G \subset H$ is an extension of monoids, H is seminormal, $x \in K(G^{-1}H)$ and $2x, 3x \in G^{-1}H$, then $2x = h - g$ for some $h \in H$ and $g \in G$, and hence $2(x+g) = h + g \in H$; analogously, $3(x+g') \in H$ for appropriate $g' \in G$; therefore $2(x+g+g')$, $3(x+g'+g) \in H$ and by seminormality of H we obtain $x+g+g' \in H \Rightarrow x \in G^{-1}H$. In particular, $L^{-1}N$ is seminormal (notations are given as above). Since $M(\Phi(-m)) = -M(\Phi(m)) = -N(\Phi(m))$ and $-N(\Phi(m)) \subset L^{-1}N$, we see that in order to show $L^{-1}N = M$, it suffices to show $x \in L^{-1}N$ for any $x \in M^+$ such that $\Phi(x) \neq \Phi(-m)$ (here for a monoid H under $-H$ we mean the monoid $\{-h \in K(H) \mid h \in H\}$). Denote by $l(x)$ the union $[\Phi(m), \Phi(x)] \cup [\Phi(x), \Phi(-m)]$ and put $(L^{-1}N)(x) = (L^{-1}N)(l(x))$ and $M(x) = M(l(x))$. The seminormality of $L^{-1}N$ and M imply that of $(L^{-1}N)(x)$ and $M(x)$. By Lemma 1.4.3. we obtain the normality of $(L^{-1}N)(x)_*$ and $M(x)_*$. We have $\Phi((L^{-1}N)(x)_*) = \mathrm{int}(l(x)) = \Phi(M(x)_*)$. Since $\dim(Q \cap \mathrm{int}(l(x)) = \dim(\mathrm{int}(l(x)))$, by Proposition 1.2.2 $K((L^{-1}N)(x)_*) = K(M(x)_*)$. Taking all this into account, we conclude that $(L^{-1}N)(x)_*$ is normal in $M(x)_*$ and the extension $(L^{-1}N)(x)_* \subset M(x)_*$ is integral. Therefore, $x \in M(x)_* = (L^{-1}N)(x)_*$. ■

We complete this section with some equalities (without proofs), relating algebraic and geometrical data on monoids. For a monoid M and its element x by $d(x)$ will be denoted the geometrical dimension of the uniquely determined face of $\Phi(M)$ for which $\Phi(x)$ is its internal point (the word "face" for some convex set is used in its obvious sense). For $x = 0$ we put $d(x) = -1$.

Proposition 1.4.4. *Let M be a monoid and x its any element. Then*
(a) $d(x) + 1 = \mathrm{rank}(U((\mathbb{Z}_+x)^{-1}M))$;
(b) *in case M is normal and finitely generated,*

$$(\mathbb{Z}_+x)^{-1}M \approx \mathbb{Z}^{d(x)+1} \oplus M'$$

for some finitely generated normal monoid M' with $u(M') = 0$.

The special case of (b) when $U(M) = 0$ and $d(x) = 0$ is proved in [G2]; if, in addition, $\Phi(M)$ is a simplex, the proof is given in [C2].

§ 1.5. Normality and seminormality conditions for monoid domains and their completions. For a finitely generated monoid M with $U(M)$ trivial and a ring (always assumed to be commutative) R the completion of $R[M]$ with respect to the $M^+R[M]$-adic topology will be denoted by $R[[M]]$. Embedding M into a free monoid $\mathbb{Z}_+^s$, we can identify $R[[M]]$ with the set of those power series $f \in R[[\mathbb{Z}_+^s]]$ in which only monomials of type am occur for some $a \in R$ and $m \in M$.

Definition 1.5.1. A monoid M will be called completely integrally closed if (writing multiplicatively) $x \in K(M)$, $m \in M$ and $mx^d \in M$ for all $d \in \mathbb{N}$ imply $x \in M$.

A completely integrally closed monoid is normal, but not conversely, however for a wide class of monoids these two conditions coincide (details will be given in the next part). A domain will be called normal if it is integrally closed in its field of fractions, and seminormal if its multiplicative monoid of all nonzero elements is such [Sw3].

Our aim here is to deduce from the structural theorems on monoids, established in the previous sections, the following

Theorem 1.5.2. *Let R be a domain and M a monoid. Then*
(a) *$R[M]$ is normal iff R and M are such;*
(b) *$R[M]$ is seminormal iff R and M are such;*
(c) *for a finitely generated monoid M with trivial $U(M)$ $R[[M]]$ is completely integrally closed iff R and M are such;*
(d) *for a finitely generated monoid M with trivial $U(M)$ $R[[M]]$ is seminormal iff R and M are such.*

Remark. (a) and (b) are well known and proved in [Gi, Corollary 12.11] and [AA, Corollary 6.2], respectively. But our approach is different and make use of the structural theorems established in the previous sections. At the same time this approach enables us to establish the statements (c) and (d) without any essential changes in the proofs.

We need the classical case of Theorem 1.5.2, i.e. the case when $M = \mathbb{Z}_+^r$.

Theorem 1.5.3. *Let R be a domain and $r \in \mathbb{N}$. Then*
(a) *$R[\mathbb{Z}_+^r]$ is normal iff R does;*
(b) *$R[\mathbb{Z}_+^r]$ is seminormal iff R does;*
(c) *$R[[\mathbb{Z}_+^r]]$ is completely integrally closed iff R does;*
(d) *$R[[\mathbb{Z}_+^r]]$ is seminormal iff R does.*

Proof. (a) is proved in [Bo, Ch.V, §1.3], (b) follows from [GH], (c) can be found in [Bo, Ch.V, §1.4], (d) is contained in [BN]. ■

We shall use

Lemma 1.5.4. *For a seminormal monoid M and its normalization $n(M)$ (i.e., $n(M) = K(M)(\Phi(M))$) we have $M_* = n(M)_*$.*

Proof follows directly from Proposition 1.2.1 and Lemma 1.4.3.

Lemma 1.5.5. *A normal (seminormal) monoid is a filtered union of finitely generated normal (seminormal) monoids.*

Proof. Represent arbitrarily M as a filtered union of finitely generated monoids, say $M = \underset{i}{\cup} M_i$. Then $M = \underset{i}{\cup} N_i$ is the desired representation for $N_i = (K(M_i) \cap M)(\Phi(M_i))$ (the finite generation of N_i follows, for example, from Gordan's Lemma). ■

Proof of Theorem 1.5.2. (a) and (b). Since filtered unions and localizations preserve normality (for domains), by Proposition 1.4.2 and Lemma 1.5.5 we can assume M to be finitely generated and $U(M)$ to be trivial. By Corollary 1.3.5 there exists a pull-back diagram of the type

$$\begin{array}{ccc} M & \xrightarrow{h} & \mathbb{Z}_+^r \\ {\scriptstyle g}\downarrow & & \downarrow \\ \mathbb{Z}_+^s & \longrightarrow & L, \end{array} \quad , \quad U(L) = 0,$$

consisting of embeddings. Clearly, the diagram

$$\begin{array}{ccc} R[M] & \xrightarrow{h_*} & R[\mathbb{Z}_+^r] \\ {\scriptstyle g_*}\downarrow & & \downarrow \\ R[\mathbb{Z}_+^s] & \longrightarrow & R[L] \end{array} \quad , \tag{*}$$

will be a pull-back one, consisting of R-embeddings, and similarly for the diagram

$$\begin{array}{ccc} R[[M]] & \xrightarrow{h_{**}} & R[[\mathbb{Z}_+^r]] \\ {\scriptstyle g_{**}}\downarrow & & \downarrow \\ R[[\mathbb{Z}_+^s]] & \longrightarrow & R[[L]] \end{array} \quad . \tag{**}$$

Of course, the crucial moment for the former diagram is that $U(L)$ is trivial.

Now assume f to be an element of the field of fractions of $R[M]$, which is integral over R[M]. Then $h_*(f) \in R[\mathbb{Z}_+^r]$ and $g_*(f) \in R[\mathbb{Z}_+^s]$ (by Theorem 1.5.3 (a)). Hence $f \in R[M]$. The same arguments applied to (**) show the truth of (c).

Now let us return to the seminormality property ((b) and (d)). Again, by Lemma 1.5.5 and Proposition 1.4.2, without loss of generality , we can assume (considering (b)) M to be finitely generated and $U(M)$ to be trivial. By Gordon's Lemma so will be the normalization of M $(= K(M)(\Phi(M)))$ denoted later on by $n(M)$. Considering the diagrams of types (*) and (**) for the monoid ring $R[n(M)]$ and its completion $R[[n(M)]]$, respectively, and using Theorem 1.5.3 (b),(d), we get $f \in R[n(M)]$ or $f \in R[[n(M)]]$ with respect to $f^2, f^3 \in R[n(M)]$ or $f^2, f^3 \in R[[n(M)]]$. By Lemma 1.5.4 we obtain $f_* \in R[M]$ or $f_* \in R[[M]]$ with respect to $f^2, f^3 \in R[M]$ or $f^2, f^3 \in R[[M]]$, where f_* is the sum (or formal sum) of all those monomials involved in the canonical form of f which have a form am for some $a \in R$ and $m \in M_*^+$.

Now let F be a face of $\Phi(M)$ (of arbitrary dimension). Then we have the R-retractions $R[n(M)(F)] \overset{p(F)}{\leftrightarrows} R[n(M)]$ and $R[[n(M)(F)]] \overset{q(F)}{\leftrightarrows} R[[n(M)]]$, where $p(F)$ $(m) = q(F)(m) = m$ for $m \in n(M)(F)$ and $p(F)(m) = q(F)(m) = 0$ otherwise. Assume $f^2, f^3 \in R[M]$ ($R[[M]]$, respectively). Then the same arguments, we have used above, show $p(F)(f_*) \in R[M(F)] \subset R[M]$ ($q(F)(f_*) \in R[[M(F)]] \subset R[[M]]$), where $p(F)(f_*)$ and $q(F)(f_*)$ are determined analogously to f_*. To complete the proof it remains only to observe that

$$f = \sum_F p(F)(f_*) \quad \Big(f = \sum_F q(F)(f)_* \quad \text{resp.} \Big),$$

where F varies over the set of all fases of $\Phi(M)$ including $\Phi(M)$ and $\varnothing$, and $p(\varnothing)(f)_* = f(0)$ (the constant term of f), and similarly for $q(\varnothing)(f)$. ■

Corollary 1.5.6. *Let R be a seminormal domain and $\{M_i\}_{i=1}^n$ and $\{N_i\}_{i=1}^n$ two systems of seminormal monoids without nontrivial units. Assume M_i to be finitely generated. Then the natural homomorphism* $\mathrm{Pic}(R) \to \mathrm{Pic}(R[[M_1]][N_1]\cdots[[M_i]][N_i]\cdots[[M_n]][N_i]$ *is an isomorphism.*

Proof of Corollary follows directly from Theorem 1.5.2, the main result of [A2] and the observation that Pic endures reduction modulo radical ideals [B].

Remark 1.5.7. Actually our arguments show a slightly general result than (b) (from Theorem 1.5.2) and (correspondingly) Corollary 1.5.6. Namely, we mean the one concerning the reduced seminormal rings (in the sense of [Sw3]).

PART II

Divisor Class Groups of Normal Monoids

§ 2.1. Definitions. Let M be a monoid. A divisorial ideal of M is defined as a nonempty intersection (in $K(M)$) of the type $\bigcap_i (x_i + M)$, where $x_i \in K(M)$ and $x_i + M$ denotes the set $\{x_i + m \,|\, m \in M\}$. The set of divisorial ideals of M will be denoted by $\mathrm{Div}(M)$. It contains the set of principal divisorial ideals, i.e. those of the type $x + M \subset K(M)$, $x \in K(M)$. Denote by $\mathrm{Pr}(M)$ the above mentioned subset of $\mathrm{Div}(M)$. As one has for domains [Bo, Ch.VII, §1], $\mathrm{Div}(M)$ can be aquipped with a monoid stucture. Namely, for $d, d' \in \mathrm{Div}(M)$ put $dd' = \bigcap_{x \in K(M), d+d' \subset x+M} (x + M)$, where $d + d' = \{a + a' \,|\, a \in d,\ a' \in d'\}$. Then $\mathrm{Pr}(M)$ will turn out to be a subgroup of $\mathrm{Div}(M)$, isomorphic to $K(M)/U(M)$. Analogously to the domains we have

Proposition 2.1.1 (see [C1]). $\mathrm{Div}(M)$ *is a group if and only if M is completely integrally closed.*

For a completely integrally closed monoid M the quotient group $Cl(M) = \mathrm{Div}(M)/\mathrm{Pr}(M)$ is called a divisor class group of M. It just measures the deviation from the unique factorization property.

Here is an alternative characterization of completely integrally closed monoids.

Proposition 2.1.2. *A finite rank monoid M is completely integrally closed if and only if it is normal and any rational point of $\overline{\Phi(M)}$ (the closure of $\Phi(M)$) belongs to $\Phi(M)$.*

Proof. Assume M to be normal and all rational points of $\overline{\Phi(M)}$ belong to $\Phi(M)$. Assume further $m \in M$, $x \in K(M)$ and $m+nx \in M$ for all $n \in M$. Since $\Phi(m+nx) \to \Phi(x)$ when $n \to \infty$ and $\Phi(x)$ is rational, we get $\Phi(x) \in \Phi(M)$, or equivalently, $ax \in M$ for some $a \in \mathbb{N}$. By normality $x \in M$. Assume now M to be completely integrally closed. If $z = x - y \in K(M)$, where $x, y \in M$ satisfies the condition $cz \in M$ for some $c \in \mathbb{N}$, then $az + cy \in M$ for all $a \in \mathbb{N}$. Hence $z \in M$. Consequently, M is normal. Consider now a rational point $X \in \overline{\Phi(M)}$. Clearly, there exists an element $x \in K(M)^+$ such that $\Phi(x) = X$. Consider any $m \in M_*^+$. Elementary geometrical observation shows that $\Phi(m + nx) \in \Phi(M)$ for all $n \in \mathbb{N}$. Because of completely integrally closedness $x \in M$. Hence $x \in \Phi(M)$. ■

Corollary 2.1.3. *A finitely generated normal monoid is completely integrally closed.*

Proof. Since the Φ-image of a finitely generated monoid is a polyhedron, we that in the finite generation case $\Phi(M) = \overline{\Phi(M)}$.

In what follows all the above considered monoids will be assumed to be of finite rank. For a monoid M by $\mathrm{Div}_0(M)$ will be denoted the intermediated subset $\Pr(M) \subset \mathrm{Div}_0(M) \subset \mathrm{Div}(M)$ consisting of the intersections of finite families of principal divisorial ideals. It is straightforward (from the cancellativity) that $\Pr(M) \cdot \mathrm{Div}_0(M) \subset \mathrm{Div}_0(M)$ and hence we can consider the subset $Cl_0(M) = \mathrm{Div}_0(M)/\Pr(M) \subset Cl(M)$. ■

Proposition 2.1.4. *Let M a completely integrally closed monoid and $d = \bigcap_i (x_i + M)$ and $d' = \bigcap_j (y_j + M)$ two arbitrary elements of* $\mathrm{Div}(M)$. *Then*

$$dd' = \bigcap_{i,j} (x_i + y_j + M).$$

Proof. The arguments very similar to those from [Bo, Ch.VII, §1.2] show that for any $a, b \in \mathrm{Div}(M)$ the set $a : b = \{m \in M \,|\, m + b \subset a\}$ belongs to $\mathrm{Div}(M)$ and $(a : b)b = a$. Hence we will be done whenever the equality

$$\bigcap_{i,j} (x_i + y_j + M) \ : \ \bigcap_i (x_i + M) = \bigcap_j (y_j + M)$$

is established. We have

$$\bigcap_{i,j} (x_i + y_j + M) \ : \ \bigcap_i (x_i + M) = \bigcap_j \Big(\bigcap_i (x_i + y_j + M) : \bigcap_i (x_i + M) \Big) =$$

$$= \text{(by cancellativity)} \ \bigcap_j \bigg(\Big(y_j + \bigcap_i (x_i + M) \Big) \ : \ \bigcap_i (x_i + M) \bigg).$$

Therefore, it suffices to show that for any j

$$(y_j + d) : d = y_j + M.$$

By the aforementioned equality $(a : b)b = a$ we obtain $((y_j + d) : d)d = y_j + d$ (of course, we use the obvious observation that $y_j + d \in Div(M)$). On the other hand $(y_j + M)d = y_j + d$. Since $Div(M)$ is a group (by Proposition 2.1.1) we arrive to the desired equality $(y_j + d) : d = y_j + M$. ■

Corollary 2.1.5. *For a completely integrally closed monoid M the set* $\mathrm{Div}_0(M)$ *is a submonoid of* $\mathrm{Div}(M)$ *and, hence,* $Cl_0(M)$ *is that of* $Cl(M)$.

Corollary 2.1.6. *Let M be a completely integrally closed monoid, $a \in \mathbb{N}$ and $d = \bigcap_i (x_i + M) \in \mathrm{Div}(M)$. Then $d^a = \bigcap_i (ax_i + M)$.*

Proof. By Proposition 2.1.4 $d^a = \bigcap_{i_1,\dots,i_a} (x_{i_1} + \cdots + x_{i_a} + M)$. Consequently, it suffices to show

$$\bigcap_{k=1}^{k=a} (ax_{i_k} + M) \subset x_{i_1} + \cdots + x_{i_a} + M$$

for any subset $\{x_{i_1}, \dots, x_{i_a}\} \subset \{x_i\}_i$. Obviously, the point $x_{i_1} + \cdots + x_{i_a} \in \mathbb{R}^r$, where $r = \mathrm{rank}(M)$, belongs to the convex hull P of the subset $\{ax_{i_k}\}_{k=1}^{k=a} \subset \mathbb{R}^r$. It is also obvious that P is a finite (convex) polyhedron. Let $\{z_j\}$ be the set of all vertices of P (hence $\{z_j\} \subset \{ax_{i_k}\}_{k=1}^{k=a}$). We are going to prove a more general statement which will imply the desired inclusion. Namely, we claim that for any point $z \in P$ the following inclusion holds: $\bigcap_j (z_j + M) \subset z + M$. Let $c(M)$ be the (convex) cone, with vertex in 0 spanned by M. By normality we have $M = c(M) \cap K(M)$. Denote by $\overline{c(M)}$ the closure of $c(M)$. Then Proposition 2.1.2 immediately implies $M = \overline{c(M)} \cap K(M)$. Therefore, we will be done whenever the inclusion $\bigcap_j (z_j + \overline{c(M)}) \subset z + \overline{c(M)}$ is established. Represent arbitrarily $\overline{c(M)}$ as an intersection of closed half spaces in $\mathbb{R}^n$ the boundary hyperplanes of which pass through 0. Say $c(M) = \bigcap_s H_s$. Now our claim follows from the following obvious inclusions $\bigcap_j (z_j + H_s) \subset z + H_s$ (for all s). ■

§ 2.2. Divisible monoids. For a monoid M by $\mathbb{Q}_+ \otimes M$ will be denoted its divisible hull in $\mathbb{Q} \otimes K(M)$, i.e. $\mathbb{Q}_+ \otimes M = \bigcup_{c \in \mathbb{N}} \{m/c \mid m \in M\} \subset \mathbb{Q}^r$, where $r = \mathrm{rank}(M)$. Clearly, $\mathbb{Q}_+ \otimes M$ is the smallest divisible monoid containing M. Since the extension $M \subset \mathbb{Q}_+ \otimes M$ is an integral, we have $\Phi(M) = \Phi(\mathbb{Q}_+ \otimes M)$.

