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CRITERIA FOR THE BOUNDEDNESS OF POTENTIAL OPERATORS IN GRAND LEBESGUE SPACES

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Abstract. It is shown that the fractional integral operators with the parameter α, $0 < \alpha < 1$, are not bounded between the generalized grand Lebesgue spaces $L^{p),\theta_1}$ and $L^{q),\theta_2}$ for $\theta_2 < \theta_1 q/p$, where $1 < p < 1/\alpha$ and $q = \frac{p}{1-\alpha p}$. It is proved that the one–weight inequality

$$\|I_\alpha(fw^\alpha)\|_{L^{q),\,\theta q/p}_w} \le c\|f\|_{L^{p),\theta}_w},$$

where I_α is the potential operator on the interval $[0,1]$, holds if and only if $w \in A_{1+q/p'}([0,1])$.

რეზიუმე. ნაჩვენებია, რომ წილადური ინტეგრალური ოპერატორი α პარამეტრით, $0 < \alpha < 1$, არ არის შემოსაზღვრული $L^{p),\theta_1}$ და $L^{q),\theta_2}$ გრანდ ლებეგის სირვრცეებს შორის, სადაც $\theta_2 < \theta_1 q/p$, $1 < p < 1/\alpha$ და $q = \frac{p}{1-\alpha p}$. ნაჩვენებია, რომ ერთწონიანი უტოლობა

$$\|I_\alpha(fw^\alpha)\|_{L^{q),\,\theta q/p}_w} \le c\|f\|_{L^{p),\theta}_w},$$

სადაც I_α პოტენციალის ოპერატორია $[0,1]$ ინტერვალზე, ძალაშია მაშინ და მხოლოდ მაშინ, როცა $w \in A_{1+q/p'}([0,1])$.

Introduction

In this paper we prove that potential operators with the parameter α, $0 < \alpha < 1$, are not bounded from $L^{p)}$ to $L^{q)}$, where $1 < p < \infty$ and q is the Hardy–Littlewood–Sobolev exponent of p: $q = \frac{p}{1-\alpha p}$. This phenomena motivates us to investigate the boundedness problem for the Riesz potential operator I_α in the generalized grand Lebesgue spaces. In particular, we study this problem in weighted $L^{p),\theta}_w$ spaces and prove that the one–weight

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inequality

$$\|I_\alpha(fw^\alpha)\|_{L_w^{q),\ \theta q/p}([0,1])} \le c\|f\|_{L_w^{p),\theta}([0,1])}$$

holds if and only if w belongs to the Muckenhoupt's class $A_{1+q/p'}$.

The unweight spaces $L^{p),\theta}$ (i.e. $L_w^{p),\theta}$ for $w \equiv$ const) were introduced by E. Greco, T. Iwaniec and C. Sbordone [6] when they studied existence and uniqueness of the nonhomogeneous $n-$ harmonic equation $divA(x,\nabla u)=\mu$.

The grand Lebesgue spaces $L^{p)} = L^{p),1}$ first appeared in the paper by T. Iwaniec and C. Sbordone [7]. In that paper the authors showed that if $f=(f_1,\dots,f_n):\Omega\to\mathbb{R}^n$ belongs to the Sobolev class $W^{1,1}$, where Ω is an open subset in $\mathbb{R}^n$, $n\ge 2$, then the Jacobian determinant $J=J(f,x)=$ $det\ Df(x)$ $(J(x,f)\ge 0$ a.e.) of f belongs to the class $L^1_{\mathrm{loc}}(\Omega)$ provided that $g\in L^{n)}$, where

$$g(x) := |Df(x)| = \{\sup|Df(x)y| : y\in S^{n-1}\}.$$

Recently necessary and sufficient conditions guaranteeing the one–weight inequality for the Hardy–Littlewood maximal operator in $L_w^{p)}(I)$, where $I=[0,1]$, were established by A. Fiorenza, B. Gupra and P. Jain [4], while the same problem for the Hilbert transform was studied in the paper [9]. In particular, it turned out that the Hardy–Littlewood maximal operator (resp. the Hilbert transform) is bounded in $L_w^{p)}(I)$ if and only if the weight w belongs to the Muckenhoupt class $A_p(I)$.

1. Preliminaries

Let Ω be a bounded subset of $\mathbb{R}^n$ and let w be an a.e. positive, integrable function on Ω (i.e. a weight). The weighted generalized grand Lebesgue space $L^{p),\theta}(\Omega)$ $(1<p<\infty)$ is the class of those $f:\Omega\to\mathbb{R}$ for which the norm

$$\|f\|_{L_w^{p),\theta}(\Omega)} = \sup_{0<\varepsilon\le p-1}\left(\frac{\varepsilon^\theta}{|\Omega|}\int\limits_\Omega |f(t)|^{p-\varepsilon}w(t)dt\right)^{1/(p-\varepsilon)}$$

is finite.

If $w\equiv$ const 1, then we denote $L^{p),\theta}(\Omega) := L_w^{p),\theta}(\Omega)$. The space $L_w^{p),\theta}(\Omega)$ is not rearrangement invariant unless $w\equiv$ const.

Hölder's inequality and simple estimates yield the following embeddings (see also [6], [4]):

$$L_w^p(\Omega)\subset L_w^{p),\theta_1}(\Omega)\subset L_w^{p),\theta_2}(\Omega)\subset L_w^{p-\varepsilon}(\Omega), \tag{1.1}$$

where $0<\varepsilon<p-1$ and $\theta_1<\theta_2$.

