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# CYCLIC REFINEMENT OF BECK'S INEQUALITIES

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**Abstract.** In this paper, we refine the discrete Jensen's inequality for vectors by the idea recently given in [2]. As a consequence, we are able to refine the inequality of E. Beck [1] with the help of cyclic generalized mixed symmetric means. This leads to the refinements of the classical Hölder and Minkowski's inequalities.

**რეზიუმე.** [2] შრომაში განვითარებულ იდეაზე დაყრდნობით დაზუსტებულია იენსენის უტოლობა ვექტორებისათვის. ამ შედეგზე დაყრდნობით ზუსტდება ე. ბეკის [1] უტოლობა ციკლური განზოგადებული შერეული საშუალოების გამოყენებით. ამ უკანსაკნელ უტოლობას მოსდევს ჰელდერისა და მინკოვსკის კლასიკური უტოლობების დაზუსტება.

## 1. Introduction and Preliminary Results

Let $U$ be a convex subset of a real linear space, and let $f : U \to \mathbb{R}$ be a convex function. If $x_i \in U$ $(1 \le i \le n)$ and $p_i \ge 0$ $(1 \le i \le n)$ such that $\sum\limits_{i=1}^{n} p_i = 1$, then the discrete Jensen's inequality

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \le \sum_{i=1}^{n} p_i f(x_i) \tag{1}$$

holds. Particularly, we have

$$f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \le \frac{1}{n}\sum_{i=1}^{n} f(x_i). \tag{2}$$

Let $I \subset \mathbb{R}$ be an interval, $h : I \to \mathbb{R}$ be a continuous and strictly monotone function, and let $\mathbf{a} = (a_1, \dots, a_n) \in I^n$. Then the quasi-arithmetic

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$h$-mean of vector $\mathbf{a}$ is defined by

$$h_n(\mathbf{a}) = h_n(a_i; 1 \le i \le n) = h(\mathbf{a}; n) := h^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} h(a_i)\right).$$

First, we extend Beck's results (see [1]). The use will be made of the following hypothesis:

($A_1$) Let $L_t : I_t \to \mathbb{R}$ $(t = 1, \ldots, m)$ and $N : I_N \to \mathbb{R}$ be continuous and strictly monotone functions whose domains are intervals in $\mathbb{R}$, and let $f : I_1 \times \cdots \times I_m \to I_N$ be a continuous function. Let $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m)} \in \mathbb{R}^n$ $(n \ge 2)$ such that $\mathbf{x}^{(t)} \in I_t^n$ for each $t = 1, \ldots, m$.

The following result is a simple consequence of the discrete Jensen's inequality (2).

**Theorem 1.1.** *Assume* ($A_1$). *If* $N$ *is an increasing function, then the inequality*

$$f\left(L_1(\mathbf{x}^{(1)}; n), \ldots, L_m(\mathbf{x}^{(m)}; n)\right) \ge$$
$$\ge N^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} N(f(x_i^{(1)}, \ldots, x_i^{(m)}))\right), \tag{3}$$

*holds for all possible* $\mathbf{x}^{(t)}$ $(t = 1, \ldots, m)$, *if and only if the function* $H$ *defined on* $L_1(I_1) \times \cdots \times L_m(I_m)$ *by*

$$H(t_1, \ldots, t_m) := N\left(f\left(L_1^{-1}(t_1), \ldots, L_m^{-1}(t_m)\right)\right) \tag{4}$$

*is concave. The inequality in* (3) *is reversed for all possible* $\mathbf{x}^{(t)}$ $(t = 1, \ldots, m)$, *if and only if* $H$ *is convex.*

*Proof.* We replace the convex function $f$ by $-H$ or $H$, and $x_i$ by $L_t(x_i^{(t)})$ in (2) and then, applying the increasing function $N^{-1}$, we get the required results. □

Beck's original result (see [4], p. 249 or [3], p. 300) was the weighted form of Theorem 1.1 (see in [10], p. 157), but with $m = 2$ and $I_1 = [k_1, k_2]$, $I_2 = [l_1, l_2]$ and $I_N = [n_1, n_2]$.

For the simplicity, in case $m = 2$ we use the following form of ($A_1$):

($A_2$) Let $K : I_K \to \mathbb{R}$, $L : I_L \to \mathbb{R}$ and $N : I_N \to \mathbb{R}$ be continuous and strictly monotone functions whose domains are intervals in $\mathbb{R}$, and let $f : I_K \times I_L \to I_N$ be a continuous function. Let $\mathbf{a}$, $\mathbf{b} \in \mathbb{R}^n$ $(n \ge 2)$ such that $\mathbf{a} \in I_K^n$ and $\mathbf{b} \in I_L^n$.