Proposition 2.2.1. (a) *For a monoid M its divisible hull $\mathbb{Q}_+ \otimes M$ is normal; if, in addition, M is finitely generated, then $\mathbb{Q}_+ \otimes M$ is completely integrally closed;*

(b) *if $\Phi(M) = \Phi(M')$, then $\mathbb{Q}_+ \otimes M = \mathbb{Q}_+ \otimes M'$;*

(c) *for a finitely generated monoid M with $U(M) = 0$ there exists a divisible submonoid $N \subset \mathbb{Q}^r_+$ ($r = \mathrm{rank}(M)$) isomorphic to $\mathbb{Q}_+ \otimes M$;*

(d) *any divisible submonoid of $\mathbb{Q}^r$ ($r \in \mathbb{N}$) is normal in $\mathbb{Q}^r$;*

(e) $\mathbb{Q}_+ \otimes M \approx \varinjlim (M \xrightarrow{2} M \xrightarrow{3} M \xrightarrow{4} M \xrightarrow{5} \cdots)$.

Proof is trivial. We only mention that (c) follows from the appropriate choice of a basis of the $\mathbb{Q}$-space $\mathbb{Q} \otimes K(M)$.

In what follows we fix a finitely generated monoid M with $U(M) = 0$. Without loss of generality we shall assume (basing on (c) from the proposition above) $\mathbb{Q}_+ \otimes M \subset \mathbb{Q}^r_+$ ($r = \mathrm{rank}(M)$). For brevity we put $N = \mathbb{Q}_+ \otimes M$. In this section it will be proved that both $Cl(N)$ and its submonoid $Cl_0(N)$ are the $\mathbb{Q}$-spaces. Actually we shall construct certain generating set of the $\mathbb{Q}$-space $Cl_0(N)$ and establish the corresponding relations. These data will be used in the next (mostly geometrical) section to compute explicitly $\operatorname{rank}(Cl_0(N))$.

First we introduce some auxiliary notations. C will denote the cone $c(M)$; for any $t \in \mathbb{Q}_+$ by H_t will be denoted the hyperplane $(X_1 + \cdots + X_r = t)$ and if, in addition, $t \neq 0$, by h_t we denote the homothetic transformation of $\mathbb{R}^r$ with centre 0 and coefficient t^{-1}; for any $z \in \mathbb{Q}^r$ $(= K(N))$ we put $P_{t,z} = (z + C) \cap H_t$ and $Q_{t,z} = h_t(P_{t,z})$; for simplicity we shall assume $P_t = P_{t,0}$ and $Q_t = Q_{t,0}$ (thus $Q_t = P_1$). It is clear that for arbitrary t and z the polyhedron $P_{t,z}$ is either empty, or degenerated into a point, or homothetic to P_t, where two polyhedra P and Q in H_t are called homothetic if there exists a homothetic transformation of H_t with a centre and a positive coefficient transforming P into Q. Of course, when t is sufficiently large (for z fixed), the polyhedra P_t and $P_{t,z}$ are homothetic and the corresponding homothetic transformation of H_t is uniquely determined and has a rational centre and a rational coefficient (a point on H_t is called rational if it belongs to $\mathbb{Q}^r$). Denote by $c_{t,z}$ the above mentioned rational coefficient. The immediate geometrical observation shows that $c_{t,z} \to 1$ for z fixed when $t \to \infty$. Moreover, the distances between the corresponding faces (i.e. between their affine hulls) of P_t and $P_{t,z}$ do not depend on t. Since, at the same time, the polyhedron P_t "blows up" proportionally to the growth of t, we conclude that $Q_t \cap Q_{t,z} = P_1 \cap Q_{t,z} \to P_1$ when z is fixed and $t \to \infty$; moreover, for t sufficiently large the numbers of codimension 1 faces of $P_1 \cap Q_{t,z}$ and P_1 coincide, there exist (for all sufficiently large t) natural bijective correspondences between the sets of these faces such that the corresponding faces are parallel and approximate each other when $t \to \infty$. Now assume we are given a finite set of points $z_1, \ldots, z_k \in \mathbb{Q}^r$. The very same arguments show that for t sufficiently large the two polyhedra P_1 and $P_1 \cap Q_{t,z_1} \cap \cdots \cap Q_{t,z_k}$ relate to each other in the same way (described above) as P_1 and $Q_{t,z}$. We have the following obvious equality

$$P_1 \cap Q_{t,z_1} \cap \cdots \cap Q_{t,z_k} = C \cap (t^{-1}z_1 + C) \cap \cdots \cap (t^{-1}z_k + C) \cap H_1.$$

For brevity put $P(t, Z) = P(t, z_1, \ldots, z_k) = P_1 \cap Q_{t,z_1} \cap \cdots \cap Q_{t,z_k}$, where $Z = (z_1, \ldots, z_k)$. Consider the element

$$d(t, Z) = C \cap (t^{-1}z_1 + C) \cap \cdots \cap (t^{-1}z_k + C) \cap \mathbb{Q}^r \in \operatorname{Div}_0(N).$$

From the elementary geometrical consideration we have

Proposition 2.2.2. *If $P(t, Z) \neq P(t', Z')$, then $d(t, Z) \neq d(t', Z')$ (notation as above). If $P(t, Z) = P(t', Z')$ and $\dim(P(t, Z)) = r - 1$, then $d(t, Z) = d(t', Z')$.*

Corollary 2.2.3. *The set of all nondegenerated polyhedra of type $P(t, Z)$ is naturally embedded in $\operatorname{Div}_0(N)$.*

Denote by G the above mentioned set of polyhedra. In what follows we shall identify G with its image in $\operatorname{Div}_0(N)$. It should be noted that a polyhedron $P(t, Z) \in G$ does not define the pair (t, Z).

For a subset $Y \subset \mathbb{Q}^r$ and a rational $t > 0$ put $tY = \{ty \mid y \in Y\}$.

Proposition 2.2.4. *For any $d_1 \in \operatorname{Pr}(N)$, $d_2 \in \operatorname{Div}_0(N)$, $d_3 \in \operatorname{Div}(N)$ and a rational $t > 0$ we have $td_1 \in \operatorname{Pr}(N)$, $td_2 \in \operatorname{Div}_0(N)$, $td_3 \in \operatorname{Div}(N)$.*

Proof immediately follows from the divisibility of N.

Proposition 2.2.5. *For any $d \in \mathrm{Div}_0(N)$ there exist $d' \in G$ and rational $t > 0$ such that $td' = d$ (modulo $\mathrm{Pr}(N)$).*

Proof. There exist elements $z_0, z_1, \ldots, z_k \in \mathbb{Q}^r$ ($k \in \mathbb{N}$) such that $d = (z_0 + C) \cap (z_1 + C) \cap \cdots \cap (z_k + C) \cap \mathbb{Q}^r$. Put $z'_i = z_i - z_0$, $i \in [1, k]$. Then $d = C \cap (z'_1 + C) \cap \cdots \cap (z'_k + C) \cap \mathbb{Q}^r$ (modulo $\mathrm{Pr}(N)$). Choose rational t sufficiently large. Then $P(t, Z')$ is not degenerated, where $Z' = (z'_1, \ldots, z'_k)$. It remains only to note that $tP(t, Z') = C \cap (z'_1 + C) \cap \cdots \cap (z'_k + C) \cap \mathbb{Q}^r$. ■

The last two propositions imply

Corollary 2.2.6. *For any $x \in Cl_0(N)$ there exist rational $t > 0$ and an element in the image of G in $Cl_0(N)$ such that $ty = x$.*

Of course, we can identify P_1 with $\Phi(N)$ (i.e. the Φ-interpretation can be assumed to be "flat"). We shall say that a polyhedron Q in H_1 is in the ε-neighbourhood of P_1 if the boundary of Q is in the ε-neighbourhood of the boundary of P_1 (for some real $\varepsilon > 0$). We recall that all the above considered polyhedra are supposed to be finite and convex. Now assume we are given a real number $\varepsilon > 0$ and a finite subset $\{z_1, \ldots, z_k\} \subset \mathbb{Q}^r$ ($k \in \mathbb{N}$). From our remarks (before Proposition 2.2.2) it becomes clear that $P(t, Z)$ is in the ε-neighbourhood of P_1 if t is sufficiently large, where $Z = (z_1, \ldots, z_k)$. Denote by G_ε the subset of G consisting of those polyhedra which fall into the ε-neighbourhood of P_1. Observe that in the proof of Proposition 2.2.5 the rational number t could be chosen sufficiently large. In particular, we can strengthen the last corollary as follows

Proposition 2.2.7. *For any $x \in Cl_0(N)$ and a real $\varepsilon > 0$ there exist rational $t > 0$ and an element y of the image of G_ε in $Cl_0(N)$ such that $ty = x$.*

For a codimension 1 face F of P_1 fix arbitrarily a rational internal point $n(F)$. Let us also fix an arbitrary internal rational point of P_1, say m. Clearly, $n(F) - s$ and m belong to N. More precisely, $n(F) \in N(F)^+_*$ and $m \in N^+_*$ (at the same time these points belong to H_1). For a natural number c consider the elements

$$v_c(F) = c^{-1}m - (1 + c^{-1})n(F),$$

where F varies over the set of all codimension 1 faces of P_1. Now consider the elements

$$d_c(F) = C \cap (v_c(F) + C) \cap K(N) \subset \mathrm{Div}_0(N).$$

We claim that $Cl_0(N)$ is a $\mathbb{Q}$-space and for c sufficiently large the images $\overline{d_c(F)}$ of $d_c(F)$ in $Cl_0(N)$ generate this $\mathbb{Q}$-space. The rest of this section is devoted to the proof of this claim.

For each codimension 1 face F of P_1 and a rational number $u > 0$ let h^u_F denote the homothetic transformation of H_1 with respect to the centre $n(F)$ and the coefficient u, and T^u_F the parallel transposition of H_1 with respect to the vector $uc^{-1}(m - n(F))$.

From the direct calculations we have

Proposition 2.2.8. *Let F be a codimension 1 face of P_1 and $u > 0$ a rational number. Then*

$$\big(uv_c(F) + C\big) \cap H_1 = T_F^u h_F^{1+u}(P_1).$$

Assume now c is sufficiently large. From the elementary geometrical observation it becomes clear that the polyhedron

$$P_{c,1}(F) = C \cap \big(v_c(F) + C\big) \cap H_1 = T_F^1 h_F^2(P_1) \cap P_1$$

is obtained as follows: it is bounded by the affine hulls of all codimension 1 faces of P_1 except the face F, and the last face of $P_{c,1}(F)$ is parallel to F, sufficiently close to F and lies on the same side of F as the polyhedron P_1.

We leave to the reader the detailed verification of the following

Proposition 2.2.9. *Let $c > 0$ be a sufficiently large rational number. Then for any rational $u \in (0,1)$ and a codimension 1 face of P_1, say F, the polyhedron*

$$P_{c,u}(F) = C \cap \big(uv_c(F) + C\big) \cap H_1 = T_F^u h_F^{1+u}(P_1) \cap P_1$$

satisfies the conditions

(1) $P_{c,1}(F) \subset P_{c,u}(F) \subset P_1$*;*

(2) *the faces (of codimension 1) of $P_{c,u}(F)$ span the same affine hulls as those of P_1 except the last (in the appropriate enumeration) face, which is parallel to F, is situated sufficiently close to F and lies on the same side of F as the polyhedron P_1.*

Conversely, any polyhedron in H_1 satisfying the conditions (1) and (2) is of type $P_{c,u}(F)$ for some $u \in (0,1)$.

Corollary 2.2.10. *For all sufficiently large rational c-s there exist real ε (depending on c) such that any element $P \in G_\varepsilon$ is of type $\bigcap_F P_{c,u_F}(F)$, where F varies over the set of all codimension 1 faces of P_1 and $u_F \in \mathbb{Q} \cap [0,1]$.*

Now we have the following passage to arithmetics.

Proposition 2.2.11. *For all sufficiently large rational c-s the following equalities are valid in $\mathrm{Div}_0(N)$,*

$$\bigcap_F P_{c,u_F}(F) = \prod_F P_{c,u_F}(F)$$

where F varies over the set of all codimension 1 faces of P_1 and $u_F \in \mathbb{Q} \cap [0,1]$.

Remark 2.2.12. Observe that for all sufficiently large c-s the polyhedra of the type $\bigcap_F P_{c,u_F}(F)$ (F and u_F are as above) belong to G which in its turn is identified with the corresponding subset of $\mathrm{Div}_0(N)$.

Proof. Due to our embedding of G into $\mathrm{Div}_0(N)$ (Corollary 2.2.3) we have to show that for all sufficiently large rational c-s the following equalities are valid in $\mathrm{Div}_0(N)$,

$$C \cap \big(\bigcap_F (u_F v_c(F) + C)\big) \cap \mathbb{Q}^r = \prod_F \big(C \cap (u_F v_c(F) + C) \cap \mathbb{Q}^r\big),$$

where F-s and u_F-s are as in the proposition. According to Proposition 2.1.4,

$$\prod_F \big(C \cap (u_F v_c(F) + C) \cap \mathbb{Q}^r\big) = \underset{S(\underline{F})}{\cap} \Big(C \cap \big(\sum_{s(\underline{F})} u_F v_c(F) + C\big) \cap \mathbb{Q}^r\Big),$$

where $S(\underline{F})$ denotes the set of all nonempty subsets of the set of all codimension 1 faces of P_1 and $s(\underline{F})$ varies over $S(\underline{F})$. Therefore, we will be done whenever the inclusions

$$C \cap \Big(\sum_{s(\underline{F})} u_F v_c(F) + C\Big) \supset \underset{s(\underline{F})}{\cap} \big(C \cap (u_F v_c(F) + C)\big)$$

are established for all $s(\underline{F}) \in S(\underline{F})$.

For simplicity the corresponding codimension 1 faces of C and P_1 will be denoted by the same symbols. There exists a system of $\mathbb{R}$-linear maps $V_F : \mathbb{R}^r \to \mathbb{R}$, where F varies over the set of all codimension 1 faces of C such that $V_F(F) = 0$ and $C = \{x \in \mathbb{R}^r \mid V_F(x) \geq 0 \text{ for all } F\}$. Fix arbitrarily such a system. In terms of $V_F - s$ we just want to show the following implication (for each $s(\underline{F}) \in S(\underline{F})$):

$$x \in \mathbb{R}^r, \; V_F(x) \geq 0 \text{ and } V_F(x) \geq V_F(u_{F'} v_c(F')) \text{ for all } F, F' \in s(\underline{F}) \Rightarrow$$
$$\Rightarrow V_F(x) \geq V_F\Big(\sum_{s(\underline{F})} u_{F'} v_c(F')\Big) \text{ for all } F \in s(\underline{F})$$
$$(F' \text{ varies over } s(\underline{F}) \text{ in the summation}).$$

In its turn the above mentioned implication follows from the following system of implications (the system is considered for each $s(\underline{F}) \in S(\underline{F})$):

For each $F \in s(\underline{F})$

$$x \in \mathbb{R}^r, \; V_F(x) \geq 0 \text{ and } V_F(x) \geq V_F(u_{F'} v_c(F')) \text{ for all } F' \in s(\underline{F}) \Rightarrow$$
$$\Rightarrow V_F(x) \geq V_F\Big(\sum_{s(\underline{F})} u_{F'} v_c(F')\Big).$$

Now observe that for $F' \neq F$ we have $V_F(n(F')) > 0$. Therefore, for c sufficiently large $V_F(v_c(F')) = V_F(c^{-1}m - (1 + c^{-1})n(F')) < 0$ for all $F' \in s(\underline{F})$ different from F. Hence, for such c-s it will suffice to show the implication

$$x \in \mathbb{R}^r, \; V_F(x) \geq 0 \text{ and } V_F(x) \geq V_F(u_{F'} v_c(F')) \text{ for all } F' \in s(\underline{F}) \Rightarrow$$
$$\Rightarrow V_F(x) \geq V_F\big(u_F v_c(F)\big)$$

(recall that $u_{F'} \geq 0$), which is obvious (since $F \in S(\underline{F})$). ∎

Proposition 2.2.13. *$Cl_0(N)$ is a $\mathbb{Q}$-space and for all sufficiently large rational c-s the images in $Cl_0(N)$ of the elements*

$$d_c(F) = C \cap (v_c(F) + C) \cap K(N) \in \mathrm{Div}_0(N),$$

where F varies over the set of all codimension 1 faces of P_1, generate this $\mathbb{Q}$-space.

Proof. **Step 1.** First we shall show that the group $Cl(N)$ is a $\mathbb{Q}$-space. Since the group $\mathrm{Pr}(N)$ is obviously a $\mathbb{Q}$-space, we only have to show that the group $\mathrm{Div}(N)$ is also a $\mathbb{Q}$-space. By Corollary 2.1.6 for arbitrary $t \in \mathbb{N}$ and $d \in \mathrm{Div}(N)$ we have $(t^{-1}d)^t = d$ (recall that in our notations $t^{-1}d = \{t^{-1}y \mid y \in d\}$). Therefore, it remains only to show the torsionfreeness of $Cl(N)$. Assume $d^t \in \mathrm{Pr}(N)$ for some $d = \bigcap_i (x_i + N) \in \mathrm{Div}(N)$ ($x_i \in K(N)$) and $t \in \mathbb{N}$. Then, since $\mathrm{Pr}(N)$ is a $\mathbb{Q}$-space, we get $t^{-1}(d^t) \in \mathrm{Pr}(N)$. Finally, by Corollary 2.1.6. $t^{-1}(d^t) = t^{-1}(\bigcap_i (tx_i + N)) = \bigcap_i (x_i + N) = d$ (of course, the crucial moment here is the divisibility of N).

Step 2. Since $Cl(N)$ turned out to be a $\mathbb{Q}$-space, the submonoid $Cl_0(N) \subset Cl(N)$ (see Corollary 2.1.5) must be torsion free. Moreover, the arguments in the previous step show that for arbitrary rational numbers $u_F \geq 0$ (F are as above) the elements $d_c(F)^{u_F} \in \mathrm{Div}(N)$ belong actually to $\mathrm{Div}_0(N)$ (in other words, $Cl_0(N)$ is a root closed in $Cl(N)$). Now, by Proposition 2.2.7 Corollary 2.2.10 and Proposition 2.2.11 we obtain that for all sufficiently large rational c-s

$$\Big\{ \prod_F d_c(F)^{u_F} \;\Big|\; u_F \in \mathbb{Q}_+ \Big\} = \mathrm{Div}_0(N).$$

Step 3. It remains only to show that for all sufficiently large rational c-s there exist strictly positive rational numbers u_F (F-s are as above) such that $\prod_F d_c(F)^{u_F} \in \mathrm{Pr}(N)$. It is clear that for any real $\varepsilon > 0$ there exists a rational number $k_\varepsilon \in (0,1)$ such that the homothetic image of P_1 with respect to the centre m (notations are as above) and the coefficient k_ε belongs to G_ε. It is also clear (from the divisibility of N) that the above mentioned image of P_1 defines an element of $\mathrm{Pr}(N)$). Therefore, Corollary 2.2.10 and Proposition 2.2.11 give the desired result. ∎

§ 2.3. Ranks of divisor class groups. We keep all the notations from the previous section. The main purpose of this section is to prove

Theorem 2.3.1. *Let M be a finitely generated normal monoid with $U(M) = 0$. Then* $\mathrm{rank}(Cl(M)) + \mathrm{rank}(M) = F(M)$, *where $F(M)$ denotes the number of all codimension 1 faces of $\Phi(M)$.*

The proof of this theorem consists of three independent propositions.