In the classical weighted Lebesgue spaces L_w^p the equality

$$\|f\|_{L_w^p} = \|w^{1/p}f\|_{L^p}$$

holds but this property fails in the case of grand Lebesgue spaces. In particular, there is $f \in L^{p)}_w$ such that $w^{1/p} f \notin L^{p)}$ (see also [4] for the details).

Let φ be positive increasing function on $(0, p-1)$ satisfying the condition $\varphi(0+) = 0$, where $1 < p < \infty$. We will also need the following auxiliary class of functions defined on Ω and associated with φ:

$$L^{p),\varphi(x)}_w(\Omega) := \left\{ f : \sup_{0<\varepsilon\le p-1} \left(\varphi(\varepsilon)^{\frac{1}{p-\varepsilon}} \|f\|_{L^{p-\varepsilon}_w} \right) < \infty \right\}.$$

The space $L^{p),\theta}_w(\Omega)$, $\theta > 0$, is the special case of $L^{p),\varphi(x)}_w(\Omega)$ taking $\varphi(x) = \frac{x^\theta}{|\Omega|}$.

Throughout the paper the symbol $\varphi(t) \approx \psi(t)$ means that there exist positive constants c_1 and c_2 such that $c_1\varphi(t) \le \psi(t) \le c_2\psi(t)$. Constants (often different constants in the same series of inequalities) will generally be denoted by c or C. By the symbol p' we denote the conjugate number of p, i.e. $p' := \frac{p}{p-1}$, $1 < p < \infty$.

2. Fractional Integrals and Fractional Maximal Functions in Unweighted Grand Lebesgue Spaces

Let

$$(I_\alpha f)(x) = \int_0^1 \frac{f(y)}{|x-y|^{1-\alpha}}\, dy, \quad 0 < \alpha < 1$$

be the Riesz potential operator defined on $[0, 1]$. We begin this section with the following result:

Theorem 2.1. *Let $0 < \alpha < 1$, $1 < p < \frac{1}{\alpha}$, θ_1 and θ_2 be positive numbers such that $\theta_2 < \theta_1 q/p$, where $q = \frac{p}{1-\alpha p}$. Then the operator I_α is not bounded from $L^{p),\theta_1}$ to $L^{q),\theta_2}$.*

Proof. First observe that $q/p = 1+\alpha q$. Suppose the contrary: I_α is bounded from $L^{p),\theta_1}$ to $L^{q),\theta_2}$ i. e. the inequality

$$\|I_\alpha f\|_{L^{q),\theta_2}} \le c\|f\|_{L^{p),\theta_1}} \tag{2.1}$$

holds, where the positive constant c does not depend on f. Taking $f = \chi_J$ in (2.1), where J is an interval in $[0, 1]$, we have

$$(I_\alpha f)(x) = \int_J \frac{dy}{|x-y|^{1-\alpha}} \ge |J|^\alpha, \ \ x \in J.$$

Consequently,

$$\|I_\alpha f\|_{L^{q),\theta_2}} \ge |J|^\alpha \|\chi_J\|_{L^{q),\theta_2}}.$$

Taking inequality (2.1) into account we have that

$$|J|^\alpha \|\chi_J\|_{L^{q),\theta_2}} \le c\|\chi_J\|_{L^{p),\theta_1}}, \tag{2.2}$$

where the positive constant c does not depend on J.

Let us define the number $\varepsilon_J \in (0, p-1]$ satisfying the condition

$$\sup_{0<\varepsilon\le p-1} \left(\varepsilon^{\theta_1}|J|\right)^{\frac{1}{p-\varepsilon}} = \left(\varepsilon_J^{\theta_1}|J|\right)^{\frac{1}{p-\varepsilon_J}}. \tag{2.3}$$

Now we claim that $\lim\limits_{|J|\to 0} \varepsilon_J = 0$. Indeed, suppose the contrary: that there is a sequence of intervals J_n and a positive number λ such that $|J_n| \to 0$ and $\varepsilon_{J_n} \ge \lambda > 0$ for all $n \in N$. It is obvious that we can choose J_{n_0} so that

$$\frac{|J_{n_0}|^{\frac{1}{\theta_1}}(p-1)}{e} < e^{-\frac{p}{\lambda/2}}.$$

Now we claim that $g'(x) < 0$ for x satisfying the condition $x \in [\lambda/2, p-1]$, where $g(x) = \left(x^{\theta_1}|J_{n_0}|\right)^{\frac{1}{p-x}}$. Indeed, it is easy to see that for such an x,

$$\frac{|J_{n_0}|^{\frac{1}{\theta_1}}x}{e} \le \frac{|J_{n_0}|^{\frac{1}{\theta_1}}(p-1)}{e} < e^{-\frac{p}{\lambda/2}} \le e^{-\frac{p}{x}}$$

hold. Hence, using the formula

$$g'(t) = g(t)\cdot\frac{1}{p-t}\left[\frac{\ln\left(t^{\theta_1}|J_{n_0}|\right)}{p-t} + \frac{\theta_1}{t}\right]$$

and the fact that

$$g'(t) < 0 \iff \frac{t|J_{n_0}|^{\frac{1}{\theta_1}}}{e} < e^{-\frac{p}{t}}$$

we conclude that $g'(x) < 0$ for $x \in [\lambda/2, p-1]$. This observation together with the equality $\lim\limits_{x\to 0} g(x) = 0$ gives that $0 < \varepsilon_{J_{n_0}} < \lambda$, where $\varepsilon_{J_{n_0}}$ is defined by

$$\sup_{0<\varepsilon\le p-1} \left(\varepsilon^{\theta_1}|J_{n_0}|\right)^{\frac{1}{p-\varepsilon}} = \left(\varepsilon_{J_{n_0}}^{\theta_1}|J_{n_0}|\right)^{1/(p-\varepsilon_{J_{n_0}})}.$$

This contradicts the assumption that $\varepsilon_{J_n} \ge \lambda > 0$ for all n.

