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THE JACOBI TRANSFORM METHOD IN APPROXIMATION THEORY

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Abstract. In this paper, the behavior of Fourier coefficients of some classes of functions on an arbitrary orthogonal system is studied. The estimations of order of convergence to zero of Fourier–Jacobi coefficients are found. These estimations are precise and of terminal character. The obtained results are used for the convergence of Fourier–Jacobi series.

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0. Introduction

The estimations of Fourier-Legendre coefficients of functions belonging to one of the classes $C\,[-1,\,1]$, $L\,[-1,\,1]$ or $L^2\,[-1,\,1]$ were given in [1]. The obtained inequalities were applied to the problems of convergence of Fourier-Legendre series. In [2], these results were generalized to ultraspherical series for $f \in L_{p,\,\mu}\,[-1,\,1]$, $1 \le p \le \infty$. In [3], the author obtained the estimations of the Fourier–Jacobi coefficient of smooth functions of bounded variation.

Unlike the above-indicated papers, in this paper we study the behavior of Fourier coefficients of some classes of functions on an arbitrary orthogonal system.

Suppose μ be a measure on $[a,b]$, such that $\mu[a,b]=1$.

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Let $\varphi_n(x)$, $n = 0, 1, \ldots$ be a system of orthogonal functions with respect to μ on the segment $[a, b]$ and let

$$\hat{f}(n) = \int_a^b f(x)\,\varphi_n(x)\,d\mu(x) \tag{0.1}$$

be the n-th Fourier coefficient of the functions f, belonging to one of the classes $L_{p,\mu}[a, b]$, $(1 \le p < \infty)$, that is to a class of summable functions of p-th degree, with respect to the measure μ. L' is a class of functions with an integrable derivative on $[a, b]$.

Denote by X one of the linear spaces $L_{p,\mu}$ or L' and by $L = L(X, X)$ the space of linear operators acting from X to X, for which the equality

$$\int_a^b (A^r f)(x)\,g(x)\,d\mu(x) = \int_a^b f(x)\,(A^r g)(x)\,d\mu(x)\,, \quad r = 1, 2, \ldots . \tag{0.2}$$

is fulfilled.

In the case $f \in L_{p,\mu}$ we assume $g \in L_{q,\mu}$, where $\frac{1}{p} + \frac{1}{q} = 1$.

We'll say that $f \in W_X^r$, if $\exists g \in X$ such that

$$f(x) = (A^r g)(x) + c, \quad r = 1, 2, \ldots, \tag{0.3}$$

where $A \in L(X, X)$, $A^0 f = f$, $A^r f = A\left(A^{r-1} f\right)$, $r = 1, 2, \ldots$ and c is some constant.

Define the norm $f \in L_{p,\mu}$ by

$$\|f\|_{L_{p,\mu}} \equiv \|f\|_{p,\mu} = \left(\int_a^b |f(x)|^p\,d\mu(x)\right)^{\frac{1}{p}} < \infty, \quad 1 \le p < \infty,$$

and the norm of $f \in L'$ by $\|f\|_{L'} \equiv \|f\|_C = \sup\limits_{a \le x \le b} |f(x)|$.

Hereafter the operator satisfying the condition (0.2) for which presentation (0.3) is true will be constructed.

In Section 1 we prove general theorems on the convergence to zero of Fourie coefficients of the functions from X on an arbitrary orthogonal system. In Section 2 we study basic properties of the Jacobi transform of the functions in X. The operator satisfying the conditions (0.2) and (0.3) is constructed in Section 3. Here we establish integral estimates for Jacobi polynomials. The results of Section 4 have auxiliary character. The results of Section 5 are realization of generalized theorems of Section 1. The order of convergence to zero of Fourier–Jacobi coefficients of the functions from X are found. In Section 6 we prove the asymptotics of theorems on the order of convergence for particular sums of Fourier–Jacobi series.

1. On Fourier Coefficients of Classes X

In this section we prove the generalized theorems on the convergence to zero of Fourier coefficients from X.