Then (3) has the form

$$f\left(K_n(\mathbf{a}), L_n(\mathbf{b})\right) \ge N_n\left(f(\mathbf{a}, \mathbf{b})\right), \tag{5}$$

where $f(\mathbf{a}, \mathbf{b})$ means $(f(a_1, b_1), \ldots, f(a_n, b_n))$.

The following results are the important special cases of Theorem 1.1 and generalize the corresponding results of Beck. We use the following hypothesis:

($A_3$) Let $K : I_K \to \mathbb{R}$, $L : I_L \to \mathbb{R}$ and $N : I_N \to \mathbb{R}$ be continuous and strictly monotone functions whose domains are intervals in $\mathbb{R}$ such that either $I_K + I_L \subset I_N$ and $f(x,y) = x+y$ $((x,y) \in I_K \times I_L)$, or $I_K$, $I_L \subset ]0,\infty[$, $I_K \cdot I_L \subset I_N$ and $f(x,y) = xy$ $((x,y) \in I_K \times I_L)$. Assume further that the functions $K$, $L$ and $N$ are twice continuously differentiable on the interior of their domains, respectively. Let $\mathbf{a}$, $\mathbf{b} \in \mathbb{R}^n$ $(n \geq 2)$ such that $\mathbf{a} \in I_K^n$ and $\mathbf{b} \in I_L^n$.

The interior of a subset $A$ of $\mathbb{R}$ is denoted by $A^\circ$.

**Corollary 1.2.** *Assume* ($A_3$) *with* $f(x,y) = x+y$ $((x,y) \in I_K \times I_L)$, *and assume that* $K'$, $L'$, $N'$, $K''$, $L''$ *and* $N''$ *are all positive. Introducing* $E := \frac{K'}{K''}$, $F := \frac{L'}{L''}$, $G := \frac{N'}{N''}$, (5) *holds for all possible tuples* $\mathbf{a}$ *and* $\mathbf{b}$, *if and only if*

$$E(x) + F(y) \leq G(x+y), \quad (x,y) \in I_K^\circ \times I_L^\circ. \tag{6}$$

**Corollary 1.3.** *Assume* ($A_3$) *with* $f(x,y) = xy$ $((x,y) \in I_K \times I_L)$. *Suppose the functions* $A(x) := \frac{K'(x)}{K'(x)+xK''(x)}$, $B(x) := \frac{L'(x)}{L'(x)+xL''(x)}$ *and* $C(x) := \frac{N'(x)}{N'(x)+xN''(x)}$ *are defined on* $I_K^\circ$, $I_L^\circ$ *and* $I_N^\circ$, *respectively. Assume further that* $K'$, $L'$, $N'$, $A$, $B$ *and* $C$ *are all positive. Then* (5) *holds for all possible tuples* $\mathbf{a}$ *and* $\mathbf{b}$, *if and only if*

$$A(x) + B(y) \leq C(xy), \quad (x,y) \in I_K^\circ \times I_L^\circ.$$

To prove these corollaries, similar arguments can be applied as in the analogous results of Beck. We just sketch the proof of Corollary 1.2.

*Proof.* By Theorem 1.1, it is enough to prove that the function

$$H : K(I_K) \times L(I_L) \to \mathbb{R}, \quad H(t,s) := N\left(K^{-1}(t) + L^{-1}(s)\right)$$

is concave. Since $H$ is continuous, and twice continuously differentiable on the interior $K(I_K^\circ) \times L(I_L^\circ)$ of its domain, we have to show that

$$h_1^2 \frac{\partial^2 H(t,s)}{\partial t^2} + 2h_1h_2 \frac{\partial^2 H(t,s)}{\partial t \partial s} + h_2^2 \frac{\partial^2 H(t,s)}{\partial s^2} \leq 0$$

for all $(t,s) \in K(I_K^\circ) \times L(I_L^\circ)$ and $(h_1,h_2) \in \mathbb{R}^2$. By computing the partial derivatives of $H$ of order 2 at the points of $K(I_K^\circ) \times L(I_L^\circ)$, we have the condition (6) (see [3] p. 303). □

The interpolations of the discrete Jensen's inequality (2) given in [13] are used in [11] (see also [12], p.195) to refine the inequality of E. Beck for a function of two variables. The similar idea is utilized in [8, 9] (see also [10], Chapter 7) for the refinements of weighted discrete Jensen's inequality (1) appeared in [5, 6, 7]. Analogously, in this paper we work out the new refinement of Beck's inequality (3) by cyclic mixed symmetric means as a consequence of the new refinement of the discrete Jensen's inequality (2) constructed in [2]. This, obviously, leads to some new refinements of the classical Hölder and Minkowski's inequalities.