Proposition 2.3.2. *Let M be a finitely generated normal monoid with $U(M) = 0$. Then $Cl_0(M) = Cl(M)$.*

Proof. We have to show that $\mathrm{Div}_0(M) = \mathrm{Div}(M)$. Let d be an element of $\mathrm{Div}(M)$. It can be represented as an intersection $\bigcap_i (x_i + C) \cap K(M)$, where x_i-s are some elements of $K(M)$. Let $C = \bigcap_F H(F)$ be the uniquely determined representation of the cone C as an intersection of certain half-spaces whose boundaries are spanned by the codimension 1 faces of C, where F varies over the set of the above mentioned faces of C. Let us show that for each F there exists an index $i(F)$ such that $x_{i(F)} + H(F) = \bigcap_i (x_i + H(F))$. Since M is finitely generated, $K(M)$ is a free group of rank r. Let $dH(F)$ denote the boundary hyperplane of $H(F)$. Then $K(M) \cap dH(F)$ will be isomorphic to $\mathbb{Z}^{r-1}$ and $K(M) = Z \oplus (K(M) \cap dH(F))$ for some subgroup Z ($\approx \mathbb{Z}$) of

$K(M)$ (recall that we had the analogous situation in the proof of Lemma 1.3.3). We can fix an isomorphism $Z \approx \mathbb{Z}$ such that the "positive part" of Z will fall in $H(F)$. Denote by z_i the natural images of x_i in Z respectively. Since $\bigcap_i (x_i + H(F) \neq \varnothing$, we get $\sup_i(z_i) \neq \infty$. But then there exists an index $i(F)$ such that $z_{i(F)} = \sup_i(z_i)$. Clearly, the index $i(F)$ satisfies the required property. For a codimension 1 face F choose arbitrarily an index of the above mentioned type. Then $d = \bigcap_{i(F)} (x_i + C) \cap K(M) = \bigcap_{i(F)} (x_i + M) \in \mathrm{Div}_0(M)$. ■

Proposition 2.3.3. *Let M be as above. Then $Cl_0(N) \approx \mathbb{Q} \otimes Cl_0(M)$ (N, as above, denotes the divisible hull of M).*

Proof. We define the map $\psi : \mathrm{Div}_0(M) \to \mathrm{Div}_0(N)$ as follows $\psi(\bigcap_i (x_i + M)) = \bigcap_i (x_i + N)$ for an arbitrary finite subset $\{x_i\}_i \subset K(M)$. Let us make sure that ψ is well defined. Let $\bigcap_i (x_i + M) = \bigcap_j (y_j + M)$ in $\mathrm{Div}_0(M)$. We claim that $\bigcap_i (x_i + N) = \bigcap_j (y_j + N)$. For any natural t put $M_t = t^{-1}M$. We have $N = \bigcap_{t \in \mathbb{N}} M_t$. Therefore it will suffice to show that $\bigcap_i (x_i + M_t) = \bigcap_j (y_j + M_t)$ for all t-s. Multiplying both sides by t (in the sense we did it in §2.2), we come to the equivalent equality $\bigcap_i (tx_i + M) = \bigcap_j (ty_j + M)$ which actually is valid, since

$$\bigcap_i (tx_i + M) = \big(\bigcap_i (x_i + M)\big)^t = \big(\bigcap_j (y_j + M)\big)^t = \bigcap_j (ty_j + M)$$

by Corollary 2.1.6. Since a $\mathbb{Q}$-span of the image of ψ is the whole $\mathrm{Div}_0(N)$, we get that when passing to the quotient groups the image of the corresponding map $\overline{\psi} : Cl_0(M) \to Cl_0(N)$ is a generating set of the $\mathbb{Q}$-space $Cl_0(N)$ (Proposition 2.2.13). Hence we will be done if the triviality of $\ker(\overline{\psi}) \otimes \mathbb{Q}$ is proved. Assume that the image of $d = \bigcap_i (x_i + M) \in \mathrm{Div}_0(M)$ in $Cl_0(M)$ belongs to $\ker(\overline{\psi})$. It means that $\psi(d) \in \mathrm{Pr}(N)$. Say, $\psi(d) = z + N$ for some $z \in K(N)$. Choose a natural number t such that $tz \in K(M)$. We have $\psi(td) = \psi(d^t) = \bigcap_i (tx_i + N) = tz + N$. Therefore, $tz + M = (tz + N) \cap M = \bigcap_i ((tx_i + N) \cap M) = \bigcap_i (tx_i + M) = d^t$ or, equivalently, $d \in Cl_0(M)_{tor}$. Consequently, $\ker(\overline{\psi}) \otimes \mathbb{Q} = 0$. ■

The following proposition will complete the proof of Theorem 2.3.1.

Proposition 2.3.4. *Let N be a divisible hull of an infinitely generated monoid M with $U(M) = 0$. Then $\mathrm{rank}(Cl_0(N)) = F(M) - \mathrm{rank}(M)$, where $F(M)$ denotes the number of codimension 1 faces of $\Phi(M)$.*

The proof needs some preparatory results.

Definition 2.3.5. Assume we are given two finite convex polyhedra P and Q in a real affine space and $\dim(P) = \dim(Q)$. We shall say that P is inscribed in Q if $P \subset Q$ and each codimension 1 face of Q contains a codimension 1 face of P. Now, assume we are given a monoid extension $K \subset L$ with trivial units such that $\Phi(K)$ and $\Phi(L)$ are finite polyhedra (we consider finite rank monoids). We shall say that K is inscribed in L if $\Phi(K)$ is inscribed in $\Phi(L)$ and K is normal in L (clearly, $\mathrm{rank}(K) = \mathrm{rank}(L)$).

Lemma 2.3.6. *Let N be as in Proposition 2.3.4. Then N can be inscribed in $\mathbb{Q}^r_+$ ($r = \mathrm{rank}(M)$), i.e. there exists a divisible submonoid $N' \subset \mathbb{Q}^r_+$ which is isomorphic to N and is inscribed in $\mathbb{Q}^r_+$.*

Proof. Each codimension 1 face F of the polyhedron $\Phi(N)$ determines uniquely a closed hemisphere $h_F \subset S^{r-1}$ such that $\Phi(N) \subset h_F$ and $F = \Phi(N) \cap h_F$ (here the Φ-correspondence is considered on S^{r-1} instead of H_1). Obviously, we can choose r many codimension 1 faces of $\Phi(N)$, say $F_1, \dots, F_r$ such that $\overset{i=r}{\underset{i=1}{\cap}} h_F$ will be a (rational) simplex in S^{r-1}. The proof of this fact is left to the reader. Denote the above mentioned simplex by D and consider the monoid $\mathbb{Q}^r(D) \approx \mathbb{Q}^r_+$. It only remains to note that N is inscribed in $\mathbb{Q}^r(D)$. ■

We define a small perurbation T of a codimension 1 face F of P_1 as a parallel transposition of the affine hull of F satisfying the conditions: (a) $T(F) \subset H_1$; (b) the polyhedron $P_1(T)$ bounded by the affine hulls of all codimension 1 faces of P_1 different from F and that of $T(F)$ has the same number of codimension 1 faces as P_1 itself; (c) $P_1(T) \subset P$ and $P_1(T) \neq P$. It can be shown that for a small perturbation T of F and any real $u \in (0,1)$ the transposition uT is likewise a small perturbation of F. A small perturbation T will be called rational if $P_1(T)$ is a rational polyhedron. A parallel transposition T of the face F will be called and ε-transposition if the distance betwen the affine hulls of F and $T(F)$ is strictly positive and less than ε (ε here is a given positive real number), and $T(F)$ lies on the same side of F as P_1. Evidently, for sufficiently small $\varepsilon > 0$ any ε-transposition of F is actually a small perturbation of F. A small perturbation of a system of codimension 1 faces of P_1, say $F_1, \dots, F_s$, is defined as a system of the corresponding small perturbations $T_1, \dots, T_s$ such that the polyhedron $\overset{i=1}{\underset{i=1}{\cap}} P_1(T_i)$ has the same number of codimension 1 faces as P_1. Again, for such a system of T_i and any real numbers $u_i \in (0,1)$ the system u_iT_i will be a small perturbation of F_i-s. A rational small perturbation of a system of faces is defined analogously. Finally, a small perturbation of a system of codimension 1 faces will be called an ε-perturbation if it consists of ε-transpositions of the corresponding faces.

Let c be a sufficiently large rational number and let $d_c(F) \in \mathrm{Div}_0(N)$ be the elements of the type mentioned in Proposition 2.2.13. Then we have

Lemma 2.3.7. *For all sufficiently small real $\varepsilon > 0$ and ε-perturbations of a codimension 1 face F of P_1, say T, the polyhedra $P_1(T)$ are of type $d_c(F)^u$ for some positive rational u. For all sufficiently small $\varepsilon > 0$ and all ε-perturbations of codimension 1 faces F of P_1, say $T(F)$, the polyhedra $\underset{F}{\cap} P_1(T(F))$ are of type $\prod_F d_c(F)^{u_F}$ for some $u_F \in \mathbb{Q}_+$.*

Proof immediately follows from Propositions 2.2.9 and 2.2.11 and Corollary 2.2.10. In what follows we fix such a rational number c and a small real $\varepsilon > 0$.

Assume we are given a nonempty system of codimension 1 faces $\{F_i\}_{i=1}^{i=s}$ which will be called d-depended (d is for divisors) if there exist a disjoint union $X \cup Y \subset \{1, \ldots, s\}$ and a small ε-perturbations of systems $\{F_i\}_{i \in X}$ and $\{F_i\}_{i \in Y}$, say $\{T(F_i)\}_{i \in X}$ and $\{T(F_i)\}_{i \in Y}$, respectively, such that the polyhedra $\bigcap_X P_1(T(F_i))$ and $\bigcap_Y P_1(T(F_i))$ are homothetic where, in addition, we do not exclude $Y = \varnothing$ for which we put $\bigcap_\varnothing P_1(T(F_i)) = P_1$. A system is called d-independent if it is not d-depended. Clearly, in these definitions we exclude the case $X = \varnothing$.

Lemma 2.3.8. *The number of elements of a maximal (with respect to the inclusion) d-independent system of faces of P_1 is equal to* $\dim_{\mathbb{Q}}(Cl_0(N))$.

Proof follows directly from the previous lemma and the observation that if $P, Q \in G_\varepsilon$ are homothetic, then their images $\overline{P}$ and $\overline{Q}$ in $Cl_0(N)$ satisfies the equality of the type $\overline{P}^u = \overline{Q}$ for some rational $u > 0$.

Proof of Proposition 2.3.4. By Lemma 2.3.6 we can assume N is inscribed in $\mathbb{Q}_+^r$. Then each codimension 1 face of the simplex $\Phi(\mathbb{Q}_+^r)$ contains exactly one codimension 1 face of $\Phi(N)$. Let $F_1, \ldots, F_r$ be these mentioned faces of $\Phi(N)$ and $K_1, \ldots, K_s$ be all other faces ($s = F(M) - r$). By Lemma 2.3.8 we will be done whenever the d-independence and the maximality of the system $\{K_1, \ldots, K_s\}$ are shown.

Step 1. The set $\{K_1, \ldots, K_s\}$ is d-independent.

Assume $X \cup Y \subset \{1, \ldots, s\}$ is a disjoint union and the polyhedra $\bigcap_X P_1(T(K_i))$ and $\bigcap_Y P_1(T(K_i))$ are homothetic for some ε-perturbations $\{T(K_i)\}_{i \in X}$ and $\{T(K_i)\}_{i \in Y}$ of the systems $\{K_i\}_{i \in X}$ and $\{K_i\}_{i \in Y}$, respectively. On the other hand, both of the above mentioned polyhedra are inscribed in the simplex $\Phi(\mathbb{Q}_+^r)$. We give here an elementary observation that two homothetic polyhedra P and Q coincide whenever they are inscribed in the same simplex. In particular,

$$\bigcap_X P_1(T(K_i)) = \bigcap_Y P_1(T(K_i)).$$

But the last equality is impossible, since $X \cap Y = \varnothing$.

Step 2. The d-independent set $\{K_1, \ldots, K_s\}$ is maximal.

In other words we have to show that the set $\{F_j, K_1, \ldots, K_s\}$ is d-depended for any $j \in [1, r]$. Let D be the simplex $\Phi(\mathbb{Q}_+^r)$ and o_j its vertex situated oppositely to the afine hull of F_j. It should be noted here (as in the previous step) the Φ-correspondence here is considered in the hyperplane H_1. Let $\delta > 0$ be a small rational number and consider the homothetic transformation t with respect to the centre o_j and the coefficient $1 - \delta$. Put $P = P_1 \cap t(P_1)$, and $Q = P_1 \cap t^{-1}(P_1)$. Now, divide the set $\{F_j, K_1, \ldots, K_s\}$ into three parts as follows: call a face from this set positive if P_1 and o_j lie on the same side of the affine hull of this face; if the above mentioned affine hull separates P_1 and o_j, we call this face negative; and, finally, if o_j belongs to this affine hull, the face is called neutral. It is an easy exercise that for all sufficiently small rational number δ the polyhedron P is of type $\bigcap_{\mathrm{POS}} P_1(T(L))$, where POS denotes the positive part of $\{F_j, K_1, \ldots, K_s\}$ (clearly $F_j \in$ POS) and $T(L)$-s constitute an ε-perturbation of $\{F_1, \ldots, F_r, K_1, \ldots, K_s\}$ (L varies over this set). Analogously, Q

is of type $\bigcap_{\mathrm{NEG}} P_1(T(L))$ for the negative part NEG of $\{F_j, K_1, \ldots, K_s\}$ (in both of these representations the systems $T(L)$ can be obviously chosen the same). Since we have an obvious equality $P = tQ$, we obtain that the disjoint union $\mathrm{POS} \cup \mathrm{NEG} \subset \{F_j, k_1, \ldots, K_s\}$, where $\mathrm{POS} \neq \varnothing$, possesses the property mentioned in the definition of d-depended systems. ■

§ 2.4. On torsion parts of divisor class groups.

Theorem 2.4.1. *Let M and N be finitely generated normal monoids with trivial units. Assume M is inscribed in N. Then there is an embedding of $Cl(M)_{tor}$ into $Cl(N)_{tor}$.*

Lemma 2.4.2. *Let L be a finitely generated normal monoid with $U(L) = 0$ and let C be the cone with vertex 0 and spanned by L. Then*

$$x_i, z \in K(L), \quad \bigcap_i (x_i + L) = z + L \Rightarrow \bigcap_i (x_i + C) = z + C.$$

Proof. Replacing x_i's by $x_i - z$'s, we see that without loss of generality it can be assumed $z = 0$. Due to the normality of L there exists a system of $\mathbb{R}$-linear maps $v_F : \mathbb{R}^r \longrightarrow \mathbb{R}$, where F varies over the set of all codimension 1 faces of C such that $L = \{x \in K(L) \mid v_F(x) \geq 0 \text{ for all } F\}$. In terms of these maps by Corollary 2.1.6 and Lemma 2.3.2 the equality $(\bigcap_i (x_i + L))^n = L$ (for each n) is equivalent to the validity of the implications

$$x \in K(L), \;\; \forall F \, \forall i \;\; v_F(x) \geq v_F(x_i) \cdot n \quad \Leftrightarrow \quad \forall F \;\; v_F(x) \geq 0. \qquad (*)_n$$

Since for each $x \in \mathbb{Q}^r$ $(r = \mathrm{rank}(L))$ there exists natural n for which $nx \in K(L)$, (*) implies the validity of (here (*) means $\bigcup_n (*)_n$)

$$x \in \mathbb{Q}^r, \;\; \forall F \, \forall i \;\; v_F(x) \geq v_F(x_i) \quad \Leftrightarrow \quad \forall F \;\; v_F(x) \geq 0. \qquad (**)$$

In its turn (**) is equivalent to the equality

$$\bigcap_i (x_i + C) \cap \mathbb{Q}^r = C \cap \mathbb{Q}^r.$$

Now our lemma easily follows from the observations that $C \subset \bigcap_i (x_i + C)$ are both convex closed sets whose rational points coincide (to complete the proof we could use, for example, separation theorems from [Be, §11.4.1]. ■

Lemma 2.4.3. *Let M and N be as in Theorem 2.4.1 and $x_i \in K(M)$. Assume $\bigcap_i (x_i + M) \in \mathrm{Pr}(M)$. Then $\bigcap_i (x_i + N) \in \mathrm{Pr}(N)$.*

Proof. We shall prove a bit stronger result which obviously will imply the lemma. Namely, we claim that $\bigcap_i (x_i + N) = N$ whenever $\bigcap_i (x_i + M) = M$, where $x_i \in K(M)$. Indeed, assume $\bigcap_i (x_i + M) = M$. Denote by $C(M)$ and $C(N)$ the cones with vertices at 0 spanned by M and N, respectively. By Lemma 2.4.2, $\bigcap_i (x_i + C(M)) = C(M)$. We have the representations of types $C(M) = \bigcap_F H(F)$ and $C(N) = \bigcap_G H(G)$, where $H(F)$'s and $H(G)$'s are the closed half-spaces in $\mathbb{R}^r$ $(r = \mathrm{rank}(M) = \mathrm{rank}(N))$ with boundaries containing 0, and F and G vary over the sets of all codimension 1 faces of $C(M)$ and $C(N)$, respectively. Let us show that $\bigcap_i(x_i + H(G)) = H(G)$ for each G. Since $\Phi(M)$ is inscribed in $\Phi(N)$, we have $\{H(G)\}_G \subset \{H(F)\}_F$. But for each F the equality $\bigcap_i(x_i + C(M)) = C(M)$ obviously implies $\bigcap_i(x_i + H(F)) = H(F)$. Now the equalities $\bigcap_i(x_i + H(G)) = H(G)$ established above imply the following one $\bigcap_i(x_i + C(N)) = C(N)$. Consequently, $\bigcap_i(x_i + N) = \bigcap_i((x_i + C(N)) \cap K(N)) = C(N) \cap K(N) = N$. ■

It should be noted that in the proof above we have not used the equality $K(M) = K(N)$ (valid for inscribed monoids), we have only used the inclusion $K(M) \subset K(N)$.

Lemma 2.4.4. *Let M and N be as in Theorem* 2.4.1 *and $x_i, y_j \in K(M)$. Assume $\bigcap_i(x_i + M) = \bigcap_j(y_j + M)$, then $\bigcap_i(x_i + N) = \bigcap_j(y_j + N)$.*

Proof. Since $\mathrm{Div}(M)$ is a group (Proposition 2.1.1 and Corollary 2.1.3), by Proposition 2.1.4 there exist elements $z_k \in K(M)$ such that $\bigcap_{i,k}(x_i + z_k + M) = \bigcap_{j,k}(y_j + z_k + M) = M$. In this situation, as it has been shown in the proof of Lemma 2.4.3, $\bigcap_{i,k}(x_i+z_k+N) = \bigcap_{j,k}(y_j+z_k+N) = N$. Hence, again by Proposition 2.1.4 (and Proposition 2.1.1 and Corollary 2.1.3) we get $\bigcap_i(x_i + N) = (\bigcap_k(z_k + N))^{-1} = \bigcap_j(y_j + N)$. ■

Clearly, Lemma 2.4.4 implies Lemma 2.4.3.

Proof of Theorem 2.4.1. By Lemma 2.4.4 the correspondence $\bigcap_i(x_i+M) \longmapsto \bigcap_i(x_i+N)$ gives us a well defined map from $\mathrm{Div}(M)$ to $\mathrm{Div}(N)$. Denote it by f. By Proposition 2.1.4, f is a group homomorphism. By Lemma 2.4.3, f naturally induces a group homomorphism $\overline{f} : Cl(M) \to Cl(N)$. We claim that the restriction of $\overline{f}$ to the torsion part $Cl(M)_{tor}$ is a monomorphism. This means that we have to show that $\bigcap_i(x_i + M) \in \mathrm{Pr}(M)$ whenever $\bigcap_i(x_i + N) \in \mathrm{Pr}(N)$ and $\bigcap_i(ax_i + M) \in \mathrm{Pr}(M)$ where $x_i \in K(M)$ and $a \in \mathbb{N}$. Since $\mathrm{rank}(M) = \mathrm{rank}(N)$ and M is normal in N, by Proposition 1.2.1 $K(M) = K(N)$. Denote this common group of quotionts by K. Assume $x_i \in K$ and $a \in \mathbb{N}$ satisfy the above-mentioned conditions. This just means that there exist $u, v \in K$ such that $\bigcap_i(x_i + N) = u + N$ and $\bigcap_i(ax_i + M) = v + M$. As it was shown in the proof of Lemma 2.4.3, the equality $\bigcap_i(ax_i + M) = v + M$ implies $\bigcap_i(ax_i + N) = v + N$. On the other hand, the equality $\bigcap_i(x_i + N) = u + N$ implies $\bigcap_i(ax_i+N) = au+N$. Therefore, $au+N = v+N$. Since $U(N)$ is trivial, we get $au = v$. By Lemma 2.4.2, $\bigcap_i(ax_i + C(M)) = au + C(M)$, where $C(M)$ is the cone spanned by M (with vertex at 0). For a subset $Y \subset \mathbb{R}^r$ ($r = \mathrm{rank}(M)$) and an element $t \in \mathbb{R}$ we put $tY = \{ty \mid y \in Y\} \subset \mathbb{R}^r$. We have $a^{-1}(\bigcap_i(ax_i + C(M))) = \bigcap_i(x_i + a^{-1}C(M)) = \bigcap_i(x_i+C(M))$ and $a^{-1}(au+C(M)) = u+C(M)$. Therefore, $\bigcap_i(x_i+C(M)) = u+C(M)$. Finally, we get $\bigcap_i(x_i+M) = \bigcap_i(x_i+C(M))\cap K = (u+C(M))\cap K = u+M \in \mathrm{Pr}(M)$. ■

§ 2.5. Examples of computations of divisor clas groups.