Theorem 1.1. *Let* $f \in W_X^r\,(X = L_{p,\mu})$, $(1 < p < \infty)$, $\frac{1}{p} + \frac{1}{q} = 1$. *If*
1^0. $\|A^r\varphi_n\|_{q,\,\mu} \le M - \text{const}$, $q > 1$, $r = 0, 1, \ldots$;
2^0. $\lim\limits_{n\to\infty} \int\limits_{\alpha}^{\beta} (A^r\varphi_n)\,(x)\,d\mu\,(x) = 0$, $a \le \alpha < \beta \le b$, $r = 0, 1, \ldots$, *where* α *and* β *are arbitrary numbers, then*

$$\lim_{n\to\infty} \hat{f}(n) = 0.$$

Proof. According to (0.2), we can write

$$\hat{f}(n) = \int_a^b f\,(x)\;\varphi_n\,(x)\,d\mu\,(x) = \int_a^b ((A^r g)\,(x) + c)\,\varphi_n\,(x)\,d\mu\,(x) =$$

$$= \int_a^b (A^r g)\,(x)\,\varphi_n\,(x)\,d\mu\,(x) + c\int_a^b \varphi_n\,(x)\,d\mu\,(x) =$$

$$= \int_a^b g\,(x)\,(A^r\varphi_n)\,(x)\,d\mu\,(x) + c\int_a^b \varphi_n\,(x)\,d\mu\,(x) = A_n + B_n. \tag{1.1}$$

By condition 2^0 of the theorem,

$$\lim_{n\to\infty} B_n = 0, \tag{1.2}$$

since $A^0\varphi_n = \varphi_n$.

Let's turn to A_n. Let $g \in L_{p,\mu}\,[a,b]$. By density of C in $L_{p,\mu}$, $\exists h \in C$, such that

$$\|h - g\|_{p,\,\mu} < \frac{\varepsilon}{M}. \tag{1.3}$$

Further,

$$|A_n| \le \left|\int_a^b (g\,(x) - h\,(x))\,(A^r\varphi_n)\,(x)\,d\mu\,(x)\right| +$$

$$+ \left|\int_a^b h\,(x)\,(A^r\varphi_n\,)\,(x)\,d\mu\,(x)\right| = |A_{n.1}| + |A_{n.2}|. \tag{1.4}$$

By condition 1^0 of the theorem, inequality (1.3) and Hőlder's inequality

$$|A_{n.1}| \leq \int_a^b |g(x) - h(x)| \, |(A^r\varphi_n)(x)| \, d\mu(x) \leq$$

$$\leq \left(\int_a^b |g(x) - h(x)|^p \, d\mu(x) \right)^{\frac{1}{p}} \left(\int_a^b |(A^r\varphi_n)(x)|^q \, d\mu(x) \right)^{\frac{1}{q}} =$$

$$= \|A^r\varphi_n\|_{q,\,\mu} \, \|f - g\|_{p,\,\mu} < \varepsilon. \tag{1.5}$$

It remains to consider $A_{n,2}$. According to the Cantour theorem, we partition the segment $[a,b]$ by the points $a = x_0 < x_1 < \cdots < x_m = b$ so that at each partial interval $[x_k,\, x_{k+1}]$, $k = 0, 1, \ldots, m-1$, the vibrations of the function h couldn't exceed the given $\varepsilon > 0$.

Then

$$|A_{n,2}| = \left| \int_a^b h(x) \, (A^r\varphi_n)(x) \, d\mu(x) \right| \leq$$

$$\leq \left| \sum_{k=1}^m \int_{x_{k-1}}^{x_k} (h(x) - h(x_{k-1})) \, (A^r\varphi_n)(x) \, d\mu(x) \right| +$$

$$+ \left| \sum_{k=1}^m h(x_{k-1}) \int_{x_{k-1}}^{x_k} (A^r\varphi_n)(x) \, d\mu(x) \right| = A^{(1)}_{n\cdot 2} + A^{(2)}_{n\cdot 2}.$$