We need another hypothesis:

($H_2$) Let $U$ be a convex set in $\mathbb{R}^m$, $\mathbf{x}_1, \dots, \mathbf{x}_n \in U$, such that $\mathbf{x}_{i+n} = \mathbf{x}_i$, and $\lambda := (\lambda_1, \dots, \lambda_n)$ be a positive $n$-tuple such that $\sum_{i=1}^{k} \lambda_i = 1$ for $2 \le k \le n$. Further, let $f : U \to \mathbb{R}$ be a convex function.

The following refinement of the discrete Jensen's inequality for functions of several variables is analogous to the refinement given in [2] for the function of one variable:

**Theorem A.** *Assume* ($H_2$), *and consider the following sum*

$$S = \frac{1}{n} \sum_{i=1}^{n} f\left( \sum_{j=0}^{k-1} \lambda_{j+1} \mathbf{x}_{i+j} \right). \tag{7}$$

*Then*

$$f\left( \frac{\sum_{i=1}^{n} \mathbf{x}_i}{n} \right) \le \frac{1}{n} \sum_{i=1}^{n} f\left( \sum_{j=0}^{k-1} \lambda_{j+1} \mathbf{x}_{i+j} \right) \le \frac{\sum_{i=1}^{n} f(\mathbf{x}_i)}{n}. \tag{8}$$

*Proof.* The idea of proof is the same as that given in [2].

First, we prove the second inequality in (8). Since $f$ is convex, by Jensen's inequality, we have

$$\sum_{i=1}^{n} f\left( \sum_{j=0}^{k-1} \lambda_{j+1} \mathbf{x}_{i+j} \right) \le \sum_{i=1}^{n} \sum_{j=0}^{k-1} \lambda_{j+1} f(\mathbf{x}_{i+j}) =$$

$$= \sum_{i=1}^{n} f(\mathbf{x}_i) \sum_{j=1}^{k} \lambda_j = \sum_{i=1}^{n} f(\mathbf{x}_i)$$

Now we prove the first inequality in (8). Since $f$ is convex, by Jensen's inequality, we have

$$\frac{1}{n} \sum_{i=1}^{n} f\left( \sum_{j=0}^{k-1} \lambda_{j+1} \mathbf{x}_{i+j} \right) \ge f\left( \frac{\sum_{i=1}^{n} \sum_{j=0}^{k-1} \lambda_{j+1} \mathbf{x}_{i+j}}{n} \right) =$$

$$= f\left( \frac{\sum_{i=1}^{n} \mathbf{x}_i \sum_{j=1}^{k} \lambda_j}{n} \right) = f\left( \frac{\sum_{i=1}^{n} \mathbf{x}_i}{n} \right) \qquad \square$$

## 2. Refinement of Beck's Inequality

In what follows, we assume ($A_1$) such that $x_{i+n}^{(t)} = x_i^{(t)}$ ($t = 1, \dots, m$) and $(\lambda_1, \dots, \lambda_n)$ is a positive $n$-tuple in the way that $\sum_{i=1}^{k} \lambda_i = 1$ for $2 \le k \le n$.

The cyclic mixed symmetric means relative to (7) are defined by

$$M(L_1,\dots,L_m;\mathbf{x}^{(1)},\dots,\mathbf{x}^{(m)}) := \\ = N^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} N\left(f\left(L_1(\mathbf{x}^{(1)};k),\dots,L_m(\mathbf{x}^{(m)};k)\right)\right)\right) \tag{9}$$

$$L_t(\mathbf{x}^{(t)};k) = L_t^{-1}\left(\sum_{j=0}^{k-1}\lambda_{j+1}L_t(x_{i+j}^{(t)})\right);\quad t=1,\dots,m.$$

Now, we get an interpolation of (3) by the direct application of Theorem A as follows.