Example 2.5.1 (Chouinard, [C1]). For a Krull domain R and a finitely generated normal monoid M the ring $R[M]$ is a Krull domain and $Cl(R[M]) = Cl(R) \oplus Cl(M)$.

Example 2.5.2 (Chouinard, [C1]). Any finitely generated abelian group can be realized as $Cl(M)$ for some finitely generated normal monoid M; furthermore, M can be chosen with trivial $U(M)$.

Example 2.5.3 ([C1, G4]). For a finitely generated normal monoid M the group $Cl(M)$ is a torsion group iff $\Phi(M)$ is a simplex. This also follows from Theorem 2.3.1.

Example 2.5.4. For finitely generated normal monoids M and N, $Cl(M \oplus N) = Cl(M) \oplus Cl(N)$. This is an easy consequence of Example 2.5.1.

Example 2.5.5. For a finitely generated normal monoid M, $Cl(M) = 0$ iff M is isomorphic to $\mathbb{Z}^r \oplus \mathbb{Z}_+^s$ for some $r, s \in \mathbb{Z}_+$. This follows from the following observations: (a) $M = U(M) \oplus M'$ for an appropriate M'; (b) polynomial and Laurent polynomial rings with coefficients in a field are factorial; (c) $K[M]$ is factorial iff M is a free monoid, where K is a field and M is a finitely generated normal monoid with $U(M) = 0$.

Example 2.5.6. In general, $\ker(\overline{f})$ is not trivial, where $\overline{f} : Cl(M) \to Cl(N)$ is the map mentioned in the proof of Theorem 2.4.1. Indeed, consider any rational polyhedron P inscribed in the symplex $\Phi(\mathbb{Z}_+^r)$ for some $r \geq 3$. We can choose $P \neq \Phi(\mathbb{Z}_+^r)$. Then, putting $M = \mathbb{Z}_+^r(P)$ and $N = \mathbb{Z}_+^r$, we get (according to Theorem 2.3.1 and Example 2.5.5) $Cl(M) \neq 0$ and $Cl(N) = 0$. Hence $\ker(\overline{f}) \neq 0$.

Example 2.5.7. For any natural n there exists a finitely generated normal monoid M of rank 3 such that $Cl(M) = \mathbb{Z}^n$. Indeed, we have only to choose the polyhedron P, mentioned in the previous example, to have $n+3$ edges, put $M = \mathbb{Z}_+^3(P)$ and then apply Theorem 2.4.1.

Example 2.5.8 (=concluding remark of [G5]). For any finitely generated abelian group A there exists a Φ-simplicial growth normal monoid M such that $Cl(M) = A$ (for definitions see Remark 3.3.8). Indeed, assume $A = \mathbb{Z}^r \oplus A_{tor}$ for some $r \in \mathbb{Z}_+$. By Example 2.5.7 there exists a finitely generated rank 3 monoid M with $U(M) = 0$ such that $Cl(M) = \mathbb{Z}^r$. By Examples 2.5.2 and 2.5.3 there exists a finitely generated normal monoid N such that $Cl(N) = A_{tor}$ and $\Phi(N)$ is a simplex. It is clear that $K \oplus L$ is of simplicial growth whenever K is so and $\Phi(L)$ is a simplex. On the other hand, all finitely generated monoids of rank 3 and with trivial units are of Φ-simplicial growth [G5]. It only remains to apply Example 2.5.4.

Example 2.5.9. Assume $r \in \mathbb{N}$ and $\Phi(\mathbb{Z}_+^r) = D \cup G$, where D is a simplex containing exactly one of the vertices of the simplex $\Phi(\mathbb{Z}_+^r)$, and $D \cap G$ is a common face of D and G. In [G4] we called the monoids of type $\mathbb{Z}_+^r(D)$ the truncated monoids. As it was shown in [G4], the divisor class groups of truncated monoids are always finite cyclic. Now we are interested in divisor class groups of monoids of type $\mathbb{Z}_+^r(G)$ (we could call these monoids co-truncated). Theorems 2.3.1 and 2.4.1 imply directly

$Cl(\mathbb{Z}_+^r(G)) = \mathbb{Z}$ (it should only to be noted that $Cl(M)$ is finitely generated when M is finitely generated [Cl]). In particular, $Cl(\mathbb{Z}_+(1,0,0) + \mathbb{Z}_+(0,1,1) + \mathbb{Z}_+(1,1,0) + \mathbb{Z}_+(0,0,1)) = \mathbb{Z}$.

Example 2.5.10. Proposition 2.3.2 can be generalized as follows. For a subset $S \subset \mathbb{N}$ and a monoid M let $S^{-1}M$ denote the smallest monoid containing M and s divisible for each $s \in S$. Then we claim that for any finitely generated monoid M with $U(M) = 0$ the normality of M implies $S^{-1}Cl(M) = Cl_0(S^{-1}M)$. As a result we obtain that for any finite abelian group A there exist $r \in \mathbb{N}$, prime p and an intermediate monoid $\mathbb{Z}_+^r \subset M \subset \mathbb{Q}_+^r$ satisfying the conditions that M is p divisible, M is normal and $Cl_0(M) = A$. Indeed, the existence of a finitely generated normal monoid N situated between $\mathbb{Z}_+^{\mathrm{rank}(N)}$ and $\mathbb{Q}_+^{\mathrm{rank}(N)}$ and satisfying the equality $Cl(N) = A$ follows from Examples 2.5.2 and 2.5.3 and [G4, Proposition 1.1]. Now, we have only to choose prime p not dividing the order of A, and apply the aforementioned claim to the subset $\{p\} \subset \mathbb{N}$.

PART III

K-THEORY OF MONOID RINGS

§ 3.1. Hemispherical Monoids. A (finite rank) monoid M will be called hemispherical if $\Phi(M)$ is an open hemisphere and, in addition, if the closure in $S^{\mathrm{rank}(M)-1}$ of $\Phi(M)$ is a rational hemisphere, then M will be called the rational one. We see that the class of monoids treated in Lemma 1.3.3 coincides with that of rational hemispherical normal monoids with finitely generated groups of fractions. We could ask whether such monoids are filtered unions of free monoids themselves (instead of monoids isomorphic to $(\mathbb{Z}_+^r)_*$). Clearly, this would imply Lemma 1.3.3. However, it can be shown that this is not the case. A bit stronger result will be proved below.

In the proof of Lemma 1.3.3 it was established that

Proposition 3.1.1. *Rational hemispherical normal monoids with finitely generated groups of fractions are exactly the monoids isomorphic to* $H_r = (\mathbb{Z}^{r-1} \oplus \mathbb{Z}_+) \backslash \{(a,0) | a \in \mathbb{Z}^{r-1},\ a \neq 0\}$ *for some* $r \in \mathbb{N}$.

Put $c^{-1}\mathbb{Z}_+ = \{z/c^s | z, s \in \mathbb{Z}_+\} \subset \mathbb{Q}_+$, where $c \in \mathbb{N}$, and $H_{r,c} = (\mathbb{Z}^{r-1} \oplus c^{-1}\mathbb{Z}_+) \backslash \{(a,0) | a \in \mathbb{Z}^{r-1},\ a \neq 0\}$. Clearly, $H_{r,c}$ is a rational hemispherical normal monoid which is a filtered union of monoids isomorphic to H_r and $\Phi(H_{r,c}) = \Phi(H_r)$.

Theorem 3.1.2. (a) *For any natural numbers r and c for $r \geq 2$ the monoid $H_{r,c}$ is not a filtered union of free monoids;*

(b) *For a (commutative) regular ring R and natural numbers r and c, such that c has at least two distinct prime divisors, the natural homomorphisms $K_i(R) \longrightarrow K_i(R[H_{r,c}])$ (for Quillen's K-groups) are isomorphisms for all $i \geq 0$.*

Proof. (a) Let r and c be as in the point (a). Fix arbitrarily a subset $G \subset \mathbb{Z}^{r-1}$ which generates $\mathbb{Z}^{r-1}$ as a monoid and consider the set $X = \{(g,1) | g \in G\} \subset H_{r,c}$. We claim that there does not exist a free intermediate monoid $X \subset F \subset H_{r,c}$ which implies (a). Suppose such a free monoid F does exist. Denote its basis by $\{m_1, \ldots, m_r\}$. Assume

$m_i = (m_{i1}, \dots, m_{ir})$ for some $m_{i1}, \dots, m_{ir-1} \in \mathbb{Z}$ and $m_{ir} \in c^{-1}\mathbb{Z}_+$. Choose among the numbers $m_{ir} (i \in [1, r])$ the minimal one, say $m_{i_0 r}$ (maybe occuring in some of m_i-s). Clearly, from the very beginning we could assume $(0, 1) \in X$. In this situation $\mathbb{Z}^{r-1} \oplus 0 \subset K(F)$. Simultaneously, some positive multiple of $(0, \dots, 0, m_{i_0 r})$ will also belong to F. Since $(0, \dots, 0, m_{i_0 r}) \in K(F)$, we obtain (by normality of free monoids) $(0, \dots, 0, m_{i_0 r}) \in F$. But then $(0, \dots, 0, m_{i_0 r}) \in \{m_1, \dots, m_r\}$, because otherwise $(0, \dots, 0, m_{i_0 r}) = x_{i_1} m_{i_1} + \dots + x_{i_k} m_{i_k}$ for some natural numbers $x_{i_1}, \dots, x_{i_k}$, where $2 \le k \le r$, which contradics the minimality of $m_{i_0 r}$ (note that by the definition of $H_{r,c}$ the numbers m_{ir} are all strictly positive). Consider now the natural projection $\pi : H_{r,c} \longrightarrow \mathbb{Z}^{r-1}$. Since $\pi(F) = \mathbb{Z}^{r-1}$, we get that the set $\{\pi(m_i) | i \ne i_0\} \subset \mathbb{Z}^{r-1}$ must generate $\mathbb{Z}^{r-1}$ as a monoid. But this is a contadiction, since $\mathbb{Z}^{r-1}$ cannot be generated as a monoid by a system consisting of $r - 1$ elements.

(b) We need some information on transfer maps in algebraic K-theory, namely, we mean the "projection formula". Let $f : A \longrightarrow B$ be a ring homomorphism of commutative rings such that B becomes a finitely generated projective A-module via this homomorphism. Then there exist group homomorphisms $f^* : K_i(B) \longrightarrow K_i(A)$ for all $i \ge 0$ such that $f^*(y \cdot f_*(x)) = f^*(y) \cdot x$ for all $y \in K_0(B)$ and $x \in K_i(A)$ (f_* denotes the functorial homomorphism $K_i(A) \longrightarrow K_i(B)$), where the multiplication is meant in the sense of K_0-module structure on K_i. Actually, a more general formula is valid. Namely, for a ring homomorphism $f : A \longrightarrow B$, making B a finitely generated A-module of finite projective dimension, there exist homomorphisms $f^* : K_i(B) \longrightarrow K_i(A)$, such that $f(y \cdot f_*(x)) = f^*(y) \cdot x$ for all $y \in K_i(B)$ and $x \in K_j(A)$ $(i, j \in \mathbb{Z}_+)$ [Ge]. f^* here means the homomorphism induced by the exact functor $P_\infty(B) \longrightarrow P_\infty(A)$ of scalar restriction from B to A. However, the special case mentioned above can be obtained quite directly. It merely follows from the application of Quillen's Q-construction [Q] to the isomorphic exact functors

$$X \to Y \underset{B}{\otimes} (B \underset{A}{\otimes} X) \quad \text{and} \quad X \to Y \underset{A}{\otimes} X$$

from $P_\infty(A)$ to $P_\infty(B)$, where $Y \in \mathbb{P}(B)$. We shall use the following consequence of the "projection formula" for a ring homomorphism $f : A \longrightarrow B$, for which B is a free A-module of rank n, there exist homomorphisms $f^* : K_i(B) \longrightarrow K_i(A)$ (for all $i \ge 0$) such that $f^* f_* : K_i(A) \longrightarrow K_i(A)$ is a multiplication by n.

Now, let R, r and c be as in the point (b). Suppose p and q are distinct prime divisors of c. Fix arbitrarily a basis $\{e_1, \dots, e_{r-1}\}$ of $\mathbb{Z}^{r-1}$ and for each natural i and k put

$$M_{ik} = \Big(\bigcup_{\substack{0 \le a_1, \dots, a_{r-1} < p^k \\ \max(a_1, \dots, a_{r-1}) > 0}} \big(H_{r,c} + \frac{a_1}{p^k} e_1 + \dots + \frac{a_{r-1}}{p^k} e_{r-1} + \frac{1}{c^i} e_r \big) \Big) \bigcup H_{r,c} \,,$$

where $\mathbb{Z}^{r-1}$ is identified with $\mathbb{Z}^{r-1} \oplus 0$, e_r denotes $(0, \dots, 0, 1)$, a_j-s are nonnegative integers satisfying the above indicated inequalities and the sums are considered in $\mathbb{Q}^r$ ($x + H_{r,c}$ means $\{x + h | h \in H_{r,c}\}$). We claim that

(1) M_{ik}-s are the submonoids of $\mathbb{Q}^r$ for all i and k;

(2) the natural inclusions $R[H_{r,c}] \to R[M_{ik}]$ make $R[M_{ik}]$-s free $R[H_{r,c}]$-modules of rank $p^{k(r-1)}$, respectively;

(3) M_{ik} is contained in M_{ik+1} and M_{i+1k} and the filtered union of M_{ik}-s $(i, k \in \mathbb{N})$ is a rational hemispherical p divisible monoid.

Let us first show how these claims imply that

$$\operatorname{coker}\big(K_n(R) \longrightarrow K_n(R[H_{r,c}])\big)$$

are the p-groups for all $n \geq 0$. Since the very same arguments imply that these groups are q-groups, we will be done, i.e. the homomorphisms $K_n(R) \to K_n(R[H_{r,c}])$ will be isomorphisms for all $n \geq 0$.

Denote by M the filtered union $\underset{i,k}{\cup} M_{ik}$ which by (3) is a hemispherical p divisible monoid. Obviously, $\Phi(M)$ is a filtered union of rational simplices, say $\Phi(M) = \underset{s}{\cup} D_s$. Then M will be a filtered union of monoids $M(D_s)_*$, which are p divisible as well. By Approximation Theorem A from [G3], $M(D_s)_*$ are the filtered unions of free monoids. Now, by the continuity of K-functors and Quillen's theorem on K-regularity of regul;ar rings [Q], we obtain $K_n(R) \xrightarrow{\sim} \varinjlim K_n(R[M(D_s)_*]) = K_n(R[M])$. Assume $[z] \in \operatorname{coker}(K_n(R) \longrightarrow K_n(R[H_{r,c}]))$ for some $n \geq 0$ and $z \in K_n(R[H_{r,c}])$. Again by continuity of K-functors the image z_* of z in $K_n(R[M_{ik}])$ will belong to the image of $K_n(R)$ for appropriate indices i and k. Therefore, there exists $x \in \operatorname{im}(K_n(R) \to K_n(R[H_{r,c}]))$ such that $z - x$ maps into 0 in $K_n(R[M_{ik}])$. Using (2) and the aforementioned consequence of "projection formula", we get $p^{k(r-1)}(z-x) = 0$ or equivalently, $p^{k(r-1)}[z] = 0$.

It remains to check (1),(2) and (3).

(1) Let $\pi : \mathbb{Q}^r \to \mathbb{Q}^{r-1}$ be the projection corresponding to the last coordinate. Obviously, $\pi(M_{ik}) = \mathbb{Z}^{r-1}/p^k$ $(= \{a/p^k | a \in \mathbb{Z}^{r-1}\} \subset \mathbb{Q}^{r-1})$. Assume $u, v \in M_{ik}$. Since $\pi(u+v) \in \pi(M_{ik})$, we have to check that

$$u_r + v_r \begin{cases} \geq 1/c^i & \text{if } \pi(u+v) = \frac{a_1}{p^k}e_1 + \cdots + \frac{a_{r-1}}{p^k}e_{r-1} \\ & \text{for some } a_j \in \mathbb{Z}_+ \text{ such that } 0 \leq a_j < p^k \\ & \text{and } \max_j(a_j) > 0, \ \ j \in [1, r-1], \\ > 1/c^i & \text{if } \pi(u+v) \text{ is not of the aforementioned type} \\ & \text{and does not belong to } \mathbb{Z}^{r-1}, \\ > 0 & \text{if } \pi(u+v) \in \mathbb{Z}^{r-1}, \end{cases}$$

where u_r and v_r denote r-th coordinates of u and v respectively. But these inequalities immediately follow from the observations that $u_r + v_r > u_r$ and $u_r + v_r > v_r$, whenever $u \neq 0 \neq v$ and that $\pi(u+v) \notin \mathbb{Z}^{r-1}$ implies at least one of $\pi(u)$ and $\pi(v)$ does not belong to $\mathbb{Z}^{r-1}$ (in this situation we necessarily have $u_r \geq 1/c^i$ or $v_r \geq 1/c^i$, respectively).

(2) Let us show that the system

$$\Big\{\frac{a_1}{p^k}e_1+\cdots+\frac{a_{r-1}}{p^k}e_{r-1}+\frac{1}{c^i}e_r\Big|a_j\in\mathbb{Z}_+,\ \ 0\le a_j<p^k,$$
$$\max_j(a_j)>0,\ \ j\in[1,r-1]\Big\}\cup\{0(\in M_{ik})\}$$

determines a free $R[H_{r,c}]$-basis of $R[M_{ik}]$. In general, we have the following observation (see also Lemma 3.3.3 below): for an extension of (commutative cancellative) monoids $G\subset H$ and a subset $\{h_1,\dots,h_s\}\subset H\backslash K(G)$, such that $h_x-h_y\notin K(G)$ for all distinct $x,y\in[1,s]$ and $H=G\cup\bigcup_{x=1}^{s}(h_x+G)$, the ring extension $A[G]\subset A[H]$ (for an arbitrary commutative ring A) makes $A[H]$ a free $A[G]$-module with basis $\{0,h_1,\dots,h_s\}$ (0 here means the neutral element of G). Now the equality $\pi(M_{ik})=\mathbb{Z}^{r-1}/p^k$ ensures us that the extension $H_{r,c}\subset M_{ik}$ and the considered system fall into the situation described above.

(3) The inclusions $M_{ik}\subset M_{ik+1}$ and $M_{ik}\subset M_{i+1k}$ are trivial. Suppose $M=\bigcup_{i,k}M_{ik}$. Then $\pi(M)=p^{-1}\mathbb{Z}^{r-1}=\bigcup_{s\in\mathbb{N}}\mathbb{Z}^{r-1}/p^s$ (notations as above). It is also easy to show that for each $a\in p^{-1}\mathbb{Z}^{r-1}$ one has $\pi^{-1}(a)=$ either $\{(a,b)|b\in(c^{-1}\mathbb{Z}_+)^+\}$ or $\{(a,b)|b\in c^{-1}\mathbb{Z}_+\}$ with respect to $a\neq 0$ or $a=0$. Then $M=(p^{-1}\mathbb{Z}^{r-1}\oplus c^{-1}\mathbb{Z}_+)\backslash\{(a,0)|a\in p^{-1}\mathbb{Z}^{r-1},\ a\neq 0\}$, which is obviously p divisible. ■

Remark 3.1.3. If we consider monoids of type $(\mathbb{Z}_+^{r-1}\oplus c^{-1}\mathbb{Z}_+)\backslash\{(a,0)|a\in\mathbb{Z}_+^{r-1},\ a\neq 0\}$ instead of monoids $H_{r,c}$, then the very same arguments would imply the analogous isomorphisms for K-groups. However, it turns out that such monoids are the filtered unions of free monoids and, thus, the above mentioned isomorphisms are actually trivial ones.