Further, we choose η_J so that

$$\alpha = \frac{1}{p} - \frac{1}{q} = \frac{1}{p-\varepsilon_J} - \frac{1}{q-\eta_J}.$$

This is equivalent to say that

$$\eta_J = q - \frac{p-\varepsilon_J}{1-\alpha(p-\varepsilon_J)}. \tag{2.4}$$

By (2.2) and (2.3) we have that

$$|J|^{\alpha}\eta_J^{\frac{\theta_2}{q-\eta_J}}|J|^{\frac{1}{q-\eta_J}} \le c\varepsilon_J^{\frac{\theta_1}{p-\varepsilon_J}}|J|^{\frac{1}{p-\varepsilon_J}}. \tag{2.5}$$

(here we used the fact that if ε_J is small, then $0 < \eta_J < q-1$). Now (2.5) yields:

$$\eta_J^{\frac{\theta_2}{q-\eta_J}} \varepsilon_J^{-\frac{\theta_1}{p-\varepsilon_J}} \le c. \tag{2.6}$$

Further, (2.4) and (2.6) imply

$$\left(\frac{q - \frac{p-\varepsilon_J}{1-\alpha(p-\varepsilon_J)}}{\varepsilon_J} \right)^{\frac{\theta_2}{p-\varepsilon_J} - \alpha\theta_2} \varepsilon_J^{-\frac{\theta_1}{p-\varepsilon_J} + \frac{\theta_2}{p-\varepsilon_J} - \alpha\theta_2} \le c. \tag{2.7}$$

Passing now to the limit as $|J| \to 0$ we see that the left-hand side of (2.7) tends to $+\infty$ because the limit of the first factor is $\left[\frac{1}{(1-\alpha p)^2}\right]^{\frac{\theta_2}{p} - \alpha\theta_2}$, and

$$\lim_{|J|\to 0} \varepsilon_J^{\frac{\theta_2-\theta_1}{p-\varepsilon_J} - \alpha\theta_2} = \lim_{|J|\to 0} \varepsilon_J^{\frac{\theta_2-\theta_1}{p} - \alpha\theta_2} = \infty$$

(Here we used the observation $\frac{\theta_2}{\theta_1} < 1 + \alpha q \Longleftrightarrow \frac{\theta_2-\theta_1}{p} - \alpha\theta_2 < 0$). □

Analysing the proof of Theorem 2.1 we have the result similar to that of the previous statement for the fractional maximal operator

$$M_\alpha f(x) = \sup_{\substack{J \ni x \\ J \subset [0,1]}} \frac{1}{|J|^{1-\alpha}} \int_J |f|, \quad x \in [0,1].$$

Theorem 2.2. *Let the conditions of Theorem* 2.1 *be satisfied. Then the operator M_α is not bounded from $L^{p),\theta_1}$ to $L^{q),\theta_2}$.*

Proof. Proof is the same as in the case of Theorem 2.1. We only need to observe that the inequality

$$M_\alpha f(x) \ge \frac{1}{|J|^{1-\alpha}} \int_J dx = |J|^\alpha, \quad x \in J,$$

holds for $f(x) = \chi_J(x)$, where J is a subinterval of $[0,1]$. Details are omitted. □

3. Sobolev's Embedding in Weighted Generalized Grand Lebesgue Spaces

This section is devoted to the investigation of the one–weight inequality for the operator I_α in $L^{p),\theta}_w$ spaces.

First we introduce the function

$$\varphi(u) = \left[\frac{u-q}{1-\alpha(u-q)} + p \right]^{1-(u-q)\alpha} \tag{3.1}$$

where $0 < \alpha < 1$, $1 < p < \frac{1}{\alpha}$, $q = \frac{p}{1-\alpha p}$.

To prove the main results we need some auxiliary statements.

Lemma 3.1. *$\varphi(x) \approx x^{1+\alpha q}$ near 0.*

The proof is straightforward and therefore is omitted.

Lemma 3.2. *Let $1 < q < \infty$ and let w be a weight. Then*

$$\|f\|_{L_w^{q),\varphi(x)}([0,1])} \approx \|f\|_{L_w^{q),1+\alpha q}([0,1])}$$

where φ is defined by (3.1).

Proof. Follows immediately from Lemma 3.1. □

Lemma 3.3. *Let $1 < q < \infty$ and let $\theta > 0$. Then*

$$\|f\|_{L_w^{q),\varphi(x^\theta)}([0,1])} \approx \|f\|_{L_w^{q),\theta(1+\alpha q)}([0,1])},$$

where φ is defined by (3.1).

The proof follows immediately from Lemma 3.1.

Lemma 3.4. *Let $1 < p < \infty$ and let Φ be a positive increasing function on $(0, p-1)$ satisfying $\Phi(0+) = 0$. Then there is a positive constant c such that for all intervals $J \subset [0,1]$ and $f \in L_w^{p,\Phi(x)}$ the inequality*

$$\|f\|_{L_w^{p),\Phi(x)}(J)} \leq c(w(J))^{-\frac{1}{p}} \left(\int_J |f(t)|^p w(t)dt \right)^{\frac{1}{p}} \|\chi_J\|_{L_w^{p),\Phi(x)}}$$

holds.