**Theorem 2.1.** *Assume* $(A_1)$ *such that* $x_{i+n}^{(t)} = x_i^{(t)}$ $(t=1,\dots,m)$, *and* $(\lambda_1,\dots,\lambda_n)$ *is a positive* $n$*-tuple such that* $\sum_{i=1}^{k}\lambda_i = 1$ *for* $2\le k\le n$. *If* $N$ *is an increasing* (*decreasing*) *function, then the inequalities*

$$f\left(L_1(\mathbf{x}^{(1)};n),\dots,L_m(\mathbf{x}^{(m)};n)\right) \le M(L_1,\dots,L_m;\mathbf{x}^{(1)},\dots,\mathbf{x}^{(m)}) \le \\ \le N^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}N(f(x_i^{(1)},\dots,x_i^{(m)}))\right), \tag{10}$$

*hold for all possible* $\mathbf{x}^{(t)}$ $(t=1,\dots,m)$, *if and only if the function* $H$ *defined in Theorem* 1.1 *is convex* (*concave*). *If* $N$ *is an increasing* (*decreasing*) *function, then the inequalities in* (10) *are reversed for all possible* $\mathbf{x}^{(t)}$ $(t=1,\dots,m)$, *if and only if* $H$ *is concave* (*convex*).

*Proof.* Suppose $N$ is increasing and the function $H : L_1(I_1)\times\cdots\times L_m(I_m) \to \mathbb{R}$,

$$H(t_1,\dots,t_m) = N\left(f\left(L_1^{-1}(t_1),\dots,L_m^{-1}(t_m)\right)\right)$$

is convex. We apply Theorem A to the function $H$ and to the vectors $(L_1(x_i^{(1)}),\dots,L_m(x_i^{(m)}))$, $i=1,\dots,n$. Then the first term in (8) gives

$$H\left(\frac{1}{n}\sum_{i=1}^{n}\left(L_1(x_i^{(1)}),\dots,L_m(x_i^{(m)})\right)\right) = \\ = H\left(\frac{1}{n}\sum_{i=1}^{n}L_1(x_i^{(1)}),\dots,\frac{1}{n}\sum_{i=1}^{n}L_m(x_i^{(m)})\right) = \\ = N\left(f\left(L_1^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}L_1(x_i^{(1)})\right),\dots,L_m^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}L_m(x_i^{(m)})\right)\right)\right) = \\ = N\left(f\left(L_1(\mathbf{x}^{(1)};n),\dots,L_m(\mathbf{x}^{(m)};n)\right)\right).$$

The last term in (8) is

$$\frac{1}{n}\sum_{i=1}^{n} H(L_1(x_i^{(1)}),\ldots,L_m(x_i^{(m)})) = \frac{1}{n}\sum_{i=1}^{n} N\left(f\left(x_i^{(1)},\ldots,x_i^{(m)}\right)\right),$$

and the middle term in (8) has the form

$$\frac{1}{n}\sum_{i=1}^{n} H\left(\sum_{j=0}^{k-1}\lambda_{j+1}\left(L_1(x_{i+j}^{(1)}),\ldots,L_m(x_{i+j}^{(m)})\right)\right) =$$

$$= \frac{1}{n}\sum_{i=1}^{n} H\left(\sum_{j=0}^{k-1}\lambda_{j+1}L_1(x_{i+j}^{(1)}),\ldots,\sum_{j=0}^{k-1}\lambda_{j+1}L_m(x_{i+j}^{(m)})\right) =$$

$$= \frac{1}{n}\sum_{i=1}^{n} N\left(f\left(L_1^{-1}\left(\sum_{j=0}^{k-1}\lambda_{j+1}L_1(x_{i+j}^{(1)})\right),\ldots,L_m^{-1}\left(\sum_{j=0}^{k-1}\lambda_{j+1}L_m(x_{i+j}^{(m)})\right)\right)\right) =$$

$$= \frac{1}{n}\sum_{i=1}^{n} N\left(f\left(L_1(\mathbf{x}^{(1)};k),\ldots,L_m(\mathbf{x}^{(m)};k)\right)\right).$$

The inequalities (10) follow from these observations and Theorem A since $N^{-1}$ is increasing.