Remark 3.1.4. It is not clear whether the groups $K_i(R)$ and $K_i(R[H_r])$ coincide for regular R and $i\ge 1$.

Remark 3.1.5. Call a hemispherical monoid M purely irrational if the hyperplane spanned by the boundary of the closure of $\Phi(M)$, is of type $(a_1X_1+\cdots+a_rX_r=0)$, where a_j-s constitute a $\mathbb{Q}$-linearly independent system. On the contrary to the rational hemispherical monoids, it turns out that all purely irrational hemispherical normal monoids are the filtered unions of free monoids [G6]. This result could also be used in the proof of Theorem 1.5.2.

§ 3.2. Excision in Algebraic K-theory of Monoid Rings. Statement of the results. A subset $I\subset M$ of a monoid M is called ideal if (writing multiplicatively) $IM\subset I$ or RI is an ideal in $R[M]$ for any ring R. An ideal I of a monoid M is called radical if $x^n\in I$ for some $x\in M$ and $n\in\mathbb{N}$ implies $x\in I$. It can be proved that for a commutative reduced ring R and an ideal I of a monoid M the ideal $RI\subset R[M]$ is radical iff I is radical. The rings are not assumed to be commutative and the term 'monoid ring' will include the commutativity property between the scalars and 'pure monomials' (i.e. the elements from monoid).

Theorem 3.2.1. (a) *Let R be a ring and M be a p divisible monoid for a prime p. Assume* $\operatorname{rank}(M) < \infty$, *$i \in \mathbb{N}$ and I is a proper radical ideal of M. Then the following implication holds*

$$K_i(R) \xrightarrow{\sim} K_i(R[M]) \Rightarrow K_i(R) \xrightarrow{\sim} K_i(R[M]/RI),$$

(b) *Assume R is a ring satisfying the condition $K_i(R) \xrightarrow{\sim} K_i(R[M])$ for all p divisible monoids M without nontrivial invertibles for a natural i and a prime p. Then $K_i(R) \xrightarrow{\sim} K_i(R[M]/RI)$ for all p divisible monoids M without nontrivial units and all proper radical ideals I of M;*

(c) *For $i = 0$ we can omit in* (a) *the p divisibility assumption on M and the radicality assumption on I (keeping the condition* $\operatorname{rank}(M) < \infty$*). For the commutative coefficient rings and the functor SK_1 we have an analogous implication (monoid and its ideal are arbitrary)*

$$\operatorname{rank}(M) < \infty, SK_1(R) \xrightarrow{\sim} SK_1(R[M]) \Rightarrow SK_1(R) \xrightarrow{\sim} SK_1(R[M]/RI).$$

We would like to avoid the restriction $\operatorname{rank}(M) < \infty$ in (a) (and (c) as well). Analysing the proof we present below, where the ideals corresponding to the interiors of $\Phi(M)$ (and its faces) arise, it becomes clear that one (and possible, main) of the difficulties in this direction is the following: if we define infinite dimensional convex polyhedra as filtered unions of the finite dimensional (i.e. ordinary) ones, then the very notion of an interior vanishes. Correspondingly, there arises

Question. Let $R[M]$ be a monoid ring with some p divisible monoid M (may be with $\operatorname{rank}(M) = \infty$) for some prime p and I be a proper radical ideal of M and $i \in \mathbb{N}$. Then does the implication similar to thast mentioned in Theorem 3.2.1 (a) hold?

A partially positive answer to the question risen in [G3] is given by the following

Theorem 3.2.2. *Let R be a K-regular (for example, regular) ring, $r \in \mathbb{N}$ and p a prime number. Then for an intermediate p divisible monoid $\mathbb{Z}_+^r \subset M \subset \mathbb{Q}_+^r$ we have the natural isomorphisms*

$$K_i(R) \xrightarrow{\sim} K_i(R[M]),$$

where $i \in \mathbb{Z}_+$.

Remark 3.2.3. Observe that a monoid M can be isomorphically situated between $\mathbb{Z}_+^r$ and $\mathbb{Q}_+^r$ if and only if $\Phi(M)$ is a simplex (see also Example 2.5.10).

Excision. We shall say that a ring I without unit satisfies excision (in algebraic K-theory) if for all rings R, containing I as a two-sided ideal the natural maps $K_i(\widetilde{I}, I) \to K_i(R, I)$ are the isomorphisms for all indices i, where $\widetilde{I} = \mathbb{Z} \ltimes I$ is the ring obtained from I by adjoining unit. Hence, for such a ring I without unit and a ring R, containing I as a two-sided ideal we have the long exact sequence

$$\cdots \to K_{i+1}(R/I) \to K_i(I) \to K_i(R) \to K_i(R/I) \to \cdots$$

($i \in \mathbb{Z}$), where $K_i(I) = K_i(\widetilde{I}, I)$.

Theorem 3.2.4 (Suslin-Wodzicki, [SW]). *A ring I without unit satisfies excision if for any finite system of elements $a_1, \dots, a_n \in I \quad (n \in \mathbb{N})$ there exist elements $b_1, \dots, b_n, c, d \in I$ such that $a_i = b_i cd$ for all $i \in [1, n]$ and left annilihators of elements c and cd coincide.*

We shall use the following notation: for a monoid M and its ideal I put $\sqrt{I} = \{x \in M | x^n \in I \text{ for some } n \in \mathbb{N}\}$. Clearly, $\sqrt{I}$ is the smallest radical ideal containing I.

Proposition 3.2.5. *Let M be a p divisible monoid for some prime p and $m \in M \backslash U(M)$. Then for an arbitrary ring R the ring $R\sqrt{mM} \quad (\subset R[M])$ without unit satisfies excision.*

Proof. In this proof we shall use additional notations for a monoid operation. We just claim that the conditions of Theorem 3.2.4 are satisfied for $\sqrt{m+M}R$. Obviously, without loss of generality we can assume $a_1, \dots, a_n \in \sqrt{m+M}$. Let us show that there exists $x_0 \in \mathbb{N}$ such that for all $x > x_0$

$$a_1 - m/p^x, \dots, a_n - m/p^x \in \sqrt{m+M} \ .$$

Let G be the group generated by $\{a_1, \dots, a_n, m\}$ and N be the normalization of $G \cap (\sqrt{m+M} \cup \{0\})$ in $\sqrt{m+M} \cup \{0\}$. Since M is p divisible, so will be $\sqrt{m+M} \cup \{0\}$ and, hence, N. In addition, we have $\operatorname{rank}(N) < \infty$. Therefore, we can consider the Φ-correspondence for N. Of course, for any natural x the elements

$$a_1 - m/p^x, \dots, a_n - m/p^x$$

belong to $K(N)$. Observe now that when x is large enough, the points $\Phi(a_i - m/p^x)$ are sufficiently close to $\Phi(a_i)$, respectively. Denote by G_i the groups generated by $\{a_i, m\}$ and let N_i be the normalizations of $G_i \cap N$ in N. N_i is p divisible for all $i \in [1, n]$. $\Phi(N_i)$ is 1− or 0-dimensional convex subset of $\Phi(N)$ for each $i \in [1, n]$. The conditions $a_i \in \sqrt{m+M}$, obviously, imply $\Phi(a_i) \in \operatorname{int}(\Phi(N_i))$. Hence, by the aforementioned remark, $\Phi(a_i - m/p^x) \in \operatorname{int}(\Phi(N_i))$, respectively (of course, taking x large, we can assume $a_i - m/p^x \neq 0$ for all $i \in [1, n]$). Equivalently, $a_i - m/p^x \in (N_i)_* \backslash \{0\}$ for x large enough, because $a_i - m/p^x \in K(N_i) = K((N_i)_*)$, and in its turn, $(N_i)_*$ is normal, since N_i is seminirmal, being p divisible (see Lemma 1.4.3): $2y, 3y \in N_i \Rightarrow py \in N_i \Rightarrow y \in N_i$. Therefore, for sufficiently large x-s

$$a_i - m/p^x \in ((N_i)_* \backslash \{0\}) \subset \sqrt{m+M}.$$

Now put $b_i = a_i - m/p^x, i \in [1, n]$ (x as above), $c = m/p^{x+1}$ and $d = (p-1)m/p^{x+1}$. Returning to the multiplicative notations, we get $a_i = b_i cd \ (i \in [1, n]), \operatorname{ann}_{\text{left}}(c) = \operatorname{ann}_{\text{left}}(cd) = 0$ in $\sqrt{mM}R$. ■

Corollary 3.2.6. *Let R be a ring and M a monoid of a finite* rank. *Then p divisibility of M implies that the ring* $\operatorname{int}(M)R \quad (\subset R[M])$ *without unit satisfies excision, where* $\operatorname{int}(M) = M_* \backslash \{1\}$ *and p is a prime number.*

Proof. immediately follows from the observation that $\mathrm{int}(M) = \sqrt{mM}$ for arbitrary $m \in \mathrm{int}(M)$.

Proof of Theorem 3.2.1 (a) *and* (b) Let us first show that (a) implies (b). Assume R, p, M and I are as in the theorem. M is a filtered union of finitely generated monoids without nontrivial units, say $M = \bigcup_s M_s$. Denote by N_s the normalizations (i.e. integral closures) of M_s-s in M, respectively. Then N_s will be p divisible, of finite ranks and without nontrivial units. Furthermore, $R[M]/RI$ will be a filtered union of $R[N_s]/RI_s$-s, where $I_s = I \cap N_s$ are radical ideals of N_s-s, respectively. Hence, the algebras treated in (b) are the filtered unions of those treated in (a). By our conditions on R we have $K_i(R) \xrightarrow{\sim} \varinjlim K_i(R[N_s]/RI_s) = K_i(R[M]/RI)$.

Let us prove (a). First some remarks on s.c. extremal submonoids (corresponding in terms of our Φ-correspondence to the faces of convex polyhedra). Recall that for a monoid extension $A \subset B$ the submonoid A is called extremal in B if $b_1, b_2 \in B$, $b_1 + b_2 \in A \Rightarrow b_1, b_2 \in A$. For any monoid B the group of its units $U(B)$ is the smallest extremal submonoid of B. Clearly, B itself is the biggest one.

Lemma 3.2.7. *Let B be a monoid and I its proper ideal. Then any maximal submonoid $A \subset B$ (with respect to the inclusion) avoiding I (i.e. $A \cap I = \varnothing$) is extremal.*

Proof. Let A be of the above mentioned type and $b_1 + b_2 \in A$ for some $b_1, b_2 \in B$. If $b_1 \notin A$, then the monoid spanned (in B) by A and b_1 meets I (in view of maximality of A). Assume $a + b_1 \in I$ for some $a \in A$. We have then a contradiction $a + b_1 + b_2 \in I \cap A$.

Lemma 3.2.8. *Bet B a monoid and I its proper ideal. Assume A is a maximal submonoid of B avoiding I and R is a ring. Then $R[A]$ is a retract of $R[B]/RI$.*

proof. Clearly, the composite map $R[A] \longrightarrow R[B] \longrightarrow R[B]/RI$ is injective. Thus we can identify $R[A]$ with its image in $R[B]/RI$. Now consider the map $f : R[B] \to R[A]$ determined by

$$f(b) = \begin{cases} b, & \text{if } b \in A, \\ 0, & \text{if } b \notin A. \end{cases}$$

Obviously, f passes through the map $R[B] \longrightarrow R[B]/RI$.

Lemma 3.2.9. *Let B be a monoid of a finite rank and I its any proper radical ideal. Then for a maximal submonoid $A \subset B$ avoiding I the set* $\mathrm{int}(A) \cup I$ *is an ideal of B. Moreover,* $\mathrm{int}(A) \cup I$ *is a radical one.*

Remark. Since $\mathrm{rank}(A) < \infty$, we can define the interior submonoid $A_* \subset A$ and, as it was in Corollary 3.2.6, we put $\mathrm{int}(A) = A_*^+$.

Proof. If $b \in A$, then $b + \mathrm{int}(A) \subset \mathrm{int}(A) \subset \mathrm{int}(A) \cup I$. Asume now $b \notin A$. We want to show that $b + c \in \mathrm{int}(A) \cup I$ for any $c \in \mathrm{int}(A)$. By maximality of A there exists $a \in A$ for which $b + a \in I$. Now assume $b + c \notin I$. By radicality $xb + xc \notin I$ as well for arbitrary $x \in \mathbb{N}$. On the other hand $xc = a + a'$ for some $a' \in A$ whenever x is large inough (the determing property of $\mathrm{int}(A)$. Now we obtain the contradiction $xb + xc = (b + a) + [(x-1)b + a'] \in I$ for x and a' as above. ■

Now suppose R, p, M and I are as in the point (a). Let N be a maximal submonoid of M avoiding I. Without loss of generality we shall assume $I \neq \varnothing$. The existence of N follows from the inequality $I \neq M$. We shall identify $R[N]$ with its natural image in $R[M]/RI$. Then by Lemma 3,2,9 $R\,\mathrm{int}(N)$ will be an ideal in the both rings $R[N]$ and $R[M]/RI$. Observe that N is p divisible. Hence by Corollary 3.2.6 we have the exact sequence

$$K_i(R\,\mathrm{int}(N)) \longrightarrow K_i(R[M]/RI) \longrightarrow K_i(R[M]/R(\mathrm{int}(N) \cup I)).$$

We claim that the map $K_i(R\,\mathrm{int}(N)) \longrightarrow K_i(R[M]/RI)$ is actually a zero-map. Again, by Corollary 3.2.6 one has the exact sequence

$$K_i(R\,\mathrm{int}(N)) \longrightarrow K_i(R[N]) \longrightarrow K_i(R[N]/R\,\mathrm{int}(N)).$$

By Lemma 3.2.8, $R[N]$ is a retract of $R[M]$ and, hence (since R is a retract of both of them) our conditions $K_i(R) \xrightarrow{\sim} K_i(R[M])$ $(i > 0)$ imply $K_i(R) \xrightarrow{\sim} K_i(R[N])$. Observe that R is a retract of $R[N]/R\,\mathrm{int}(N)$ (consider the map $g : R[N]/R\,\mathrm{int}(N) \longrightarrow R$ determined by $g([n]) = 0$ if $n \in N \backslash U(N)$ and $g([n]) = 1$ otherwise). Therefore, the map $K_i(R[N]) \longrightarrow K_i(R[N]/R\,\mathrm{int}(N))$ coincides with the split embedding $K(R) \longrightarrow K_i(R[N]/R\,\mathrm{int}(N))$. Thus $K_i(R\,\mathrm{int}(N)) \longrightarrow K_i(R[N])$ is a zero-map. Now the triviality of the map $K_i(R\,\mathrm{int}(N)) \longrightarrow K_i(R[M]/RI)$ follows from the commutative triangle

$$\begin{tikzcd}
 & K_i(R[N]) \arrow[d] \\
K_i(R\,\mathrm{int}(N)) \arrow[ur] \arrow[r] & K_i(R[M]/RI)\ ,
\end{tikzcd}$$

induced by the commutative triangle

$$\begin{tikzcd}
 & R[N] \arrow[d] \\
R\,\mathrm{int}(N) \arrow[ur] \arrow[r] & R[M]/RI\ .
\end{tikzcd}$$

Consider the chains of finite lengths of the type

$$A_1 \longrightarrow A_2 \longrightarrow \cdots \longrightarrow A_n\ ,$$

where n is a natural number, $A_1 = R[M]/RI$, each of A_j-s ($j \in [1, n]$) is of type $R[M]/RJ$ for some radical proper ideal J of M and the ring homomorphisms $A_j \longrightarrow A_{j+1}$ (for all $j \in [1, n-1]$) are of type $R[M]/RJ \longrightarrow R[M]/R(J \cup \mathrm{int}(L))$ for some submonoid $L \subset M$ satisfying the condition: L is maximal in the system of submonoids of M avoiding J. The very same arguments we have used above show that for any homomorphism $A_j \longrightarrow A_{j+1}$ of the above mentioned type the corresponding homomorphism $K_i(A_j) \longrightarrow K_i(A_{j+1})$ (for each i) is injective.

Let D_1 denote the diagram obtained by the union of all chains of the aforementioned type. We claim that D_i is directed. This claim is an immediate consequence of the following observation: for the two arbitrary maximal submonoids $N_1, N_2 \subset M$

avoiding I the intersection $\operatorname{int}(N_1) \cap \operatorname{int}(N_2)$ is empty and (hence) the rings A_j , involved in our diagram, are of the type $R[M]/R(I \cup \operatorname{int}(N_1) \cup \ldots \cup \operatorname{int}(N_{j-1}))$ for some maximal submonoids $N_1, \ldots, N_{j-1} \subset M$ avoiding I. In its turn this observation follows from Lemma 3.2.7 and the basic (and easy) observation that interiors of two extremal submonoids meet if and only if they do coincide. Taking all these remarks into account, we conclude that the natural homomorphism $K_i(R[M]/RI) \longrightarrow \varinjlim_{D_1} K_i(A_j) = K_i(\varinjlim D_1)$ is injective.

Remark. It may happen that $\varinjlim D_1 = R[U(M)]$. But, in general, this is not the case. For example, if $\Phi(M)$ is a finite polyhedron, then $\varinjlim D_1 \neq R[U(M)]$ providing $M \neq U(M)$. On the other hand, if $U(M)$ is trivial and $\Phi(M)$ is a convex set spanned by all rational points of same closed ball in $S^{\operatorname{rank}(M)-1}$ (with respect to the Euclidean metric), then $\varinjlim D_1 = R$.

Clearly, $\varinjlim D_1$ is of the type $R[M]/RJ$ for some intermediate radical ideal $I \subset J \subset M$. Now apply to $\varinjlim D_1$ the same procedure which has been carried out for $R[M]/RI$. Then we shall get a new directed diagram with similar properties. Denote it by D_2, then pass to $\varinjlim D_2$ and so on. In this manner we shall obtain the embeddings

$$K_i(R[M]/RI \to K_i(\varinjlim D_1) \to K_i(\varinjlim D_2) \to \ldots$$

We claim that our procedure must stop after some finite steps, moreover, the number k of these steps does not exceed $\operatorname{rank}(M) - 1$. In other words we claim that $\varinjlim D_k = R[U(M)]$ for some $k \leq \operatorname{rank}(M) - 1$. Indeed, let N be any extremal submonoid of M of rank $= r - 1$, where $r = \operatorname{rank}(M)$ (if such a monoid exists). It is clear that the image of $R\operatorname{int}(N)$ into $\varinjlim D_1$ is 0 (because, if N avoids I it is automatically maximal). Analogously, for any extremal submonoid $L \subset M$ with rank $= r - 2$ the image of $R\operatorname{int}(L)$ into $\varinjlim D_2$ is 0, and so on. We see that after some finite k $(\leq r-1)$ steps the image of $R\operatorname{int}(K)$ into $\varinjlim D_k$ is 0 whenever K is proper extremal submonoid of M, equivalently $\varinjlim D_k = R[U(M)]$ (we use the following natural conversation: for a group G of finite rank $\Phi(G)$ is the only face of the convex set $\Phi(G)$ without proper faces).

Since $R[U(M)]$ is a retract of $R[M]$ (and since $K_i(R[M]/RI)$ is embedded in $K_i(\varinjlim D_k)$), we obtain

$$K_i(R) \longrightarrow K_i(R[M]/RI) \longrightarrow K_i(R[U(M)]) \longrightarrow K_i(R[M]) \xrightarrow{\sim} K_i(R).$$

Hence, $K_i(R) \xrightarrow{\sim} K_i(R[M]/RI)$.