Proof. We have

$$\|f\|_{L_w^{p),\Phi(x)}(J)} = \sup_{0<\varepsilon\leq p-1} \left(\Phi(\varepsilon) \int_J |f(x)|^{p-\varepsilon} w(x)dx \right)^{\frac{1}{p-\varepsilon}} =$$

$$= \sup_{0<\varepsilon\leq p-1} \left(\Phi(\varepsilon) \int_J |f(x)|^{p-\varepsilon} w(x)^{\frac{p-\varepsilon}{p}} w(x)^{\frac{\varepsilon}{p}} dx \right)^{\frac{1}{p-\varepsilon}} \leq$$

$$\leq \sup_{0<\varepsilon\leq p-1} \Phi(\varepsilon)^{\frac{1}{p-\varepsilon}} \left(\int_J \left(|f(x)|^{p-\varepsilon} w(x)^{\frac{p-\varepsilon}{p}} \right)^{\frac{p}{p-\varepsilon}} dx \right)^{\frac{1}{p}} \times$$

$$\times \left(\int_J \left[w^{\frac{\varepsilon}{p}}(x) \right]^{\frac{p}{\varepsilon}} dx \right)^{\frac{\varepsilon}{p(p-\varepsilon)}} =$$

$$= \sup_{0<\varepsilon\leq p-1} \Phi(\varepsilon)^{\frac{1}{p-\varepsilon}} \left(\int_J |f(x)|^p w(x)dx \right)^{\frac{1}{p}} \left(\int_J w(x)dx \right)^{\frac{\varepsilon}{p(p-\varepsilon)}} =$$

$$= \left(\int_J |f(x)|^p w(x)dx \right)^{\frac{1}{p}} \left(\int_J w(x)dx \right)^{-\frac{1}{p}} \sup_{0<\varepsilon\leq p-1} \left(\Phi(\varepsilon) \int_J w(x)dx \right)^{\frac{1}{p-\varepsilon}} =$$

$$= \left(\int_J |f(x)|^p w(x) dx \right)^{\frac{1}{p}} (w(J))^{-\frac{1}{p}} \|\chi_J\|_{L_w^{p),\Phi(x)}(J)}. \qquad \square$$

Lemma 3.5. *Let $\theta > 0$, $1 < p < \infty$, $0 < \alpha < 1/p$ and let $q = \frac{p}{1-\alpha p}$. Let ψ be a positive increasing function on $(0, q-1)$ satisfying the condition $\psi(0+) = 0$. Suppose that the inequality*

$$\|I_\alpha(fw^\alpha)\|_{L_w^{q),\psi(x)}([0,1])} \le c\|f\|_{L_w^{p),\theta}([0,1])} \tag{3.2}$$

holds. Then

$$\int_0^1 w^{-p'/q}(x) dx < \infty.$$

Proof. Suppose the contrary: $\int_0^1 w^{-p'/q}(x)dx = \|w^{\alpha-1}\|_{L_w^{p'}} = \infty$. This means that there is a function $g \in L_w^p$ such that $\int_0^1 gw^\alpha = \infty$.

On the other hand,

$$I_\alpha(gw^\alpha)(x) = \int_0^1 \frac{g(t)w^\alpha(t)}{|x-t|^{1-\alpha}} dt \ge \int_0^1 g(t)w^\alpha(t) dt = \infty, \quad x \in [0,1].$$

Further, Lemma A implies that $g \in L_w^{p),\theta}([0,1])$. This contradicts (i). $\square$

Definition 3.1. Let $1 < r < \infty$. We say that a weight function w belongs to the Muckenhoupt's class $A_r([0,1])$ ($w \in A_r([0,1])$) if

$$A_r(w) := \sup_{J \subset [0,1]} \left(\frac{1}{|J|} \int_J w(t) dt \right)^{1/r} \left(\frac{1}{|J|} \int_J w^{1-r'}(t) dt \right)^{1/r'} < \infty,$$

where the supremum is taken over all subintervals J of $[0,1]$.

Lemma 3.6. *Let $0 < \alpha < 1$, $1 < p < 1/\alpha$. We set $q = \frac{p}{1-\alpha p}$. Suppose that $w \in A_{1+q/p'}([0,1])$, i.e.,*

$$\sup_{J \subset [0,1]} \left(\frac{1}{|J|} \int_J w(t) dt \right)^{1/q} \left(\frac{1}{|J|} \int_J w^{-p'/q}(t) dt \right)^{1/p'} < \infty.$$

Then there are positive constants σ_1, σ_2 and L satisfying the conditions:

$$\frac{1}{p-\sigma_2} - \frac{1}{q-\sigma_1} = \alpha, \quad w \in A_{1+\frac{q-\sigma_1}{(p-\sigma_2)'}},$$
$$\|K_\alpha\|_{L_w^{p-\eta} \to L_w^{q-\varepsilon}} \le L$$

for all $0 \le \varepsilon \le \sigma_1$, $0 \le \eta \le \sigma_2$ with $\frac{1}{p-\eta} - \frac{1}{q-\varepsilon} = \alpha$, where K_α is the operator defined as follows $K_\alpha f = I_\alpha(fw^\alpha)$.

Proof. Since $w \in A_{1+q/p'}$ by the openness property of Muckenhoupt's classes (see [11]) we have that there are small positive numbers σ_1 and σ_2 such that $\frac{1}{p-\sigma_2} - \frac{1}{q-\sigma_1} = \alpha$ and $w \in A_{1+(q-\sigma_1)/(p-\sigma_2)'}$.