The converse is obtained by Theorem 1.1. □

Assume ($A_2$) such that $a_{i+n} = a_i$, $b_{i+n} = b_i$, and $(\lambda_1,\ldots,\lambda_n)$ is a positive $n$-tuple such that $\sum_{i=1}^{k}\lambda_i = 1$ for $2 \le k \le n$. Then, for $m = 2$, the reverse of (10) can be written as

$$f\left(K_n(\mathbf{a}), L_n(\mathbf{b})\right) \ge M(K,L;\mathbf{a},\mathbf{b}) \ge N^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} N(f(a_i,b_i)).\right) \tag{11}$$

**Example 2.2.** Let $f(x)=xy$ and $N(x)=x$, then $H(s,t)=K^{-1}(s)L^{-1}(t)$. If $H$ is concave, then (11) gives the following refinement of Hölder's inequality,

$$\frac{1}{n}\sum_{i=1}^{n} a_ib_i \le \frac{1}{n}\sum_{i=1}^{n} K(\mathbf{a};k)L(\mathbf{b};k) \le K_n(\mathbf{a})L_n(\mathbf{b}). \tag{12}$$

In particular, if $H(s,t) = s^{1/q}t^{1/r}$, then $H$ is concave for $q,r > 1$ and $q^{-1} + r^{-1} = 1$; we get the following refinement of the classical Hölder's inequality for positive $n$-tuples $\mathbf{a}$ and $\mathbf{b}$.

$$\sum_{i=1}^{n} a_ib_i \le \sum_{i=1}^{n}\left(\sum_{j=0}^{k-1}\lambda_{j+1}a_{i+j}^{q}\right)^{\frac{1}{q}}\left(\sum_{j=0}^{k-1}\lambda_{j+1}b_{i+j}^{r}\right)^{\frac{1}{r}} \le \left(\sum_{i=1}^{n} a_i^q\right)^{\frac{1}{q}}\left(\sum_{i=1}^{n} b_i^r\right)^{\frac{1}{r}}.$$

**Example 2.3.** If $H(s,t) = (s^{1/p} + t^{1/p})^p$, then $H$ is concave for $p > 1$, and (11) reduces to the following refinement of the classical Minkowski's inequality for positive $n$-tuples $\mathbf{a}$ and $\mathbf{b}$.

$$\left(\sum_{i=1}^{n}(a_i+b_i)^p\right)^{\frac{1}{p}} \leq$$
$$\leq \left(\sum_{i=1}^{n}\left(\left(\sum_{j=0}^{k-1}\lambda_{j+1}a_{i+j}^p\right)^{\frac{1}{p}}+\left(\sum_{j=0}^{k-1}\lambda_{j+1}b_{i+j}^p\right)^{\frac{1}{p}}\right)^p\right)^{\frac{1}{p}} \leq$$
$$\leq \left(\sum_{i=1}^{n}a_i^p\right)^{\frac{1}{p}}+\left(\sum_{i=1}^{n}b_i^p\right)^{\frac{1}{p}}.$$

On the analogy of Corollary 1.2 and Corollary 1.3, we have the following consequences of Theorem 2.1.

**Corollary 2.4.** *Assume* $(A_3)$ *such that* $a_{i+n}=a_i$, $b_{i+n}=b_i$, *and* $(\lambda_1,\dots,\lambda_n)$ *is a positive* $n$*-tuple such that* $\sum_{i=1}^{k}\lambda_i=1$ *for* $2\leq k\leq n$. *Suppose* $f(x,y)=x+y$ $((x,y)\in I_K\times I_L)$, *and assume that* $K'$, $L'$, $N'$, $K''$, $L''$ *and* $N''$ *are all positive. Introducing* $E:=\frac{K'}{K''}$, $F:=\frac{L'}{L''}$, $G:=\frac{N'}{N''}$, (11) *holds for all possible* $\mathbf{a}$ *and* $\mathbf{b}$, *if and only if*

$$E(x)+F(y)\leq G(x+y),\quad (x,y)\in I_K^\circ\times I_L^\circ.$$

*In this case,*

$$M(K,L;\mathbf{a},\mathbf{b})=N^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}N\left(K(\mathbf{a};k)+L(\mathbf{b};k)\right)\right). \tag{13}$$

**Corollary 2.5.** *Assume* $(A_3)$ *such that* $a_{i+n}=a_i$, $b_{i+n}=b_i$, $(\lambda_1,\dots,\lambda_n)$ *is a positive* $n$*-tuple such that* $\sum_{i=1}^{k}\lambda_i=1$ *for* $2\leq k\leq n$, *and* $f(x,y)=xy$ $((x,y)\in I_K\times I_L)$. *Suppose the functions* $A(x):=\frac{K'(x)}{K'(x)+xK''(x)}$, $B(x):=\frac{L'(x)}{L'(x)+xL''(x)}$ *and* $C(x):=\frac{N'(x)}{N'(x)+xN''(x)}$ *are defined on* $I_K^\circ$, $I_L^\circ$ *and* $I_N^\circ$, *respectively. Assume further that* $K'$, $L'$, $M'$, $A$, $B$ *and* $C$ *are all positive. Then* (11) *holds for all possible* $\mathbf{a}$ *and* $\mathbf{b}$, *if and only if*

$$A(x)+B(y)\leq C(xy),\quad (x,y)\in I_K^\circ\times I_L^\circ.$$

*In this case,*

$$M(K,L;\mathbf{a},\mathbf{b})=N^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}N\left(K(\mathbf{a};k)L(\mathbf{b};k)\right)\right). \tag{14}$$