Remark 3.2.10. Actually we have proved a bit stronger result. Namely, our arguments show that if a monoid M satisfies the following conditions: every proper extremal submonoid $L \subset M$ is divisible by some prime $p(L)$ (depending on L) and the natural maps $K_i(R) \longrightarrow K_i(R[L])$ (R and i as in the theorem) are isomorphisms for all L-s, then $K_i(R) \xrightarrow{\sim} K_i(R[M]/RI)$ whenever I is a nonempty proper radical ideal of M. Here is an example of such a monoid M which itself is not divisible by some prime number but satisfies the aforementioned conditions.

Let r be a natural number ≥ 2 and $p_1, \ldots, p_r$ distinct arbitrary prime numbers. For each $j \in [1, r]$ let $\operatorname{mult}(j)$ denote the monoid homomorphism $\mathbb{Z}_+^r \longrightarrow \mathbb{Z}_+^r$ determined by

$$(a_1, \ldots, a_{j-1}, a_j, a_{j+1}, \ldots, a_r) \to (p_j a_1, \ldots, p_j a_{j-1}, a_j, p_j a_{j+1}, \ldots, p_j a_r)$$

Observe that $\operatorname{mult}(k) \circ \operatorname{mult}(j) = \operatorname{mult}(j) \circ \operatorname{mult}(k)$. Therefore, we can consider the direct diagram of monoid homomorphisms consisting of the iterated maps $\operatorname{mult}(j)$ $(j \in [1, r])$. Denote this diagram by D. Then $\varinjlim D$ is the example. More precisely, $K_i(R) \xrightarrow{\sim} K_i(R[\varinjlim D]/RI)$ for all K_i-regular rings R $(i \in \mathbb{N})$ and all proper radical ideals $I \subset \varinjlim D$.

Proof of Theorem 3.2.1 (c). Let F denote one of the functors K_0 and SK_1 (the former is considered for commutative rings). Since F endures reductions modulo central nilpotent elements, we can replace I by $\sqrt{I}$ and, hence, assume that the ideal under consideration is radical. For each maximal submonoid $N \subset M$ avoiding I, we have the following Cartesian square

$$\begin{array}{ccc} R[N_*] & \longrightarrow & RM/RI \\ \big\downarrow & & \big\downarrow \\ R & \longrightarrow & R[M]/R(\operatorname{int}(N) \cup I) \end{array}$$

with surjective vertical maps (we have used here Lemma 3.2.9). Hence we have the exact sequence [B]

$$F(R[N_*]) \longrightarrow F(R) \oplus F(R[M]/RI) \longrightarrow F(R[M]/R(\operatorname{int}(N) \cup I))$$

Since $F(R)$ is a natural direct summand of $F(R[N_*])$ and the map $F(R[N_*]) \longrightarrow F(R) \oplus F(R[M]/RI$, passes through $F(R[N])$, the similar arguments we used in the proof of (a) show that the map

$$F(R[M]/RI) \longrightarrow F(R[M]/(R\operatorname{int}(N) \cup I))$$

is injective. Now we can complete the proof in the same way as we did it for (a). ■

Remark. Previously, the nonstable analogue of the claim (b) for K_0, involving arbitrary monoids and their ideals, was mentioned in Bull.Georg. Acad.Sci., 130(1988). The detailed proof of this result was published in R.Swan's pure algebraic exposition of [G2], where he used the approach similar to that we have presented above.

Proof of Theorem 3.2.2. Let R, r, p, i and M be as in the theorem. By Corollary 3.2.6 we have the following long exact sequences

$$\ldots \longrightarrow K_i(R\operatorname{int}(M)) \longrightarrow K_i(R[M]) \longrightarrow K_i(R[M]/R\operatorname{int}(M)) \longrightarrow$$
$$\longrightarrow K_{i-1}(R\operatorname{int}(M)) \longrightarrow \ldots$$

and

$$\ldots \longrightarrow K_{i+1}(R[M_*]) \longrightarrow K_{i+1}(R) \longrightarrow K_i(R\,\mathrm{int}(M)) \longrightarrow$$
$$\longrightarrow K_i(R[M_*]) \longrightarrow K_i(R) \longrightarrow \cdots$$

By Approximation Theorem (A) from [G3] M_* is a filtered union of free monoids. Hence (since R is K-regular) the homomorphisms $K_i(R[M_*]) \longrightarrow K_i(R)$ are isomorphisms for all $i \in \mathbb{Z}$. From the second exact sequence we obtain that $K_i(R\,\mathrm{int}(M)) = 0$ for all $i \in \mathbb{Z}$. Therefore, the first exact sequence show that

$$K_i(R[M]) \xrightarrow{\sim} K_i(R[M]/R\,\mathrm{int}(M))$$

for all indices i. It just means that (in the terminology of [G2] and [G3]) we 'annuled' the interior of the simplex $\Phi(M)$. At the next step (arguing as in the proof of Theorem 3.2.1 (c)) we annule the interior of any $\dim\Phi(M) - 1$ dimensional face of $\Phi(M)$, and so on. Then we turn to $\dim\Phi(M) - 2$ dimensional faces of $\Phi(M)$, and so on. Finally we shall descend to the coefficient ring R. This completes the proof. ■

§ 3.3. On passing from p divisible monoids to arbitrary monoids. Admissible pairs of functors. Let F and G be two functors from the category of commutative rings to abelian groups. We shall say that the pair (F, G) is admissible if the following conditions are satisfied:

(a) F commuts with directed limits;

(b) For a Cartesian square of rings

$$\begin{array}{ccc} A & \longrightarrow & A_1 \\ \big\downarrow & & \big\downarrow f \\ A_2 & \longrightarrow & A' \end{array},$$

involving a surjective homomorphism f, the natural sequence

$$F(A) \longrightarrow F(A_1) \oplus F(A_2) \longrightarrow F(A')$$

is exact (i.e. F is semiexact);

(c) For a ring extension $A \subset B$, such that $B = A[x]$ for some $x \in B$ for which $x^2, x^3 \in A$, the corresponding map $F(A) \longrightarrow F(B)$ is injective;

(d) For a ring extension $A \subset B$, making B a free A-module of a finite rank n, there exists a group homomorphism $t_{B,A} : G(B) \longrightarrow G(A)$ such that the corresponding composite map

$$G(A) \longrightarrow G(B) \xrightarrow{t_{B,A}} G(A)$$

is a multiplication on some power of n;

(e) There exists a natural mono-transformation from F to G.

Main example. The pair (SK_0, K_0) is admissible. Indeed, (a),(b), (d) and (e) are 'classical' [B] ($t_{B,A}$ in this situation is just the transfer map for K_0). The validity of

(c) for this pair directly follows from [I], where it is show that for a ring extension $A \subset B$ of the above mentioned type one has the following diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & K_0(A) & \longrightarrow & K_0(B) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\approx} & & \downarrow{\scriptstyle\det} & & \downarrow{\scriptstyle\det} & & \\ 0 & \longrightarrow & Y & \longrightarrow & \mathrm{Pic}(A) & \longrightarrow & \mathrm{Pic}(B) & \longrightarrow & 0 \end{array}$$

(for the definition of det see [B]). Actually, in [I] even nonstable version of this statement is proved.

Main theorem and its corollaries.

Theorem 3.3.1. *Let (F, G) be an admissible pair and R a (commutative) ring.*

(1) *Assume p and q two distinct prime numbers and $F(R[M]) = 0$ whenever monoid M is p divisible or q divisible. Then $F(R[M]) = 0$ for all monoids M. The same statement restricted to the class of monoids with trivial groups of units is also valid;*

(2) *Assume $F(R[\mathbb{Z}_+^r]) = 0$ for some $r \in \mathbb{N}$. Then $F(R[M]) = 0$ for any intermediate monoid $\mathbb{Z}_+^r \subset M \subset \mathbb{Q}_+^r$.*

Corollary 3.3.2. (I) *For a ring R the following implication holds: $SK_0(R[M]) = 0$ for all p divisible and all q divisible monoids M (p and q are distinct primes) $\Rightarrow$ $SK_0(R[M]) = 0$ for all monoids M;*

(II) *For a* ring R *and any intermediate monoid $\mathbb{Z}_+^r \subset M \subset \mathbb{Q}_+^r$ the following implication holds ($r \in \mathbb{N}$):*

$$SK_0(R[\mathbb{Z}_+^r]) = 0 \Rightarrow SK_0(R[M]) = 0,$$

(III) *For a* ring R *and any intermediate seminormal monoid $\mathbb{Z}_+^r \subset M \subset \mathbb{Q}_+^r$ the implication*

$$K_0(R[\mathbb{Z}_+^r]) = \mathbb{Z} \Rightarrow K_0(R[M]) = \mathbb{Z}$$

holds ($r \in \mathbb{N}$);

(IV) *For a Noetherian* ring R *with Krull* $\dim R = 1$ *and M as in* (III) *the implication*

$$\begin{gathered} \textit{finitely generated projective } R[\mathbb{Z}_+^r]\textit{-modules are free } \Rightarrow \\ \textit{finitely generated projective } R[M]\textit{-modules are free} \end{gathered}$$

holds ($r \in \mathbb{N}$);

(V) *For R as in* (IV) *and two distinct primes p and q the following implication holds*

$$\begin{gathered} K_0(R[M]) = \mathbb{Z} \ \textit{ for all } p \textit{ and } q \textit{ divisile monoids } M \Rightarrow \\ \textit{finitely generated projective } R[M] - \textit{modules are free} \\ \textit{for all seminormal monoids } M. \end{gathered}$$

Proof of Corollary 3.3.2. Clearly, Theorem 3.3.1 (1) and implies Corollary 3.3.1 (I) and Theorem 3.3.1 (2) implies Corollary 3.3.2 (II).

(III). We already know that $SK_0(R[M]) = 0$ (by (II)). Therefore we have only to show that $\mathrm{Pic}(R[M]) = 0$. Since M is a filtered union of finitely generated seminormal monoids (Lemma 1.5.5), we can assume that M itself is finitely generated. But then M can be embedded in $\mathbb{Z}_+^r$ (because by Theorem 1.3.2 the normalization of M can be embedded in $\mathbb{Z}_+^r$). Hence $R[M]$ can be graded $R[M] = R_0 \oplus R_1 \oplus \dots$, so that $R_0 = R$. Using the well kknown Weibel's homotopic trick, we see that it suffices to show that $\mathrm{Pic}(R[M]) \xrightarrow{\sim} \mathrm{Pic}(R[M][\mathbb{Z}_+])$. By [Sw3] we have to show that the reduced ring of $R[M]$ is seminormal (in the sense of Swan). Since the grading $R[M]_{red} = R_{red}[M]$, by condition $\mathrm{Pic}(R) \xrightarrow{\sim} \mathrm{Pic}(R[\mathbb{Z}_+^r])$ $(= 0)$ we obtain the seminormality of R_{red} (in the sense of Swan). Now the proof is completed by Remarkk 1.5.7.

Let us note that for domains R the triviality of $\mathrm{Pic}(R[M])$ immediatele follows from [A2].

(IV). The aforementioned arguments show that $K_0(R[M]) = \mathbb{Z}$. Now the freeness of all stably free $R[M]$-modules directly follows from the main result of [G4].

(V). Using Mayer-Vietoris sequences for K_0 and Pic corresponding to the Cartesian square

$$\begin{array}{ccc} R[(M \backslash U(M)) \cup \{1\}] & \longrightarrow & R[M] \\ \big\downarrow & & \big\downarrow \\ R & \longrightarrow & R[U(M)]\ , \end{array}$$

one immediately reduces the general case to the following two special cases:

(*) M is a group;

(**) $U(M)$ is trivial.

First consider (*). We can assume that M is finitely generated and, hence, $M = \mathbb{Z}^r$ for some $r \in \mathbb{N}$. By (I), $SK_0(R[\mathbb{Z}^r]) = 0$ and by [Su], all stably free $R[\mathbb{Z}^r]$-modules are free. Therefore, we have only to show that $\mathrm{Pic}(R[\mathbb{Z}^r])$=0. Consider the diagram

$$D: \quad \mathbb{Z}^r \xrightarrow{p\cdot} \mathbb{Z}^r \xrightarrow{p\cdot} \mathbb{Z}^r \longrightarrow \dots .$$

and the corresponding diagrams

$$D': \quad R[\mathbb{Z}^r] \longrightarrow R[\mathbb{Z}^r] \longrightarrow R[\mathbb{Z}^r] \longrightarrow \dots ,$$

$$D'': \quad \mathrm{Pic}(R[\mathbb{Z}^r]) \longrightarrow \mathrm{Pic}(R[\mathbb{Z}^r]) \longrightarrow \mathrm{Pic}(R[\mathbb{Z}^r]) \longrightarrow \dots .$$

Since $\varinjlim D$ is p divisible, by our conditions we have $\varinjlim D'' = 0$. Choose arbitrarily $P \in \mathrm{Pic}(R[\mathbb{Z}^r])$. We see that P 'dies' after scalars extension of type $R[\mathbb{Z}^r] \longrightarrow R[\mathbb{Z}^r]$ induced by $\mathbb{Z}^r \xrightarrow{p^n\cdot} \mathbb{Z}^r$ for some $n \in \mathbb{N}$. Since $R[\mathbb{Z}^r]$ is a free $R[\mathbb{Z}^r]$-module with respect to the above mentioned map $R[\mathbb{Z}^r] \to R[\mathbb{Z}^r]$, we come to the conclusion that $\underset{p^{nr}}{\oplus} P \approx \underset{p^{nr}}{\oplus} R[\mathbb{Z}^r]$ (clearly, rank of $R[\mathbb{Z}^r]$ over $R[\mathbb{Z}^r]$ at this map is p^{nr}). Analogously, $\underset{q^{mr}}{\oplus} P \approx \underset{q^{mr}}{\oplus} R[\mathbb{Z}^r]$ for some $m \in \mathbb{N}$. Put $x = p^{nr}$ and $y = q^{mr}$. Assume $ax + by = 1$

for some integers $a > 0$ and $b < 0$ (the case $a < 0$ and $b > 0$ can be considered analogously). Then

$$\underset{ax}{\oplus} R[\mathbb{Z}^r] \approx \underset{ax}{\oplus} P \approx P \oplus (\underset{-by}{\oplus} P) \approx P \oplus (\underset{-by}{\oplus} R[\mathbb{Z}^r]).$$

Hence $P \in \mathrm{Pic}(R[\mathbb{Z}^r])$ is stably free. But it is calssical that all stably free rank 1 modules (over commutative rings) are free [L].

Consider now the case (**). Similarly to the proof of(III) we get $\mathrm{Pic}(R[M]) = 0$ for all seminormal monoids M. By (I), the groups $SK_0(R[M])$ are also trivial (for arbitrary monoids). Hence, for any seminormal monoid M we have $K_0(R[M]) = \mathbb{Z}$ (here the considered monoids have no nontrivial units). It only remains to show that stably free $R[M]$-modules are free. According to [G2], for the normal domains R there is nothing to prove (recall that $\dim R = 1$), for monoids of Φ-simplicial growth the above mentioned property of $R[M]$ directly follows from [G5] and, finally, the general case easily follows from R. Swan's algebraization of [G2] (see the remark after the proof of Theorem 3.2.1(c)). ■

Below we shall use

Lemma 3.3.3. *Assume we are given a monoid M and a subset $X \subset \mathbb{Q} \otimes K(M)$ satisfying the conditions*

(α) $K(M) \cap X = \varnothing$,

(β) $x_1 - x_2 \notin K(M)$ for arbitrary distinct $x_1, x_2 \in X$;

(γ) The submonoid of $\mathbb{Q} \otimes K(M)$ generated by $M \cup X$ coincides with $M \cup (\bigcup_X (x + M))$ $(x + M = \{x + m | m \in M\})$.

Then for a commutative ring R the inclusion $R[M] \longrightarrow R[\langle M \cup X\rangle]$ makes $R[\langle M \cup X\rangle]$ a free $R[M]$-module of rank $\#(X) + 1$, *where $\langle M \cup X\rangle$ means the submonoid of $\mathbb{Q} \otimes K(M)$ generated by $M \cup X$ and $\#$ is the cardinal number.*

Proof. We claim that just $X \cup \{1\}$ is the basis of $R[\langle M \cup X\rangle]$ over $R[M]$, where 1 denotes the neutral element of M. The proof of this statement is exactly the same as that of the'classical' observation that for a group extension $G \subset H$ any system of representatives of G-conjugation calsses (right or left) determines a basis of the free $R[G]$-module $R[H]$ (considered as left ot right $R[G]$-module, respectively). ■

Proof of Theorem 3.3.1.. First consider the case of normal monoids. Since normal monoids are filtered unions of finitely generated normal monoids (Lemma 1.5.5), we will assume that M is finitely generated and normal. In this situation there exists a free submonoid L having the same quotient group, i.e. $K(L) = K(M)$. Indeed, consider a rational point $x \in \mathrm{int}(\Phi(M))$. By normality of M we have $M(\{x\}) \approx \mathbb{Z}_+$. Say m is a generator of $M(\{x\})$. Since $\mathbb{Z}_+ m$ and M are normal in M and $K(M)$, respectively, we obtain that $K(M) = (\mathbb{Z}m) \oplus F$ for some free subgroup $F \subset K(M)$. In other words, the element m can be completed to a basis of $K(M)$. Fix arbitrarily such a basis $\{m, m_2, \ldots, m_r\}$. Now choose $c \in \mathbb{N}$ large enough and consider the following basis of $K(M)$: $\{m, m_2 + cm, \ldots, m_r + cm\}$. Since $m \in \mathrm{int}(M)$ for large c-s, we shall have $m_i + cm \in \mathrm{int}(M)$ $(i \in [2, r])$. Finally, put $L = \mathbb{Z}_+ m + \mathbb{Z}_+(m_2 + cm) + \ldots + \mathbb{Z}_+(m_r + cm)$. For simplicity the basis of L will be denoted by $\{n_1, \ldots, n_r\}$.