Now we use the idea from [8]. By the result of B. Muckenhoupt and R. L. Wheeden [12] we have that the operator K_α is bounded from L^p_w to L^q_w and from $L^{p-\sigma_2}_w$ to $L^{q-\sigma_1}_w$. Let $0 < t < 1$ and let us define positive numbers η and ε so that

$$\frac{1}{p-\eta} = \frac{t}{p} + \frac{1-t}{p-\sigma_2}, \quad \frac{1}{q-\varepsilon} = \frac{t}{q} + \frac{1-t}{q-\sigma_1}.$$

Then by applying the Riesz–Thorin theorem (see e.g. [2], p. 16) we have that K_α is bounded from $L^{p-\eta}_w$ to $L^{q-\varepsilon}_w$ and moreover,

$$\|K_\alpha\|_{L^{p-\eta}_w \to L^{q-\varepsilon}_w} \le \|K_\alpha\|^t_{L^p_w \to L^q_w} \|K_\alpha\|^{1-t}_{L^{p-\sigma_2}_w \to L^{q-\sigma_1}_w}.$$

Observe now that

$$\frac{1}{p-\eta} - \frac{1}{q-\varepsilon} = \frac{t}{p} - \frac{t}{q} + \frac{1-t}{p-\sigma_2} - \frac{1-t}{q-\sigma_1} =$$
$$= t\Big(\frac{1}{p} - \frac{1}{q}\Big) + (1-t)\Big(\frac{1}{p-\sigma_2} - \frac{1}{q-\sigma_1}\Big) = t\alpha + (1-t)\alpha = \alpha.$$

The lemma is proved since we can take $L = \|K_\alpha\|_{L^p_w \to L^q_w} \|K_\alpha\|_{L^{p-\sigma_2}_w \to L^{q-\sigma_1}_w}$ (since without loss of generality we can assume that each factor in the latter expression is greater or equal to 1). □

Theorem 3.1. *Let $1 < p < \infty$ and let $0 < \alpha < 1/p$. Suppose that $\theta > 0$. We set $q = \frac{p}{1-\alpha p}$. Then the inequality*

$$\|I_\alpha(fw^\alpha)\|_{L^{q),\theta q/p}_w([0,1])} \le c\|f\|_{L^{p),\theta}_w([0,1])} \tag{3.3}$$

holds if and only if $w \in A_{1+q/p'}([0,1])$.

Proof. By Lemma 3.1 we have that (3.3) is equivalent to the inequality

$$\|I_\alpha(fw^\alpha)\|_{L^{q),\psi(x)}_w([0,1])} \le c\|f\|_{L^{p),\theta}_w([0,1])}, \tag{3.4}$$

where

$$\psi(x) = \varphi(x^\theta), \quad \varphi(x) = \left[\frac{x-q}{1-\alpha(x-q)} + p\right]^{1-(x-q)\alpha}. \tag{3.5}$$

Necessity. Let (3.3) and hence (3.4) hold. By Lemma 3.5 we have that $\int_0^1 w^{-p'/q} < \infty$. Let us take $f = \chi_J w^{-\alpha-p'/q}$. Then for $x \in J$, we get that

$$I_\alpha(w^\alpha f)(x) \ge \frac{1}{|J|^{1-\alpha}} \int_J w^\alpha f = \frac{1}{|J|^{1-\alpha}} \int_J w^{-p'/q}.$$

Hence,

$$\|I_\alpha(w^\alpha f)\|_{L_w^{q),\psi(x)}([0,1])} \geq |J|^{\alpha-1}\bigg(\int\limits_J w^{-p'/q}\bigg)\|\chi_J\|_{L_w^{q),\psi(x)}([0,1])}.$$

Further, by Lemma 3.4 with $\Phi(x) = x^\theta$ we find that

$$|J|^{\alpha-1}\bigg(\int\limits_J w^{-p'/q}\bigg)\|\chi_J\|_{L_w^{q),\psi(x)}([0,1])} \leq$$

$$\leq c\|f\|_{L^{p),\theta}([0,1])} \leq c(w(J))^{-\frac{1}{p}}\bigg(\int\limits_J |f(t)|^p w(t)dt\bigg)^{\frac{1}{p}}\|\chi_J\|_{L_w^{p),\theta}([0,1])} =$$

$$= cw(J)^{-\frac{1}{p}}\bigg(\int\limits_J w^{-p'/q}\bigg)^{1/p}\|\chi_J\|_{L_w^{p),\theta}([0,1])}.$$

It is easy to see that there is a number η_J depending on J such that $0 < \eta_J \leq p-1$ and

$$|J|^{\alpha-1}w(J)^{\frac{1}{p}}\bigg(\int\limits_J w^{-p'/q}\bigg)^{\frac{1}{p'}}\|\chi_J\|_{L_w^{q),\psi(x)}([0,1])} \leq c\left(\eta_J w(J)\right)^{\frac{1}{p-\eta_J}}.$$

For such an η_J we choose ε_J so that

$$\frac{1}{p-\eta_J} - \frac{1}{q-\varepsilon_J} = \alpha.$$

Then $0 < \varepsilon_J \leq q-1$ and

$$|J|^{\alpha-1}w(J)^{\frac{1}{p}-\frac{1}{p-\eta_J}}\eta_J^{-\frac{\theta}{p-\eta_J}}\psi(\varepsilon_J)^{\frac{1}{q-\varepsilon_J}}w(J)^{\frac{1}{q-\varepsilon_J}}\bigg(\int\limits_J w^{-p'/q}\bigg)^{\frac{1}{p'}} \leq c.$$