## 3. Refinement of Minkowski's Inequality

($A_4$) Let $I$ be an interval in $\mathbb{R}$, and let $M:I\to\mathbb{R}$ be a continuous and strictly monotone function. Let $\mathbf{x}_i\in I^m$ be such that $\mathbf{x}_{i+n}=\mathbf{x}_i$ $(i=1,\dots,n)$, $(\lambda_1,\dots,\lambda_n)$ be a positive $n$-tuple such that $\sum_{i=1}^{k}\lambda_i=1$ for $2\leq k\leq n$, and let $\mathbf{w}=(w_1,\dots,w_m)$ be a nonnegative $m$-tuple such that $\sum\limits_{i=1}^{m}w_i=1$.

We give a refinement of Minkowski's inequality by using Theorem A.

**Theorem 3.1.** *Assume* $(A_4)$, *and let the quasi-arithmetic mean function*

$$\mathbf{x} \to M_m(\mathbf{x};\mathbf{w}) := M^{-1}\left(\sum_{i=1}^{m} w_i M(x_i)\right), \quad \mathbf{x} \in I^m$$

*be convex. Then*

$$M_m\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i;\mathbf{w}\right) \leq$$

$$\leq \frac{1}{n}\sum_{i=1}^{n} M_m\left(\sum_{j=0}^{k-1}\lambda_{j+1}\mathbf{x}_{i+j};\mathbf{w}\right) \leq \frac{1}{n}\sum_{r=1}^{n} M_m(\mathbf{x}_r;\mathbf{w}). \tag{15}$$

*Proof.* This is obtained by applying Theorem A to the function $M_m(\cdot;\mathbf{w})$ and to the vectors $\mathbf{x}_i$ $(i = 1,\dots,n)$. □

The following necessary and sufficient condition for the quasi-arithmetic mean function to be convex is given in [12], p. 197:

**Theorem B.** *If* $M : [m_1, m_2] \to R$ *has continuous derivatives of second order and it is strictly increasing and strictly convex, then the quasi-arithmetic mean function* $M_m(\cdot;w)$ *is convex, if and only if* $M'/M''$ *is a concave function.*

$(A_5)$ Let $M : ]0,\infty[ \to ]0,\infty[$ be a continuous and strictly monotone function such that $\lim_{x\to 0} M(x) = \infty$ or $\lim_{x\to\infty} M(x) = \infty$. Let $\mathbf{x}_i \in I^m$ be such that $\mathbf{x}_{i+n} = \mathbf{x}_i$ $(i = 1,\dots,n)$, $(\lambda_1,\dots,\lambda_n)$ be a positive $n$-tuple such that $\sum_{i=1}^{k}\lambda_i = 1$ for $2 \leq k \leq n$. Let $\mathbf{w} = (w_1,\dots,w_m)$ be positive $m$-tuple such that $w_i \geq 1$ $(i = 1,\dots,m)$.

Then we define

$$\widetilde{M}_m(\mathbf{x};\mathbf{w}) = M^{-1}\left(\sum_{i=1}^{m} w_i M(x_i)\right). \tag{16}$$

The following result is also given in ([12], page 197):

**Theorem C.** *If* $M : ]0,\infty[ \to ]0,\infty[$ *has continuous derivatives of second order and it is strictly increasing and strictly convex, then* $\widetilde{M}_m(\cdot;w)$ *is a convex function if* $M/M'$ *is a convex function.*

By using (16), we have

**Theorem 3.2.** *Assume* $(A_5)$. *If the function*

$$\mathbf{x} \to \widetilde{M}_m(\mathbf{x};\mathbf{w}), \quad \mathbf{x} \in ]0,\infty[^m$$

*is convex, then Theorem* 3.1 *remains valid for* $\widetilde{M}_m(\mathbf{x};\mathbf{w})$ *instead of* $M_m(\mathbf{x};\mathbf{w})$.

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