For any $a \in \mathbb{N}$ denote by $M_{p,a}$ the submonoid of $\mathbb{Q} \otimes K(M)$ generated by $M \cup \{n_1/p^a, \ldots, n_r/p^a\}$. Put

$$X_{p,a} = \{c_1 n_1/p^a + \ldots + c_r n_r/p^a | 0 \le c_1, \ldots, c_r < p^a\} \backslash \{0\} \subset \\ \subset \mathbb{Q} \otimes K(M).$$

It can be easily checked (using the equality $K(L) = K(M)$) that the monoid M and the set $X_{p,a}$ fall in the situation of Lemma 3.3.3. It is also obvious that $M_{p,a}$ is generated by $M \cup X_{p,a}$. In particular, we obtain that the inclusion $R[M] \longrightarrow R[M_{p,a}]$ makes $R[M_{p,a}]$ a free $R[M]$-module of rank p^{ar}. Now consider the diagram

$$D: \quad M \subset M_{p,1} \subset M_{p,2} \subset \ldots$$

and put $M_p = \underset{a}{\cup} M_{p,a} = \underset{\rightarrow}{\lim} D$. Denote by $sn(M_p)$ the seminormalization of M_p,i.e. a seminormal hull of M_p. Now assume we have already proved that $F(R[N]) = 0$ for finitely generated normal monoids of rank $< r$. Choose arbitrarily $x \in F(R[M])$. We claim that the image of x in $F(R[M_p])$ is 0. Since $R[sn(M_p)]$ is a filtered union of ring extensions of the type mentioned in the property (c) of admissible pairs, we see that it suffices to show the triviality of the image of x in $R[sn(M_p)]$. Observe now that $K(sn(M_p)) = K(M_p)$ coincides with the p divisible hull of $K(L)$ (in $\mathbb{Q} \otimes K(L)$). By Lemma 1.5.4 we obtain that $sn(M_p)_* = n(M_p)_*$. On the other hand, $n(M_p) = K(M_p)(\Phi(M))$. Consequently, $sn(M_p)_*$ is p divisible. If M is a group, then $sn(M_p)_* = sn(M_p)$ (is also a group) and, therefore, the above image of x is 0 (by conditions of the theorem). Assume M is not a group. Then we can consider the following Cartesian square

$$\begin{array}{ccc} R[sn(M_p)_*] & \longrightarrow & R[sn(M_p)] \\ \downarrow & & \downarrow \\ R & \longrightarrow & R[sn(M_p)]/(Rsn(M_p)_*^+) \end{array}$$

(clearly, M_p is a group iff M is). By property (b) mentioned in the definition of admissible pairs, it will suffice to establish the triviality of x in $F(R[sn(M_p)]/Rsn(M_p)_*^+)$. Clearly, $R[U(sn(M_p))]$ is naturally embedded in $R[sn(M_p)]/Rsn(M_p)_*^+$. If this embedding is an isomorphism, then there is nothing to prove (since $U(sn(M_p))$ is a group of rank $< r$ and, hence, it is a filtered union of finitely generated normal monoids of rank $< r$; here we use the condition (a) of admissible pairs). Suppose $R[U(sn(M_p))]$ is a proper subalgebra of $R[sn(M_p)]/Rsn(M_p)_*^+$. Let d be any codimension 1 face of $\Phi(M) = \Phi(sn(M_p))$. Then, by Lemma 3.2.9 we have the Cartesian square

$$\begin{array}{ccc} R[sn(M_p)(d)_*] & \longrightarrow & R[sn(M_p)]/Rsn(M_p)_*^+ \\ \downarrow & & \downarrow \\ R & \longrightarrow & R[sn(M_p)]/(sn(M_p)_*^+ \cup (sn(M_p)(d)_*^+) \end{array}$$

(for a monoid ring $A[H]$ and a subset $Y \subset H$ by (Y) we denote the ideal $AY \subset A[H]$). Since $sn(M_p)(d)$ is seminormal, the monoid $sn(M_p)(d)_*$ is normal (Lemma 1.4.3) and, obviously, it can be represented as a filtered union of finitely generated normal monoids of rank $< r$. Again, by property (b) we will be done whenever the triviality of the image of x in $F(R[sn(M_p)]/(sn(M_p)^+_* \cup sn(M_p)(d)^+_*))$ is established. Thus we have the same procedure of 'killings' of interiors which we described in §3.2. Finally we shall descend to the ring $R[U(sn(M_p))]$ over which (as remarked above) everything is OK.

The aforementioned arguments show that x has a zero image in $F(R[M_p])$. By property (a) of F there exists $a \in \mathbb{N}$ such that the image of x in $F(R[M_{p,a}])$ is 0. Consider the following commutative diagram

$$\begin{array}{ccc} F(R[M]) & \longrightarrow & F(R[M_{p,a}]) \\ \downarrow & & \downarrow \\ G(R[M]) & \longrightarrow & G(R[M_{p,a}]) \end{array},$$

where the vertical maps are injective ((e)). Since we know that $R[M_{p,a}]$ is a free $R[M]$-module of rank p^{ra} (see above), by(d) there exists $c \in \mathbb{N}$ such that $p^{rac} \cdot x = 0$. Now, repeating all these arguments with respect to the number q, we conclude that $q^{ra'c'} \cdot x = 0$ for some $a', c' \in \mathbb{N}$. Since p and q are coprime $x = 0$. To complete the proof of Theorem 3.3.1 (1) (by inductive procedure), it remains to check the special case of finitely generated normal monoids of rank 1 and then pass to arbitrary monoids. Assume M is finitely generated and of rank 1. We have only two possible cases: either $M \approx \mathbb{Z}_+$ or $M \approx \mathbb{Z}$. Again, consider the diagram $M \xrightarrow{p\cdot} M \xrightarrow{p\cdot} M \xrightarrow{p\cdot} \dots$. In the both possible cases $R[M]$ will turn out (via the corresponding embedings $R[M] \longrightarrow R[M]$) a free $R[M]$-module of rank p. Clearly, the same arguments which we used in the final step of the proof of $F(R[N]) = 0$ for all finitely generated normal monoids N of rank r are also applicable (providing the same triviality for rank $< r$ normal and finitely generated monoids).

Now we pass to the general case. By (a) we can assume that the monoid under consideration is finitely generated. By (c) we can pass to the corresponding seminormalization which, by Gordan's lemma, is finitely generated as well. Then, using the pull-backk diagrams of the above type (and using (b)), we pass to the 'interior' submonoids. By Lemma 1.4.3 these submonoids are normal. Therefore, they are filtered unions of finitely generated normal monoids. Finally, by (a) we are done.

Clearly, the same proof is applicable to the class of monoids with trivial groups of units.

Let us turn to the statement (2). Arguing in the same manner as before, we easily see that there arise pull-back diagrams of rings whose initial terms will be the rings of type $R[M]$, where: either $\Phi(M)$ is an open simplex and M is p divisible for a preliminarily fixed arbitrary natural number p (recall that we have also fixed q, coprime to p); or $\Phi(M)$ is an open simplex of dimension strictly less than $r - 1$ (r is the rank of the original monoid) and M is normal. In the first case by Approximation Theorem (A) from [G3] M is a filtered union of free monoids and in the second case M is filtered union of finitely generated Φ-simplicial monoids. Hence (a) and the equality $F(R[\mathbb{Z}^r_+]) = 0$ will imply the triviality of $F(R[M])$ for the monoids appearing in the first case, and the induction hypothesis will imply $F(R[M]) = 0$ for the second case. To complete the consideration of normal monoids, we have only to check the case $\mathrm{rank}(M) = 1$. This can be done similarly to the claim (1). Finally, basing on conditions (a) and (c) (as in the proof of (1)), we pass to the monoids of general type. ■

On unimodular rows over monoid rings.

Let P.Sets be the category of pointed sets and Groups* be the category of pairs of groups (G, H) such that $G \subset H$ (a morphism from (G, H) to (G', H') is a group homomorphism $f : H \longrightarrow H'$ for which $f(G) \subset G'$). A pair of functors F : Com.Rings $\longrightarrow$ P.Sets and E : Com. Rings $\longrightarrow$ Groups* will be called admissible if the following conditions are satisfied

(A) F commutes with directed limits;

(B) For a Cartesian square of rings

$$\begin{array}{ccc} A & \longrightarrow & A_1 \\ \Big\downarrow & & \Big\downarrow f \\ A_2 & \xrightarrow{g} & A' \end{array},$$

involving a surjective homomorphisms f or g and an element $x \in F(A_1)$ which maps into a basepoint of $F(A')$, there exists $y \in F(A)$ mapping into x;

(C) For a ring extension $A \subset B$ of type $B = A[b]$ for some $b \in B$ such that $b^2, b^3 \in A$, the map $F(A) \to F(B)$ satisfies the condition: if $x \in F(A)$ maps into a basepoint of $F(B)$, then x itself is a basepoint of $F(A)$;

(D) For a ring extension $A \subset B$ making B a free A-module of a finite rank n, and $x \in H_A$, such that $E(i)(x) \in G_B$, there exists $c \in \mathbb{N}$ for which $x^{n^c} \in G_A$, where i is the inclusion $A \subset B$, $E(A) = (G_A, H_A)$ and $E(B) = (G_B, H_B)$,

(E) There exists a natural transformation from E_r to F such that for any A the preimage of a basepoint of $F(A)$ in $E_r(A)$ consists of the basepoint of $E_r(A)$ only, where E_r : Com. Rings $\longrightarrow$ P.Sets is defined as follows: $E_r(A) = (H_A/G_A, G_A)$ for the right conjugation classes of G_A in H_A denoted by H_A/G_A (here we mean that $E(A) = (G_A, H_A)$).

Theorem 3.3.4. *Assume we are given a pair of admissible functors F : Com.Rings $\longrightarrow$ P.Sets and E : Com.Rings $\longrightarrow$ Groups*, a commutative ring R and a (commutative, cancellative, torsion free) monoid M such that the corresponding map $E_r(R[M])$ $\longrightarrow F(R[M])$ is surjective.*

(1) *Assume p and q are two distinct prime numbers and $F(R[L]) = *$ whenever L is p- or q-divisible. Then $F(R[M]) = *$.*

(2) *Assume $F(R[\mathbb{Z}_+^r]) = *$ for $r = \mathrm{rank}(M)$ and $\mathbb{Z}_+^r \subset M \subset \mathbb{Q}_+^r$. Then $F(R[M]) = *$.*

Proof goes analogously to that of Theorem 3.3.1 and we leave the details for the reader.

Conjectural example of an admissible pair. Let $n \in \mathbb{N}$ Consider the functors UE_n : Com.Rings $\longrightarrow$ P.Sets and GE_n : Com.Rings $\longrightarrow$ Groups*, where $UE_n(A) =$

($Um_n(A)/$ right action of $E_n(A)$, class of $(1,0,\dots,0)$) and $GE_n(A)$ = (the subgroup of $GL_n(A)$ generated by $E_n(A)$ and the matrices of type $\begin{pmatrix}1 & 0\\ 0 & x\end{pmatrix}$ for $x \in Gl_{n-1}(A), GL_n(A)$).

Theorem 3.3.5. *For any $n \geq 3$ the pair of functors (UE_n, GE_n) satisfies the conditions (A),(B),(C) and (E).*

Conjecture 3.3.6. *For any $n \geq 3$ the pair of functors (UE_n, GE_n) is admissible.*

By normality of $E_n(-)$ in $GL_n(-)$ for commutative rings ($n \geq 3$) [Su] and by Theorem 3.3.5 this conjecture is equivalent to the following

Conjecture. *Assume $A \subset B$ is a ring extension making B a free A-module of a finite* rank r, *and $(a_{ij}) \in GL_n(A)$ for some $n \geq 3$. Assume further $(a_{ij}) \underset{E_n(B)}{\sim} \begin{pmatrix}1 & 0\\ 0 & x\end{pmatrix}$ for some $x \in GL_{n-1}(B)$. Then there exists $c \in \mathbb{N}$ such that*

$$(a_{ij})^{r^c} \underset{E_n(A)}{\sim} \begin{pmatrix}1 & 0\\ 0 & y\end{pmatrix}$$

for some $y \in GL_{n-1}(A)$.

Before proving Theorem 3.3.5 let us state one more

Proposition 3.3.7. *The natural transformation from $(GE_n)_r$ to UE_n given in condition (E) (and described below) satisfies the surjectivity condition for the map $(GE_n)_r$ $(R[M]) \longrightarrow UE_n(R[M])$, where R is a commutative Noetherian ring with Krull dimension $d < \infty$, M is a seminormal monoid and $n \geq \max(d+1, 2)$.*

Remark 3.3.8. The confirmation of our conjecture would imply the positive answer to the question risen in [G4] about the transitivity of the action $E_n(R[M]) \times Um_n(R[M]) \longrightarrow Um_n(R[M])$ where R is a commutative Noetherian ring of Krull dimension $d < \infty, n \geq \max(d+2, 3)$ and M is an arbitrary monoid. Indeed, in Remark 3.3.10 below we show that the general case reduces to the case of seminormal (even normal) monoids and by Theorem 3.3.4 and Proposition 3.3.7 in the situation of seminormal monoids the task can be reduced at once to the consideration of p divisible monoids for p prime. But the aforementioned transitivity was proved in [Sc] (basing on [G3]) for p divisible monoids. We recall that in [G5] the transitivity of the action $E_n(R[M]) \times Um_n(R[M]) \longrightarrow Um_n(R[M])$ for the class of monoids M admitting the sequences of convex polyhedra of type $P_0 \subset P_1 \subset \ldots \subset P_k = \Phi(M)$ (k is a natural number), where P_0 and the closures of the sets $P_{i+1} \backslash P_i$ $\quad (i \in [0, k-1])$ are all simplices.

Proof of Theorem 3.3.6 (A) is clear.

(B). Assume we are given a Cartesian square

$$\begin{array}{ccc} A & \longrightarrow & A_1 \\ \big\downarrow & & \big\downarrow{\scriptstyle k} \\ A_2 & \xrightarrow{g} & A' \end{array}$$

and elements $f \in Um_n(A_1)$ and $e \in E_n(A')$ such that $he = (1, 0, \dots, 0)$, where h is the image of f in $Um_n(A')$.

Consider first the case when k is surjective. Let E be a lifting of e in $E_n(A_1)$. Then the elements $fE \in Um_n(A_1)$ and $(1, 0, \dots, 0) \in Um_n(A_2)$ have the same images in $Um_n(A')$. Hence, there exists $d \in A^n$ mapping in fE and $(1, 0, \dots, 0)$, respectively. It is an easy exercise that $d \in Um_n(A)$.

Suppose now g is surjective and $E' \in E_n(A_2)$ is a lifting of e^{-1} in $E_n(A_2)$. Then the elements $(1, 0, \dots, 0)E' \in Um_n(A_2)$ and f have the same images in $Um_n(A')$. Again, there exists $d' \in A^n$ mapping in f and $(1, 0, \dots, 0)E'$, respectively. As we remarked above, actually $d' \in Um_n(A)$.

(C). We claim that for an extension of (commutative) rings $A \subset B$ such that $B = A[b]$, where $b^2, b^3 \in A$ and $n \geq 3$ the following implication holds: $f = (f_1, \dots, f_n) \in Um_n(A), f \underset{E_n(B)}{\sim} (1, 0, \dots, 0) \Rightarrow f \underset{E_n(A)}{\sim} (1, 0, \dots, 0)$.

In the proof we shall use the following special case of the main result of [G4].

Proposition 3.3.9. *Let $r, n \in \mathbb{N}$ and $n \geq 3$. Assume M is an intermediate monoid $\mathbb{Z}_+^r \subset M \subset \mathbb{Q}_+^r$. Then the natural action $E_n(\mathbb{Z}[M]) \times Um_n(\mathbb{Z}[M]) \longrightarrow Um_n(\mathbb{Z}[M])$ is transitive.*

Let us turn to the proof of the aforementioned claim.

Put $S = A \cup \{b\}$ and consider the polynomial algebra $\mathbb{Z}[\{t_s\}_S]$. By the conditions on b the map $g : \mathbb{Z}[\{t_s\}_S] \longrightarrow B$, sending t_s into s, is surjective (notations as in the claim). Consider the pull-back square

$$\begin{array}{ccc} D & \longrightarrow & A \\ \downarrow & & \downarrow \\ \mathbb{Z}[\{t_s\}_S] & \longrightarrow & B\,. \end{array}$$

By the validity of the condition (B) for the pair of functors being considered we will be done whenever the transitivity of the natural action $E_n(D) \times Um_n(D) \longrightarrow Um_n(D)$ is established. Let N denote the multiplicative monoid consisting of those monomials in t_s-s which are either trivial ($= 1$) or involve t_b^a for some $a \geq 2$. We claim that N is a filtered union of finitely generated monoids without nontrivial units whose Φ-images are simplices (s.c. Φ-simplicial monoids, [G4]). Assume $S = \underset{i}{\cup} S_i$ is a filtered union of all finite subsets of S containing b. Put $N_i =$ (the monoid of monomials involving t_s-s only for $s \in S_i) \cap N$. Obviously, N is a filtered union of N_i-s. So it will suffices to show that each N_i is a filtered union of monoids of the aforementioned type. We have $\Phi(N_i) = \Phi(F_i) \backslash B_i$, where F_i-s are free monoids spanned by $\{t_s\}_{S_i}$ and B_i-s are the bases of simplices $\Phi(F_i)$ (respectively) situated oppositly to the vertex $\Phi(t_b)$. Clearly, $\Phi(F_i) \backslash B_i$ can be represented as filtered unions of rational simplices and, hence N_i-s are filtered unions of finitely generated (by Gordan's lemma) monoids to whom correspond rational simplices. It should be noted that N_i-s are not normal or even seminormal. By Proposition 3.3.9, E_n acts transitively on Um_n for each monoid ring

$\mathbb{Z}[N_i]$. Hence $E_n(\mathbb{Z}[N])$ acts transitively on $Um_n(\mathbb{Z}[N])$. Observe now that we have the following Cartesian square

$$\begin{pmatrix} N^+ \\ \big\downarrow \\ 0 \in \mathbb{Z} \end{pmatrix} \quad \begin{array}{ccc} \mathbb{Z}[N] & \longrightarrow & D \\ \big\downarrow & & \big\downarrow \\ \mathbb{Z} & \longrightarrow & D/N^+D \end{array},$$

where, as usual, $N^+ = N\backslash\{1\}$. So we will be done whenever the transitivity of the action $E_n(D/N^+D) \times Um_n(D/N^+D) \longrightarrow Um_n(D/N^+D)$ is established (here we again use (B)). By the definition of (D) we see that $\mathbb{Z}[\{t_s\}_A] \subset D$. Now represent any element $d \in D$ as a sum $d_0 + t_b d_1$ for some $d_0 \in \mathbb{Z}[\{t_s\}_A]$ and $d_1 \in \mathbb{Z}[\{t_s\}_S]$. Since $(t_b d_1)^2 \in N^+$ and $(t_b\mathbb{Z}[\{t_s\}_S])\cap\mathbb{Z}[\{t_s\}_A]$, we easily see that $(D/N^+D)/(\text{nilpotents}) = \mathbb{Z}[\{t_s\}_A]$ Since the transitivity of the action $E_n \times Um_n \longrightarrow Um_n$ is an invariant property modulo nilpotents (and since $E_n(\mathbb{Z}[\{t_s\}_A])$ obviously acts transitively on $Um_n(\mathbb{Z}[\{t_s\}_A])$), the proof is completed.

Remark. In this proof we have used the observation that a finitely generated monoid M without nontrivial units whose Φ-image is a simplex can isomorphically be situated between $Z_+^{\text{rank}(M)}$ and $\mathbb{Q}_+^{\text{rank}(M)}$ (see Example 2.5.10).

(E). We have to define a natural transformation from $(GE_n)_r$ to UE_n which will possess the required properties. Let A be a commutative ring and $x \in (GE_n)_r(A)$. Asume $(a_{ij}) \in GL_n(A)$ is a representative of x. Denote by a the element $(a_{11}, a_{12}, \dots, a_{1n}) \in Um_n(A)$ and by $[a]$ the class of a in $UE_n(A)$. We define a natural transformation from $(GE_n)_r$ to UE_n as the system of maps (for all A) $(GE_n)_r(A) \longrightarrow UE_n(A)$ determined by $x \longrightarrow [a]$. First we have to show that this definition is correct. In other words, we have to show that for arbitrary $E \in E_n(A)$ any $y \in GL_{n-1}(A)$ the first rows of $E \cdot (a_{ij})$ and $\begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix} \cdot (a_{ij})$ represent the same element $[a]$ in $UE_n(A)$. For $E \cdot (a_{ij})$ this claim dirrectly follows from the normality of $E_n(A)$ in $GL_n(A)$ [Su], and for $\begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix} \cdot (a_{ij})$ there is nothing to prove, since the first row merely coincides with $(a_{11}, \dots, a_{1n})$. Now the validityy of (E) follows from the easy observation that $x \in GL_n(A)$ has the first row belonging to $E_n(A)$-orbit of $(1, 0, \dots, 0)$ if and only if x belongs to the subgroup of $GL_n(A)$ generated by $E_n(A)$ and the matrices of type $\begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix}$ for $y \in GL_{n-1}(A)$ (the part 'if' follows from the aforementioned correctness). ■

Remark 3.3.10. From the claim we stated at the verification of (C) it follows that the question from [G4] (see Remark 3.3.8) reduces to the case of finitely generated normal monoids without nontrivial units. Indeed, the passage to the finite generation case is clear and, using Cartesian squares of the type mentioned in the proof of Corollary 3.3.2 (V), we are able (basing on (B)) to avoid nontrivial units in a monoid. Here, of course, we ned the positive answer to our question for the group rings [Su]. Finally, by the suitable filtered diagrams we pass to the seminormalizations of the corresponding monoids and then 'kill' the interiors as in §3.2 which (according to Lemma 1.4.3) determine normal submonoids.