Observe that by Lemma 3.1 we have that

$$\eta_J^{-\frac{\theta}{p-\eta_J}}\psi(\varepsilon_J)^{\frac{1}{q-\varepsilon_J}} = \eta_J^{-\frac{\theta}{p-\eta_J}}\varphi\left(\varepsilon_J^\theta\right)^{\frac{1}{q-\varepsilon_J}} \approx \eta_J^{-\frac{\theta}{p-\eta_J}}\varepsilon_J^{\frac{\theta(1+\alpha q)}{q-\varepsilon_J}} =$$

$$= \left(\eta_J^{-\frac{1}{p-\eta_J}}\varepsilon_J^{\frac{1+\alpha q}{q-\varepsilon_J}}\right)^\theta \approx \left(\eta_J^{-\frac{1}{p-\eta_J}}\varphi(\varepsilon_J)^{\frac{1}{q-\varepsilon_J}}\right)^\theta = 1$$

and also,

$$\frac{1}{p} - \frac{1}{p-\eta_J} + \frac{1}{q-\varepsilon_J} = \frac{1}{p} - \alpha = \frac{1}{q}.$$

Finally, we have that

$$|J|^{\alpha-1}w(J)^{\frac{1}{q}}\bigg(\int\limits_J w^{-p'/q}\bigg)^{1/p'} \leq c.$$

Necessity is proved.

Sufficiency. Using Lemma 3.6 we have that there are positive constants σ_1, σ_2 and L satisfying the conditions: $\frac{1}{p-\sigma_2}-\frac{1}{q-\sigma_1}=\alpha$, $w\in A_{1+\frac{q-\sigma_1}{(p-\sigma_2)'}}$, $\|K_\alpha\|_{L^{p-\eta}_w\to L^{q-\varepsilon}_w}\le L$ for all $0\le\varepsilon\le\sigma_1$, $0\le\eta\le\sigma_2$ with $\frac{1}{p-\eta}-\frac{1}{q-\varepsilon}=\alpha$, where K_α is the operator defined by $K_\alpha f=I_\alpha(fw^\alpha)$.

Let σ be a small positive number such that $\sigma<\sigma_1<q-1$ and let us fix $\varepsilon\in(\sigma,q-1]$. Then $\frac{q-\sigma}{q-\varepsilon}>1$. By Hölder's inequality we have that

$$\|I_\alpha(fw^\alpha)\|_{L^{q-\varepsilon}_w([0,1])}\le\left(\int_0^1|I_\alpha(fw^\alpha)(x)|^{q-\sigma}w(x)dx\right)^{\frac{1}{q-\sigma}}w([0,1])^{\frac{\varepsilon-\sigma}{(q-\sigma)(q-\varepsilon)}}$$

because $\left(\frac{q-\sigma}{q-\varepsilon}\right)'=\frac{q-\sigma}{\varepsilon-\sigma}$.

Further, the conditions $\sigma<q-1$ and $\sigma<\varepsilon<q-1$ yield

$$0<\frac{\varepsilon-\sigma}{(q-\sigma)(q-\varepsilon)}<\frac{q-1-\sigma}{q-\sigma}.$$

Consequently, using the well–known result by B. Muckenhoupt and R. L. Wheeden [12] for the classical weighted Lebesgue spaces:

$$\|I_\alpha(fw^\alpha)\|_{L^q_w([0,1])}\le c\|f\|_{L^p_w([0,1])}\Longleftrightarrow w\in A_{1+q/p'}([0,1]),\ \ q=\frac{p}{1-\alpha p},$$

we find that

$$\begin{aligned}
\|I_\alpha(fw^\alpha)\|_{L^{q),\psi(x)}_w([0,1])}&=\max\Big\{\sup_{0<\varepsilon\le\sigma}\psi(\varepsilon)^{\frac{1}{q-\varepsilon}}\|I_\alpha(fw^\alpha)\|_{L^{q-\varepsilon}_w([0,1])},\\
&\qquad\sup_{\sigma<\varepsilon\le q-1}\psi(\varepsilon)^{\frac{1}{q-\varepsilon}}\|I_\alpha(fw^\alpha)\|_{L^{q-\varepsilon}_w([0,1])}\Big\}\le\\
&\le\max\Big\{\sup_{0<\varepsilon\le\sigma}\psi(\varepsilon)^{\frac{1}{q-\varepsilon}}\|I_\alpha(fw^\alpha)\|_{L^{q-\varepsilon}_w([0,1])},\\
&\qquad\|I_\alpha(fw^\alpha)\|_{L^{q-\sigma}_w}\sup_{\sigma<\varepsilon\le q-1}\psi(\varepsilon)^{\frac{1}{q-\varepsilon}}w([0,1])^{\frac{\varepsilon-\sigma}{(q-\sigma)(q-\varepsilon)}}\Big\}\le\\
&\le\max\Big\{1,\sup_{\sigma<\varepsilon\le q-1}\psi(\varepsilon)^{\frac{1}{q-\varepsilon}}\psi(\sigma)^{-\frac{1}{q-\sigma}}w([0,1])^{\frac{\varepsilon-\sigma}{(q-\sigma)(q-\varepsilon)}}\Big\}\times\\
&\qquad\times\sup_{0<\varepsilon\le\sigma}\psi(\varepsilon)^{\frac{1}{q-\varepsilon}}\|I_\alpha(fw^\alpha)\|_{L^{q-\varepsilon}_w([0,1])}\le\\
&\le c\max\Big\{1,\Big[\sup_{\sigma<\varepsilon\le q-1}(\psi(\varepsilon))^{\frac{1}{q-\varepsilon}}\Big]\psi(\sigma)^{-\frac{1}{q-\sigma}}(1+w([0,1])^{\frac{q-1-\sigma}{q-\sigma}}\Big\}\times\\
&\qquad\times\sup_{0<\eta\le\sigma_0}\eta^{\frac{\theta}{p-\eta}}\|f\|_{L^{p-\eta}_w([0,1])}\le\\
&\le c\Big(\sup_{\sigma<\varepsilon\le q-1}\psi(\varepsilon)^{\frac{1}{q-\varepsilon}}\Big)\psi(\sigma)^{-\frac{1}{q-\sigma}}(1+w([0,1]))^{\frac{q-1-\sigma}{q-\sigma}}\|f\|_{L^{p),\theta}_w([0,1])}.
\end{aligned}$$