But

$$A^{(1)}_{n\cdot 2} \leq \sum_{k=1}^m \int_{x_{k-1}}^{x_k} |h(x) - h(x_{k-1})| \, |(A^r\varphi_n)(x)| \, d\mu(x) <$$

$$< \varepsilon \int_a^b |(A^r\varphi_n)(x)| \, d\mu(x) < \varepsilon \|A^r\varphi_n\|_{q,\mu} < \varepsilon \cdot M. \tag{1.6}$$

And the sum $A^{(2)}_{n\cdot 2}$ by condition 2^0 tends to zero as $n \to \infty$ and therefore for the great enough numbers $n > n_0(\varepsilon)$ turns out lesser than $\varepsilon > 0$, i.e.,

$$\left| A^{(2)}_{n\cdot 2} \right| < \varepsilon, \quad \text{for} \quad n > n_0$$

This and (1.6) imply that

$$|A_{n\cdot 2}| < \varepsilon(M + 1). \tag{1.7}$$

Taking into account (1.5) and (1.7) in (1.4), we get

$$|A_n| < \varepsilon(M + 2). \tag{1.8}$$

Using (1.2) and (1.8) on (1.1), we get the assertion of the above theorem. □

Theorem 1.2. *Let* $f \in W_X^r\,(X = L_{1,\mu})$. *If*

1^0. $|(A^r \varphi_n)(x)| \le M -$ const, $r = 0, 1, \dots,\ x \in [a, b]$;

2^0. $\lim\limits_{n\to\infty} \int\limits_{\alpha}^{\beta} (A^r \varphi_n)(x)\, d\mu(x) = 0$, $a \le \alpha < \beta \le b$, *where* α *and* β *are arbitrary numbers, then*

$$\lim_{n\to\infty} \hat{f}(n) = 0.$$

Proof. As in Theorem 1.1, the proof is reduced to the study of the integral A_n. Let first $g \in C[a, b]$, then by the Cantour theorem we partition the segment $[a, b]$, $a = x_0 < \dots < x_m = b$ so that $\forall \varepsilon > 0$, $\exists \delta(\varepsilon) > 0$, such that $\forall x \in [x_k,\ x_{k+1}]$, $\max\limits_k |x_k - x_{k+1}| < \delta$,

$$|f(g) - g(x_k)| < \varepsilon. \tag{1.9}$$

Therefore

$$|A_n| = \left| \int\limits_a^b (g(x) - g(x_k))(A^r \varphi_n)(x)\, d\mu(x) + \right.$$

$$\left. + \int\limits_a^b g(x_k)(A^r \varphi_n)(x)\, d\mu(x) \right| \le$$

$$\le \int\limits_a^b |g(x) - g(x_k)|\, |(A^r \varphi_n)(x)|\, d\mu(x) + \left| \int\limits_a^b g(x_k)(A^r \varphi_n)(x)\, d\mu(x) \right| =$$

$$= \sum_{k=1}^{m} \int\limits_{x_{k-1}}^{x_k} |g(x) - g(x_k)|\, |(A^r \varphi_n)(x)|\, d\mu(x) +$$

$$+ \left| \sum_{k=1}^{m} g(x_k) \int\limits_{x_{k-1}}^{x_k} (A^r \varphi_n)(x)\, d\mu(x) \right| = A_{n.1} + A_{n.2}. \tag{1.10}$$

From (1.9) and condition 1^0 of the theorem, we have

$$A_{n.1} < \varepsilon \cdot M \int\limits_a^b d\mu(x) = \varepsilon \cdot \mu. \tag{1.11}$$

From condition 2^0 of the theorem,

$$\lim_{n\to\infty} A_{n.2} = 0. \tag{1.12}$$

Taking into account (1.10) and (1.11) in (1.9), we get

$$\lim_{n\to\infty} A_n = 0. \tag{1.13}$$

Now let g be a measurable bounded function

$$|g(x)| \le M_1 - \text{const},\ \ x \in [a,b]\,. \tag{1.14}$$

By N. Lusin's theorem (see [11], p. 118), $\varepsilon > 0\ \exists \nu(x) \in C[a,b]$ such that