Proof of Proposition 3.3.7. By the definition of the natural transformation $(GE_n)_r \longrightarrow UE_n$ we have only to show that any $f \in Um_n(R[M])$ (notations as in the proposition) is completable to an element of $GL_n(R[M])$. Equivalently, we have to prove that any finitely generated projective $R[M]$-module of rank $> d$ is extended from R whenever it is stably extended from R (see[L]). For PID-s this follows from [G2]; or more generally, for regular rings this result is explicitly stated in R.Swan's aforementioned notes on [G2]; and finally, the general case can be obtained by combining the results from these notes and [Li]. ■

§ 3.4. On monoids with torsion. In this section we no longer require the torsion freeness of monoids (which again will be assumed to be commutative and cancellative). Obviously, the inclusion functor COM. CANC. TOR. FREE MONOIDS $\longrightarrow$ COM.CANC.MONOIDS has a left adjoint functor determined by $M \longrightarrow$ (the image of M in $K(M)/K(M)_{\text{tor}}$). We denote this functor by G.

Definition 3.4.1. We shall say that a monoid M is

(a) Φ-simplicial if $G(M)$ is such (i.e. $\Phi(G(M))$ is a simplex and $G(M)$ is finitely generated (see [G4]));

(b) of Φ-simplicial growth if $G(M)$ is such (i.e. $G(M)$ is of the type described in Remark 3.3.8 (see [G5]);

(c) normal or seminormal if the obvious conditions for M are satisfied.

Theorem 3.4.2. *Let M be a monoid and R a (commutative) ring.*

(1) *Assume R is Noetherian and of Krull dimension $d < \infty$ and $K(M)_{\text{tor}}$ is cyclic. If M is of Φ-simplicial growth, then E_n acts transitively on Um_n for $R[M]$ whenever $n \geq \max(d+2, 3)$;*

(2) *Assume R is a Noetherian and of Krull dimension $d < \infty$, $p \cdot 1 = 0$ in R for some prime p and $K(M)_{\text{tor}}$ is a p-group. If M is of Φ-simplicial growth, then E_n acts transitively on Um_n for $R[M]$ whenever $n \geq \max(d+2, 3)$;*

(3) *Assume R is regular, $p \cdot 1 = 0$ in R for some prime p and $K(M)_{\text{tor}}$ is a p-group. Then $SK_0(R[M]) = SK_0(R)$.*

In the rest of this section we shall sketch the proof of this theorem. Observe that if M satisfies one of the conditions given in Theorem 3.4.2, then it is a filtered union of finitely generated monoids satisfying the same condition. Consequently, without loss of generality we shall assume that the monoid mentioned in the theorem is finitely generated. In this situation $M \subset G(M) \oplus K(M)_{\text{tor}}$. Put $T = K(M)_{\text{tor}}$ and $M_* = (G(M)_* \oplus T) \cap M$. the notations $sn(-), -^+$ and $\text{int}(-)$ will be used in the obvious sense (in particular, $\text{int}(-) = -^+_*$).

Lemma 3.4.3. *For a finitely generated monoid L we have* $\text{int}(sn(L)) = \text{int}(sn(G(L))) \times T$.

Lemma 3.4.4. *For a regular ring R and a torsion free (commutative and cancellative) monoid L we have $SK_0(R[L]) = SK_0(R)$ and $K_{-1}(R[L]) = K_{-1}(R) = 0$.*

We note only that Lemma 3.4.4 can be proved by the stable version of [G2] for arbitrary regular rings and by [I].

Lemma 3.4.5. *Without loss of generality we can assume that the monoid M given in Theorem* 3.4.2 *has no nontrivial units.*

Proof. Consider the Cartesian square with surjective vertical maps

$$\begin{array}{ccc} R[(M\backslash U(M))\cup\{1\}] & \longrightarrow & R[M] \\ \Big\downarrow & & \Big\downarrow \\ R & \longrightarrow & R[U(M)]\,. \end{array}$$

In the situations of claims (1) and (2) $G(M)$ is a finite polyhedron and, hence, $U(G(M))$ is trivial. Therefore, for (1) and (2) we have $U(M) = M \cap T$. Since $U(M)$ turned out to be a finite group, $R[U(M)]$ is again Noetherian with Krull dimension d. Therefore $E_n(R[U(M)])$ acts transitively on $Um_n(R[U(M)])$. Now let us show that in the situation of claim (3) $SK_0(R[U(M)]) = SK_0(R)$. We have $U(M) = \mathbb{Z}^s \oplus T'$ for some $s \in \mathbb{N}$ and $T' \subset T$. Assume $T' = \mathbb{Z}_{p^{a_1}} \oplus \ldots \oplus \mathbb{Z}_{p^{a_k}}$ for some $a_k, \ldots, a_1 \in \mathbb{N}$ $(k \in \mathbb{N})$. Consider the Cartesian square

$$\begin{array}{ccc} D & \longrightarrow & R[\mathbb{Z}^s][t_1,\ldots,t_k] \\ \Big\downarrow & & {\scriptstyle g}\Big\downarrow \\ R[\mathbb{Z}^s] & \longrightarrow & R[\mathbb{Z}^s \oplus T']\,, \end{array}$$

where t_i-s are variables and g is an $R[\mathbb{Z}^s]$-homomorphism determined by $t_i \longrightarrow e_i$ for the generators e_i-s of $\mathbb{Z}_{p^{a_i}}$-s, respectively. Obviously, D is an $R[\mathbb{Z}^s]$-subalgebra of $R[\mathbb{Z}^s][t_1,\ldots,t_k]$ generated by the ideal $(t_1^{p^{a_1}} - 1, \ldots, t_k^{p^{a_k}} - 1)R[\mathbb{Z}^s][t_1,\ldots,t_k] \subset R[\mathbb{Z}^s][t_1,\ldots,t_k]$. Changing the variables t_i by $u_i = t_i - 1$, D will turn out to be an $R[\mathbb{Z}^s]$-subalgebra of $R[\mathbb{Z}^s][u_1,\ldots,u_k]$ generated by a certain set of 'pure'monomials. In other words, $D = R[\mathbb{Z}^s][L]$ for an appropriate torsion free monoid L. Consider now the Mayer-Vietoris sequence corresponding to our Cartesian square $K_0(R[\mathbb{Z}^s]) \oplus K_0(R[\mathbb{Z}^s][u_1,\ldots,u_k]) \longrightarrow K_0(R[\mathbb{Z}^s \oplus T']) \longrightarrow K_{-1}(D)$ Using regularity of R and Lemma 3.4.4, we conclude that $K_0(R) = K_0(R[\mathbb{Z}^s \oplus T'])$ and, hence, $SK_0(R) = SK_0(R[\mathbb{Z}^s \oplus T'])$.

The proof of Lemma 3.4.5 is now completed by the observation that $(M\backslash U(M)) \cup \{1\}$ and M satisfy the same conditions in all the situations (of claims (1),(2) and (3)). To keep the finite generation condition, we note that in the situations of (1) and (2) $(M\backslash U(M)) \cup \{1\}$ itself is finitely generated (and, of course, of Φ-simplicial growth), while in the situation of (3) $(M\backslash U(M))\cup\{1\}$ should be represented as a filtered union of its finitely generated submonoids. ■

Proof of Theorem 3.4.2

By the remarks above and Lemma 3.4.4 without loss of generality we shall assume M to be finitely generated and $U(M)$ trivial. In this situation we are able to carry out the procedure of 'killings' of the interiors of the monoid itself and its extremal submonoids, as described in the proof of Theorem 3.3.1. Hence, it will suffice to check our claims for monoid rings of type $R[M_*]$, where $U(M)$ is supposed to be trivial. By Lemma 3.4.3, $\mathrm{int}(sn(M)) = \mathrm{int}(sn(G(M))) \times T$ $(T = K(M)_{\mathrm{tor}})$. By the claims we stated in the verification of Theorem 3.3.5 (C) and the main example of §3.3, we can (in addition) assume that $M = sn(M)$. Consequently, we have to check the claims (1),(2) and (3) for the monoids of type $L = (N_* \times T)\backslash\{(1,t)|t \in T,\ \ t \neq 1\}$ where N is a commutative, cancellative, torsion free, normal and finitely generated monoid with $U(N)$ trivial, T is a finite group and 1 denotes a neutral element of the corresponding monoid for which, with respect to the claims (1),(2) and (3), the following conditions are satisfied:

(1) N is of Φ-simplicial growth and T is cyclic;

(2) N is of Φ-simplicial growth and T is a p-group ($p \cdot 1 = 0$ in the coefficient ring R);

(3) T is a p-group ($p \cdot 1 = 0$ in the coefficient ring R).

Observe that the natural projection from L into N_* sends into $1 \in N$ the element $1 \in L$ only. On the other hand, the finite generation of N and the triviality of $U(N)$ enables us to embed N into $\mathbb{Z}_+^{\mathrm{rank}(N)}$. But then the composite map $L \to N_* \subset N \longrightarrow \mathbb{Z}_+^{\mathrm{rank}(N)}$ will send into the neutral element of $\mathbb{Z}_+^{\mathrm{rank}(N)}$ the element $1 \in L$ only. This observation implies the existence of grading $R[L] = R_0 \oplus R_1 \oplus \dots$ for which $R_0 = R$. Hence, using the local-global principle for the elementary action on unimodular rows over a graded ring [G4], for the claim (1) without loss of generality we shall suppose that R is local.

Consider the pull-back diagram

$$\begin{array}{ccc} D & \longrightarrow & R[L] \\ \big\downarrow & & \big\downarrow \\ R[N_*][t_1,\dots,t_k] & \xrightarrow{g} & R[N_* \oplus T] \end{array},$$

where t_i-s are variables, k is equal to the number of cyclic summands of T when T is a p-group (for (2) and (3)) and 1 when T itself is cyclic (for (l)), $g(t_i) = e_i$ for the corresponding generators e_i of the above mentioned cyclic direct summands of T (in all three cases (1),(2) and (3)), and the vertical maps are embeddings.

Let us first consider the cases (2) and (3). Since $R[T]$ is a Noetherian (commutative) ring of Krull dimension d and N_* is obviously a filtered union of monoids of Φ-simplicial growth, E_n acts transitively on Um_n for $R[N_* \oplus T] = R[T \oplus N_*]$ [G5]. Arguing analogously to the proof of Lemma 3.4.5 (where $\mathbb{Z}^s$ is substituted by N_*), the regularity of R in the situation of (3) will imply the equality $SK_0(R[N_* \oplus T]) = SK_0(R)$. Therefore, we will be done whenever the conclusions of the claims (2) and (3) are established for D (at the appropriate conditions).

Remark. When we substitute $\mathbb{Z}^s$ by N_* in the proof of Lemma 3.4.5 we have to use the isomorphisms of type $K_0(R) \xrightarrow{\sim} K_0(R[H])$ for arbitrary normal monoids H without a torsion. These isomorphisms are the results of the stable version of [G2] for arbitrary regular rings.

Put $I = \mathrm{int}(N)R[N_*][t_1,\dots,t_k]$ and observe that I is an ideal of D as well. Therefore we can consider the following Cartesian square

$$\begin{array}{ccc} R[I] & \longrightarrow & D \\ \big\downarrow & & \big\downarrow \\ R & \longrightarrow & D/I\ , \end{array}$$

where $R[I]$ is an R-subalgebra of $R[N_*][t_1,\dots,t_k]$ generated by I. Clearly, $R[I]$ is an R-subalgebra of $R[N_*][t_1,\dots,t_k]$ generated by 'pure' monomials of $N_*\cup\{t_1^{c_1}\dots t_k^{c_k}|c_i\geq 0\}$, i.e. $R[I]$ is a monoid ring. Easy geometrical arguments show that the corresponding monoid is a filtered union of monoids of Φ-simplicial growth. Hence, basing on this remark and Lemma 3.4.4, we see that it will sufice to to establish the transitivity of the action $E_n(D/I)\times Um_n(D/I)\longrightarrow Um_n(D/I)$ and the equality $SK_0(R)=SK_0(D/I)$ in the situations of (2) and (3), respectively. It remains only to note that the ring D/I is of type D in the proof of Lemma 3.4.5 (in which $\mathbb{Z}^s$ is omitted). Thus the corresponding substitution of variables t_i-s (as in the above mentioned lemma) completes the proof (it should be noted that after the substitution D/I will turn out to be a Φ-simplicial monoid ring and, hence, we are able to apply [G4] to the situation of (2)).

Now, consider (1). The very same arguments show that we have to show the transitivity of the action $E_n(D/I)\times Um_n(D/I)\longrightarrow Um_n(D/I)$ under the appropriate conditions. In this situation $D/I = R[(t_1^m-1)R[t_1]]$ (R-subalgebra of $R[t_1]$ generated by the ideal $(t_1^m-1)R[t_1]$), where m is an order of the cyclic group T. Suppose $m\geq 1$ and $f=(f_1,\dots,f_n)\in Um_n(D/I)$. By Lemma 3.4 from [L] we can assume $ht_{D/I}((f_1,\dots,f_{n-1}))\geq n-1$. We have the integral ring extension $R[t_1^m]\subset D/I$. By Lemma 6.5, step 2 from [G4],

$$ht_{R[t_1^m]}((f_1,\dots,f_{n-1}))\cap R[t_1^m]\geq n-1$$

(here, as above, $((f_1,\dots,f_{n-1}))$ denotes the ideal of D/I generated by $\{f_1,\dots,f_{n-1}\}$). Since $n-1>d$, by Lemma 3.2 from [L] we obtain that the ideal of the leading coefficients $\gamma(((f_1,\dots,f_{n-1}))\cap R[t_1^m])$ coincides with R. Consequently, $((f_1,\dots,f_{n-1}))$ contains a monic polynomial, say u. Clearly, if c is a sufficiently large natural number, then f_n+u^c is monic as well. Since $f \underset{E_n(D/I)}{\sim} (f_1,\dots,f_{n-1},f_n+u^c)$, without loss of generality from the very beginning we could assume that f_n itself is monic. Observe now that D/I is a finitely generated $R[t_1^m]$-algebra (more precisely, it is generated by the elements $t_1^{m+1}-t_1, t_1^{m+2}-t_1^2,\dots,t_1^{2m-1}-t_1^{m-1}$). In this situation it is not difficult to show that the quotient algebra $(D/I)/(f_n)$ is a finitely generated R-module. Since we supposed the locality of R, the aforementioned remarks imply the semilocality of $(D/I)/(f_n)$. Let bar denote the reduction modulo f_n. We have

$$(\overline{f}_1,\overline{f}_2,\dots,\overline{f}_{n-1}) \underset{E_{n-1\overline{(D/I)}}}{\sim} (\overline{1},\overline{0},\dots,\overline{0})$$

Therefore,

$$(f_1, f_2, \dots, f_{n-1}, f_n) \underset{E_n(D/I)}{\sim} (1 + h_1, h_2, \dots, h_{n-1}, f_n) \underset{E_n(D/I)}{\sim} (1, 0, \dots, 0)$$

for appropriate elements $h_1, \dots, h_{n-1} \in (f_n)$. Q.E.D.

In general, we may expect that $E_n(R[M])$ acts transitively on $Um_n(R[M])$ for arbitrary commutative cancellative monoids M, for R is a (commutative) Noetherian with Krull dimension $d < \infty$ and $n \geq \max(d + 2, 3)$.

References

[Al] D. F. Anderson, Projective modules over subrings of $k[X, Y]$ generated by monomials. *Pacific J. Math.* **79**(1978), 5–17.

[A2] D. F. Anderson, The picard group of a monoid domain. *J. Algebra* **115**(1988), 342–351 .

[AA] D. D. Anderson and D. F. Anderson, Divisorial ideals and invertible ideals in a graded domain. *J. Algebra* **76**(1982), 549–569.

[B] H. Bass, Algebraic K-theory. *W. A. Benjamin, New York*, 1968.

[Be] M. Berger, Géométrie. *Cedic/Fernand Nathan, Paris,* 1977–1978.

[BN] J. W. Brewer and W. Nichols, Seminormality in power series rings. *J. Algebra* **82**(1983), 282–284.

[Bo] N. Bourbaki, Algébre Commutative, Ch.7. *Herman, Paris,* 1965.

[Br] A. Bröndsted, An introduction to Convex Polytopes. *Springer-Verlag, New York-Heidelberg-Berlin,* 1983.

[C1] L. Chouinard, Krull semigroups and divisor class groups. *Canad. J. Math.* **33**(1981), 1459–1468.

[C2] L. Chouinard, Projective modules over Krull semigroup rings. *Mich. Math. J.* **29**(1982), 143–148.

[D] V. I. Danilov, The geometry of toric varieties. *Russian Math. Surveys* **33**(1978), 99–154.

[Gi] R. Gilmer, Commutative Semigroup Rings. *Chicago Lectures Math., Chicago U. P.,* 1984.

[GH] R. Gilmer and R. C. Heitmann, On Pic$(R[X])$ for R seminormal. *J. Pure Appl. Algebra* **16**(1980), 251–257.

[Ge] S. M. Gersten, Higher K-theory of rings. *Lect. Notes Math.* **341**(1973), 3–42, *Springer-Verlag, Berlin-Heidelberg-New York,* 1973.

[G1] J. Gubeladze, Serre's Generalized Problem for Affine Rings Generated by Monomials. *Tbilisi Univ. Press,* 1982.

[G2] J. Gubeladze, Andersons's conjecture and the maximal monoid class over which projective modules are free. *Math. Sb.* **63**(1989), 165–180.

[G3] J. Gubeladze, Classical algebraic K-theory of monoid algebras. *Lect. Notes Math.* **1437**(1990), 36–94, *Springer-Verlag, Berlin-Heidelberg-New York,* 1990.

[G4] J. Gubeladze, The elementary action on unimodular rows over a monoid ring. *J. Algebra* **148**(1992), 135–161.

[G5] J. Gubeladze, The elementary action on unimodular rows over a monoid ring, II. *J. Algebra* **155**(1993), 171–194.

[G6] J. Gubeladze, Monoidal variations. *Tbilisi,* 1992 (preprint).

[H] M. Hochster, Rings of invariants of tori, Cohen-Macaulay rings generated by monomials, and polytopes. *Ann. Math.* **96**(1972), 318–337.

[I] F. Ischebek, Subintegral ring extensions and some K-theoretical functors. *J. Algebra* **121**(1989), 323–338.

[L] T. Y. Lam, Serre's Conjecture. *Lect. Notes Math.* **635**, *Springer-Verlag, Berlin-Heidelberg-New York,* 1978.

[Li] H. Lindel, On large projective modules over polynomial rings and Laurant polynomial rings. *Preprint, Münster,* 1986.

[LLPS] The Lake Louise Problem Session in: Algebraic K-theory, connections with geometry and topology (proc.). *NATO Adv. Sci. Institutes Serries, Dordrecht,* 1989.

[Mu] Z. Mushkudiani, K_2-groups of monoid algebras over regular rings. *These proceedings.*

[O] T. Oda, Convex Bodies and Algebraic Geometry - An Introduction to the Theory of Toric Varieties. *Ergebnisse der. Math.* **(3)15**, *Springer-Verlag, Berlin-Heidelberg-New York-London-Paris-Tokyo,* 1988.

[Q] D. Quillen, Higher algebraic K-theory I. *Lect. Notes Math.* **341** (1973), 85–147.

[Sc] G. Schabhüser, Kürzungseigenschaften projectiver Moduln über Monoidringen. *Thesis, Münster,* 1991.

[Su] A. A. Suslin, On the structure of the special linear group over polynomial rings. *Math. USSR Izvestija* **11**(1977), 221–238.

[SW] A. Suslin and M. Wodzicki, Excision in algebraic K-theory. *Ann. Math.* **136**(1992), No. 1, 51–122.

[Sw1] R. G. Swan, Algebraic K-Theory. *Lect. Notes Math.* **76**, *Springer-Verlag, Berlin-Heidelberg-New York,* 1968.

[Sw2] R. G. Swan, Excision in algebraic K-theory. *J. Pure Appl. Algebra* **1**(1971), 221–252.

[Sw3] R. G. Swan, On seminormality. *J. Algebra* **67**(1980), 210–229.

[V] T. Vorst, The Serre problem for discrete Hodge algebras. *Math. Z.* **184**(1983), 425–433.

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