Here σ_0 is small positive number chosen so that if $0<\varepsilon\le\sigma$, then $0<\eta\le\sigma_0<\sigma_1<p-1$. Also, we used the estimates:

$$\psi(\varepsilon)^{\frac{1}{q-\varepsilon}}\approx\varepsilon^{\frac{\theta(1+\alpha q)}{q-\varepsilon}}\approx\varphi(\varepsilon)^{\frac{\theta}{q-\varepsilon}}=\eta^{\frac{\theta}{p-\eta}},\quad\text{as}\quad\varepsilon\to0,$$

where $\frac{1}{p-\eta}-\frac{1}{q-\varepsilon}=\alpha$. □

Remark 3.1. Theorem 3.1 implies that if $1<p<\infty$, $0<\alpha<1/p$, $q=\frac{p}{1-\alpha}$ and $\mu>0$, then the one-weight inequality

$$\sup_{0<\varepsilon<q-1}\varepsilon^{\mu}\Bigg(\int_0^1|I_\alpha(fw^\alpha)(x)|^{q-\varepsilon}w(x)dx\Bigg)^{\frac{1}{q-\varepsilon}}\le$$

$$\le C\sup_{0<\eta<p-1}\eta^{\mu}\Bigg(\int_0^1|f(x)|^{p-\eta}w(x)dx\Bigg)^{\frac{1}{p-\eta}}$$

with the positive constant C independent of f holds if and only if $w\in A_{1+q/p'}([0,1])$.

This follows from the following easily verifiable relation

$$\|g\|_{L_w^{r),\theta}([0,1])}\approx\sup_{0<\varepsilon<r-1}\varepsilon^{\frac{\theta}{r}}\Bigg(\int_0^1|g(x)|^{r-\varepsilon}w(x)dx\Bigg)^{\frac{1}{r-\varepsilon}},$$

which holds for weighted grand Lebesgue space $L_w^{r),\theta}([0,1])$, where $1<r<\infty$ and $\theta>0$.

Corollary 3.1. *Let $\theta>0$ and let $1<p<\infty$. Suppose that $0<\alpha<1/p$. We set $q=\frac{p}{1-\alpha p}$. Then I_α is bounded from $L^{p),\theta_1}([0,1])$ to $L^{q),\theta_2}([0,1])$ provided that $\theta_2>\theta_1 q/p$.*

Proof. follows immediately from Theorem 3.1 (in the unweighted case $w(x)\equiv$ const) and (1.1). □

4. One-sided potentials

In this section we show that the unboudedness result in grand Lebesgue spaces is also true for the one–sided potentials:

$$(R_\alpha f)(x)=\int_0^x\frac{f(t)}{(x-t)^{1-\alpha}}dt,\quad x\in(0,1);$$

and

$$(W_\alpha f)(x)=\int_x^1\frac{f(t)}{(t-x)^{1-\alpha}}dt,\quad x\in(0,1),$$

where $0<\alpha<1$. In particular, we claim that R_α and W_α are not bounded from $L^{p),\theta_1}$ to $L^{q),\theta_2}$, where $q=\frac{p}{1-\alpha p}$, $1<p<\infty$, θ_1, $\theta_2>0$, $\theta_2<\frac{\theta_1 q}{p}$. Indeed, let us show the result first for R_α.

Suppose the contrary:

$$\|R_\alpha f\|_{L^{q),\theta_2}([0,1])}\le c\|f\|_{L^{p),\theta_1}([0,1])},\qquad \theta_2<\frac{\theta_1 q}{p}, \tag{4.1}$$

where c does not depend on f. Let $f_n(x)=\chi_{(0,1/2n)}(x)$ in (4.1). Then taking the following inequality

$$(R_\alpha f_n)(x)\ge\int\limits_0^{\frac{1}{2n}}\frac{1}{(x-t)^{1-\alpha}}dt\ge\Big(\frac{1}{2n}\Big)^\alpha,\quad x\in\left(\frac{1}{2n},\frac{1}{n}\right), \tag{4.2}$$

into account, (4.1) yields that

$$(2n)^{-\alpha}\left\|\chi_{\left(\frac{1}{2n},\frac{1}{n}\right)}\right\|_{L^{q),\theta_2}([0,1])}\le c\left\|\chi_{(0,1/2n)}\right\|_{L^{p),\theta_1}([0,1])}. \tag{4.3}$$

Now we choose ε_n positive number so that

$$\sup_{0<\varepsilon\le p-1}\left(\varepsilon^{\theta_1}\frac{1}{2n}\right)^{\frac{1}{p-\varepsilon}}=\left(\varepsilon_n^{\theta_1}\frac{1}{2n}\right)^{\frac{1}{p-\varepsilon_n}}. \tag{4.4}$$