$$mE(g \ne \nu) < \varepsilon,\ \ |\nu(x)| \le M_1. \tag{1.15}$$

Then

$$|A_n| = \left| \int\limits_b^a g(x)\,(A^r\varphi_n)(x)\,d\mu(x) \right| \le$$

$$\le \left| \int\limits_a^b [g(x) - \nu(x)]\,(A^r\varphi_n)(x)\,d\mu(x) \right| +$$

$$+ \left| \int\limits_a^b \nu(x)\,(A^r\varphi_n)(x)\,d\mu(x) \right|.$$

But by (1.15) and condition 1^0 of the theorem,

$$\left| \int\limits_a^b [g(x) - \nu(x)]\,(A^r\varphi_n)(x)\,d\mu(x) \right| =$$

$$= \left| \int\limits_{E(f\ne\nu)} [g(x) - \nu(x)]\,(A^r\varphi_n)(x)\,d\mu(x) \right| < 2MM_1\varepsilon.$$

On the other hand, by (1.13) for great enough n one has $\beta_n < \varepsilon$. Thus we have

$$|A_n| < (2MM_1 + 1)\,\varepsilon. \tag{1.16}$$

From this follows the assertion of the theorem for measurable bounded function.

Finally, let $g \in L_{1,\mu}$. Taking $\varepsilon > 0$ and using the absolute continuity of the integral, we find $\delta > 0$ such that for any measurable set $e \in [a,b]$ with measure $me < \delta$ (see [11], p. 165),

$$\int\limits_e |g(x)|\,d\mu(x) < \varepsilon. \tag{1.17}$$

We find a bounded measurable function $\nu(x)$, so that (see [11], p. 113)

$$mE(g \ne \nu) < \delta, \quad |\nu(x)| \le M_2 - \text{const}\,. \tag{1.18}$$

Then by (1.15)–(1.18) and condition 1^0 of the theorem,

$$|A_n| \le \left| \int_a^b (g-\nu)(x)(A^r\varphi_n)(x)\,d\mu(x) \right| +$$

$$+\left| \int_a^b \nu(x)(A^r\varphi_n)(x)\,d\mu(x) \right| \le \left| \int_{E(g\neq\nu)} (g-\nu)(x)(A^r\varphi_n)(x)\,d\mu(x) \right| + \varepsilon <$$

$$< M\varepsilon + \varepsilon = (M+1)\,\varepsilon,$$

Thus the proof of the theorem is complete. □

Note that to essence for $\mu(x) \equiv 1$ this theorem was proved by Henri Lebesgue (see [11], p. 300). We present the proof for completeness of explanation.

Theorem 1.3. *Let* $f \in W_X^r$ $(X = L')$. *If*

1^0. $\left| \int_a^x (A^r\varphi_n)(t)\,d\mu(t) \right| \le M$, $r = 0, 1, \dots,$ $x \in [a,b]$;

2^0. $\lim\limits_{n\to\infty} \int_\alpha^\beta (A^r\varphi_n)(x)\,d\mu(x) = 0$, $a \le \alpha < \beta \le b$, *then almost everywhere*

$$\lim_{n\to\infty} \hat{f}(n) = 0.$$

Proof. Since $f \in L'$, then ([11], p. 292)

$$f(x) = f(a) + \int_a^x f'(t)\,dt.$$

Then

$$\int_a^b f(x)(A^r\varphi_n)(x)\,d\mu(x) =$$

$$= \int_a^b \left(f(a) + \int_a^x f'(t)\,dt \right) (A^r\varphi_n)(x)\,d\mu(x) =$$

$$= f(a) \int_a^b (A^r\varphi_n)(x)\,d\mu(x) +$$

$$+ \int_a^b \int_a^x f'(t)\,dt\,(A^r\varphi_n)(x)\,d\mu(x) = A_{n.1} + A_{n.2}. \tag{1.19}$$

By condition 2^0 of the theorem,

$$\lim_{n\to\infty} A_{n.1} = 0. \tag{1.20}$$

By 1^0 ([11], p. 113), almost everywhere

$$A_{n.2} = \int_a^b \left(\int_a^x f'(t)\,dt \right) d \int_a^x (A^r \varphi_n)(t)\,d\mu(t) =$$

$$= \int_a^x f'(t)\,dt \int_a^x (A^r \varphi_n)(t)\,d\mu(t) \Big|_a^b -$$

$$- \int_a^b \left(\int_a^x (A^r \varphi_n)(t)\,d\mu(t) \right) f'(x)\,dx = \int_a^b f'(t)\,dt \int_a^b (A^r \varphi_n)(t)\,d\mu(t) -$$

$$- \int_a^b \left(\int_a^x (A^r \varphi_n)(t)\,d\mu(t) \right) f'(x)\,dx =$$

$$= (f(b) - f(a)) \int_a^b (A^r \varphi_n)(t)\,d\mu(t) -$$

$$- \int_a^b \left(\int_a^x (A^r \varphi_n)(t)\,d\mu(t) \right) f'(x)\,dx = A'_{n.2} + A''_{n.2}. \tag{1.21}$$

By condition 2^0 of the theorem,

$$\lim_{n\to\infty} A'_{n.2} = 0.$$

And by the conditions of the theorem and Lebesgue theorem ([11] p. 139),

$$\lim_{n\to\infty} A''_{n.2} = 0.$$

Taking into account (1.22) and (1.23) in (1.21), we get

$$\lim_{n\to\infty} A_{n.2} = 0.$$

Using (1.20) and (1.24) in (1.19), we get the assertion of the theorem. □

Remark. Theorems 1.1–1.3 are just for arbitrary linear operator satisfying the condition (0.3).

2. Basic Properties of the Jacobi Transform

In this section, we study the properties of Jacobi's transform of some classes of functions. We introduce the concept of a strong derivative and of the Jacobi integral. The connection between them is established. Owing to this concept, becomes clear structural description of classes of functions. The obtained results are analogues to some theorems proved in [4] for Legendre transform.

Next, let X be one of the spaces $L_{p,\alpha}[-1, 1]$, $1 \le p < \infty$ or $C[-1, 1]$ endowed with the norms

$$\|f\|_{L_{p,\alpha}} \equiv \|f\|_{p,\alpha} = \left(\int_{-1}^{1} |f(x)|^p \, d\mu_\alpha(x)\right)^{\frac{1}{p}} < \infty, \quad 1 \le p < \infty,$$

$$\|f\|_{L'} \equiv \|f\|_C = \sup_{-1 \le x \le 1} |f(x)|,$$

where $d\mu_\alpha(x) = c_1(\alpha)(1-x)^\alpha (1+x)^{-\frac{1}{2}} dx$, $-\frac{1}{2} < \alpha < \frac{1}{2}$,

$$c_1(\alpha) = \frac{\Gamma\left(\alpha + \frac{3}{2}\right)}{2^{\alpha+\frac{1}{2}} \Gamma\left(\frac{1}{2}\right) \Gamma(\alpha+1)} = \left(\int_{-1}^{1} (1-x)^\alpha (1+x)^{-\frac{1}{2}} dx\right)^{-1}.$$

We consider the Jacobi polynomials $P_n^{\left(\alpha, -\frac{1}{2}\right)}(x)$, for $-\frac{1}{2} < \alpha < \frac{1}{2}$, $n = 0, 1, \dots$, which form the orthogonal system of functions on the segment $[-1, 1]$ with weight $(1-x)^\alpha (1+x)^{-\frac{1}{2}}$, that is (see [5], p. 80)

$$\int_{-1}^{1} P_n^{\left(\alpha,-\frac{1}{2}\right)}(x) P_k^{\left(\alpha,-\frac{1}{2}\right)}(x) (1-x)^\alpha (1+x)^{-\frac{1}{2}} dx = \\ = \begin{cases} 0, & k \ne n, \\ h_n(\alpha), & k = n, \end{cases} \tag{2.1}$$

where

$$h_n(\alpha) = \frac{2^{\alpha+\frac{1}{2}} \Gamma(n+\alpha+1) \Gamma\left(n+\frac{1}{2}\right)}{\left(\alpha + \frac{1}{2} + 2n\right) \Gamma(n+1) \Gamma\left(n+\alpha+\frac{1}{2}\right)}.$$