We now observe that $\lim\limits_{n\to 0}\varepsilon_n=0$ (see the proof of Theorem 2.1 for the similar arguments). Choose now η_n so that

$$\alpha=\frac{1}{p}-\frac{1}{q}=\frac{1}{p-\varepsilon_n}-\frac{1}{q-\eta_n}.$$

Hence,

$$\eta_n=q-\frac{p-\varepsilon_n}{1-\alpha(p-\varepsilon_n)}. \tag{4.5}$$

By (4.3)–(4.5) we conclude that

$$(2n)^{-\alpha}\eta_n^{\frac{\theta_2}{q-\eta_n}}\left(\frac{1}{2n}\right)^{\frac{1}{q-\eta_n}}\le c\varepsilon_n^{\frac{\theta_1}{p-\varepsilon_n}}(2n)^{-1/(p-\varepsilon_n)}. \tag{4.6}$$

From (4.6) we have that

$$\eta_n^{\frac{\theta_2}{q-\eta_n}}\varepsilon_n^{-\frac{\theta_1}{p-\varepsilon_n}}\le c_p,\ \text{ for all } n\in N \tag{4.7}$$

because

$$\frac{1}{2}\le\left(\frac{1}{2}\right)^{\frac{1}{p-\varepsilon_n}}\le\left(\frac{1}{2}\right)^{\frac{1}{p}},$$
$$\frac{1}{2}\le\left(\frac{1}{2}\right)^{\frac{1}{q-\eta_n}}\le\left(\frac{1}{2}\right)^{\frac{1}{q}}.$$

Now (4.5) yields

$$\left[\frac{q-\frac{p-\varepsilon_n}{1-\alpha(p-\varepsilon_n)}}{\varepsilon_n}\right]^{\frac{\theta_2}{p-\varepsilon_n}-\alpha\theta_2} \varepsilon_n^{-\frac{\theta_1}{p-\varepsilon_n}+\frac{\theta_2}{p-\varepsilon_n}-\alpha\theta_2} \le c_p.$$

Hence,

$$\left[\frac{q-\frac{p-\varepsilon_n}{1-\alpha(p-\varepsilon_n)}}{\varepsilon_n}\right]^{\frac{\theta_2}{p-\varepsilon_n}-\alpha\theta_2} \varepsilon_n^{\frac{\theta_2-\theta_1}{p-\varepsilon_n}-\alpha\theta_2} \le c_p,$$

which is impossible, because $\lim\limits_{n\to\infty} \varepsilon_n^{\frac{\theta_2-\theta_1}{p-\varepsilon_n}-\alpha\theta_2} = \infty$ (recall that $\frac{\theta_2-\theta_1}{p}-\alpha\theta_2 = \frac{\theta_2}{q}-\frac{\theta_1}{p} < 0$).

Analogously, we have that W_α is not bounded from $L^{p),\theta_1}$ to $L^{q),\theta_2}$. This follows from the inequalities

$$(W_\alpha)(x) \ge \int_x^{1-\frac{1}{3n}} \frac{f(t)}{(t-x)^{1-\alpha}}dt \ge \left(\frac{2}{3n}\right)^{\alpha-1}\cdot\frac{1}{6n} = c_\alpha n^{-\alpha},\quad x\in\left(1-\frac{1}{n},1-\frac{1}{2n}\right),$$

where $f(t) = \chi_{(1-\frac{1}{2n},1-\frac{1}{3n})}(t)$. Hence,

$$c_\alpha n^{-\alpha}\left\|\chi_{\left(1-\frac{1}{n},1-\frac{1}{2n}\right)}\right\|_{L^{q),\theta_2}([0,1])} \le c\left\|\chi_{\left(1-\frac{1}{2n},1-\frac{1}{3n}\right)}\right\|_{L^{p),\theta_1}([0,1])}.$$

Choosing now ε_n so that

$$\left[\varepsilon_n^{\theta_1}\frac{1}{6n}\right]^{\frac{1}{p-\varepsilon_n}} = \sup_{0<\varepsilon_n\le p-1}\left[\varepsilon_n^{\theta_1}\frac{1}{6n}\right]^{\frac{1}{p-\varepsilon}},\quad 0<\varepsilon_n\le p-1,$$

and observing that $\lim\limits_{n\to\infty}\varepsilon_n = 0$ (see the proof of Theorem 2.1 for the similar arguments) we find that the conclusion similar to the case of R_α is valid.

4.1. **Conclusions and Remarks.**

Let $0<\alpha<1$ and let I_α, R_α, W_α be potential operators defined above. In the sequel we denote by T_α one of these operators.

Corollary 4.1. *Let $1<p<\infty$ and let $0<\alpha<1/p$. We set $q=\frac{p}{1-\alpha p}$. Suppose that θ_1 and θ_2 be positive numbers. Then:*
(i) *If $\theta_2 < \theta_1 q/p$, then T_α is not bounded from $L^{p),\theta_1}$ to $L^{q),\theta_2}$.*
(ii) *If $\theta_2 \ge \theta_1 q/p$, then T_α is bounded from $L^{p),\theta_1}$ to $L^{q),\theta_2}$.*

Remark 4.1. There is a function f from $L^{p)}\backslash L^p$ such that $T_\alpha f\in L^{q)}\backslash L^q$.

Indeed, let $f(t)=t^{-\frac{1}{p}}$, $t\in(0,1)$. Then $f\in L^{p)}\backslash L^p$. On the other hand, (see e. g. [13]), $T_\alpha f \approx t^{-\frac{1}{q}}$. Hence $T_\alpha f\in L^{q)}\backslash L^q$.

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