Further (see [6], p. 250),

$$\max_{|x| \le 1} \left| P_n^{\left(\alpha, -\frac{1}{2}\right)}(x) \right| = P_n^{\left(\alpha, -\frac{1}{2}\right)}(1) = \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+1)}, \tag{2.2}$$

$$\frac{d}{dx} P_n^{\left(\alpha, -\frac{1}{2}\right)}(x) = \frac{1}{2}\left(n + \alpha + \frac{1}{2}\right) P_{n-1}^{\left(\alpha+1, \frac{1}{2}\right)}(x). \tag{2.3}$$

Assume $R_n^{\left(\alpha, -\frac{1}{2}\right)}(x) = P_n^{\left(\alpha,-\frac{1}{2}\right)}(x) / P_n^{\left(\alpha,-\frac{1}{2}\right)}(1)$.

According to (1.2),

$$\max_{|x|\le 1}\left|R_n^{\left(\alpha,\,-\frac{1}{2}\right)}(x)\right| = 1. \tag{2.4}$$

The Jacobi transform (the Fourier–Jacobi coefficients) is defined for $f\in X$ by

$$\hat{f}_{\left(\alpha,-\frac{1}{2}\right)}(n) = \hat{f}(n) = \int_{-1}^{1} f(x)\, R_n^{\left(\alpha,-\frac{1}{2}\right)}(x)\, d\mu_\alpha(x)\,.$$

Lemma 2.1. *Assuming $f,\, g\in X$ and $c\in R=(-\infty,\,\infty)$, we have*

(a) $\left|\hat{f}(n)\right| \le \|f\|_X\,,\quad n\in P:=\{0,\,1,\,2,\,\dots\}\,;$

(b) $(f+g)^\wedge(n) = \hat{f}(n)+\hat{g}(n),\quad (cf)^\wedge(n) = cf^\wedge(n)\,;$

(c) $\left(R_n^{\left(\alpha,\,-\frac{1}{2}\right)}\right)^\wedge(n) = \begin{cases} 0, & k\ne n,\\ \frac{2^{\alpha+\frac{1}{2}}\Gamma^2(\alpha+1)\Gamma(n+\frac{1}{2})\Gamma(n+1)}{(\alpha+\frac{1}{2}+2n)\Gamma(n+\alpha+\frac{1}{2})\Gamma(n+\alpha+1)}, & k=n;\end{cases}$

(d) *for all $n\in P$, the relation*

$$\hat{f}(n) = 0 \Leftrightarrow f(x) = 0 \quad (a.e),$$

is true and means that the assertion holds for all $x\in[-1,\,1]$ if $X = C\,[-1,\,1]$, and for almost all $x\in[-1,\,1]$ if $X = L_{p,\,\alpha}[-1,\,1]$, $1\le p<\infty$.

Proof. We prove (a). Let $f\in L'[-1,1]$, then by (1.4),

$$\left|\hat{f}(n)\right| \le \sup_{|x\le 1|}|f(x)|\int_{-1}^{1}\left|R_n^{\left(\alpha,\,-\frac{1}{2}\right)}(x)\right| d\mu_\alpha(x) \le \|f\|_C\,. \tag{2.5}$$

Now let $f\in L_{p,\,\alpha}$. For $p=1$, by (2.4), we have

$$\left|\hat{f}(n)\right| \le \int_{-1}^{1}|f(x)|\left|R_n^{\left(\alpha,\,-\frac{1}{2}\right)}(x)\right| d\mu_\alpha(x) \le \|f\|_{1,\,\alpha}\,, \tag{2.6}$$

and for $p>1$, by Hölder's inequality

$$\left|\hat{f}(n)\right| \le \|f\|_{p,\,\alpha}\left\|R_n^{\left(\alpha,\,-\frac{1}{2}\right)}\right\|_{q,\,\alpha} \le \|f\|_{p,\,\alpha}\,. \tag{2.7}$$

Thus from (2.5)–(2.7) it follows that for all $f\in X$

$$\left|\hat{f}(n)\right| \le \|f\|_X\,.$$

The properties (b) are obvious and (c) follows by (2.1).

We prove (d). The direct assertion is obvious. The conserve assertion follows from the uniquesness theorem for the Jacobi transform.

In this way, for all $n \in P$, we have

$$f^{\wedge}(n) = g^{\wedge}(n) \Leftrightarrow f(x) = g(x) \quad (a.e.).$$

Thus Lemma is proved. □

Corollary 2.1. *For all $n \in N$*

$$\hat{f}(n)(n) = 0 \Rightarrow f(x) = constant \quad (a.e.).$$

Proof. Really, let $n \in N$. From property (c) it follows that $c^{\wedge}(n) = 0$ for $n \in N$, where (c) is an arbitrary constant. But then we have

$$f^{\wedge}(n) = f^{\wedge}(n) - c^{\wedge}(n) = (f - c)^{\wedge}(n) =$$

$$= \int_{-1}^{1} (f(x) - c) R_n^{\left(\alpha, -\frac{1}{2}\right)}(x)\, d\mu_\alpha(x).$$

This implies that

$$f^{\wedge}(n) = 0 \Rightarrow f(x) - c = 0 \quad \text{(a.e.)}.$$

Thus Corollary 2.1 is proved. □

For $\alpha > \beta = -\frac{1}{2}$, the generalized Jacobi shift operator is of the form (see [7])

$$(\tau_t f)(x) = \int_{-1}^{1} f(x, t, r)\, dm_\alpha(r), \tag{2.8}$$

where

$$f(x, t, r) = f\Big(xt + r\sqrt{1 - x^2}\sqrt{1 - t^2} - \frac{1}{2}\left(1 - r^2\right)(1 - x)(1 - t)\Big),$$

$$dm_\alpha(r) = c_2(\alpha)\left(1 - r^2\right)^{a - \frac{1}{2}} dr \quad \text{and}$$

$$c_2(\alpha) = \frac{\Gamma(\alpha + 1)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\alpha + \frac{1}{2}\right)} = \left(\int_{-1}^{1} \left(1 - r^2\right)^{\alpha - \frac{1}{2}} dr\right)^{-1}.$$

The following important equality

$$\left(\tau_t R_n^{\left(\alpha, -\frac{1}{2}\right)}\right)(x) = R_n^{\left(\alpha, -\frac{1}{2}\right)}(x)\, R_n^{\left(\alpha. -\frac{1}{2}\right)}(t) \tag{2.9}$$

follows from the "multiplication theorem" for Jacobi polynomials (see [8], p. 130):

$$P_n^{\left(\alpha, -\frac{1}{2}\right)}(x)\, P_n^{\left(\alpha, -\frac{1}{2}\right)}(t) = \frac{\Gamma(n + \alpha + 1)}{\Gamma(\alpha + 1)\Gamma(n + 1)} \int_{-1}^{1} P_n^{\left(\alpha, -\frac{1}{2}\right)}(x, t, r)\, dm_\alpha(r).$$

Lemma 2.2. *The operator τ_t is linear from X into itself, satisfying*
(a) $\|\tau_t 1\|_{[X,\, X]} = 1$, $(t \in [-1,\, 1])$;
(b) $\lim\limits_{t\to 1-0} \|\tau_t f - f\|_X = 0$, $(f \in X)$;
(c) $(\tau_t f)^\wedge(n) = \hat{f}(n)\, R_n^{\left(\alpha,\, -\frac{1}{2}\right)}(t)$, $(f \in X,\ \ t \in [-1,\, 1]\,,\ n \in P)$;
(d) $(\tau_t f)(x) = (\tau_x f)(t)$, $(f \in X,\ \ x, t \in [-1,\, 1])$.
(e) $\lim_{n\to\infty} \hat{f}(n) = 0$.

Proof. We prove (a). First, we'll show that

$$\|\tau_t f\|_{p,\,\alpha} \le \|f\|_{p,\,\alpha}\,,\ \ 1 \le p < \infty. \tag{2.10}$$

By (2.8) and Hölder's inequality, we have

$$|(\tau_t f)(x)|^p = \left(\left|\int_{-1}^{1} f(x,\, t,\, r)\, dm_\alpha(r)\right|\right)^p \le$$

$$\le \left(\int_{-1}^{1} dm_\alpha(r)\right)^{p-1} \int_{-1}^{1} |f(x,\, t,\, r)|^p\, dm_\alpha(r) = \int_{-1}^{1} |f(x,\, t,\, r)|^p\, dm_\alpha(r)\,.$$

Hence we have

$$\|\tau_t f\|_{p,\,\alpha}^p = \int_{-1}^{1} |(\tau_t f)(x)|^p\, d\mu_\alpha(x) \le$$

$$\le c_1(\alpha)\, c_2(\alpha) \int_{-1}^{1}\int_{-1}^{1} (1-x)^\alpha (1+x)^{-\frac{1}{2}} \left(1-r^2\right)^{\alpha-\frac{1}{2}} \times$$

$$\times \left| f\left(xt + r\sqrt{1-x^2}\sqrt{1-t^2} - \frac{1}{2}\left(1-r^2\right)(1-x)(1-t)\right)\right|^p dxdr.$$

Assuming $t = \cos u$ and $y = \cos\frac{u}{2}$, we obtain

$$\cos u = 2y^2 - 1, \quad \sin u = 2y\sqrt{1-y^2},$$

then

$$\|\tau_t f\|_{p,\,\alpha} \le c_1(\alpha)\, c_2(\alpha) \int_{-1}^{1}\int_{-1}^{1} (1-x)^\alpha (1+x)^{-\frac{1}{2}} \left(1-r^2\right)^{\alpha-\frac{1}{2}} \times$$

$$\times \left| f\left[x\left(2y^2-1\right) + 2ry\sqrt{1-y^2}\sqrt{1-x^2} - \left(1-r^2\right)(1-x)\left(1-y^2\right)\right]\right|^p drdx.$$

Making substitution $x=\cos\theta$ and denoting $z=\cos\frac{\theta}{2}$ we get

$$\|\tau_t f\|_{p,\,\alpha}\le c_1(\alpha)\,c_2(\alpha)\,2^{\alpha+\frac{3}{2}}\int_0^1\int_{-1}^1\left(1-z^2\right)^{\alpha}\left(1-r^2\right)^{\alpha-\frac{1}{2}}\times$$

$$\times\left|f\left[\left(2z^2-1\right)\left(2y^2-1\right)+\right.\right.$$

$$\left.\left.+4ryz\sqrt{1-y^2}\sqrt{1-z^2}-2\left(1-r^2\right)\left(1-y^2\right)\left(1-z^2\right)\right]\right|^p drdz=$$

$$=2^{\alpha+\frac{3}{2}}c_1(\alpha)\,c_2(\alpha)\int_0^1\int_{-1}^1\left(1-z^2\right)^{\alpha}\left(1-r^2\right)^{\alpha-\frac{1}{2}}\left|f\Big[2\left(yz+\right.\right.$$

$$\left.\left.+r\sqrt{1-y^2}\sqrt{1-z^2}\right)^2-1\Big]\right|^p drdz.$$

Substituting the inner integral, putting $v=yz+r\sqrt{1-y^2}\sqrt{1-z^2}$ and taking into account that

$$r=(v-yz)\left(1-y^2\right)^{-\frac{1}{2}}\left(1-z^2\right)^{-\frac{1}{2}},\quad dr=\left(1-y^2\right)^{-\frac{1}{2}}\left(1-z^2\right)^{-\frac{1}{2}}dv,$$

we obtain

$$\|\tau_t f\|_{p,\,\alpha}\le 2^{\alpha+\frac{3}{2}}c_1(\alpha)\,c_2(\alpha)\left(1-y^2\right)^{-\alpha}\int_0^1 dz\times$$

$$\times\int_{yz-\sqrt{1-y^2}\sqrt{1-z^2}}^{yz+\sqrt{1-y^2}\sqrt{1-z^2}}\left|f\left(2v^2-1\right)\right|^p\left(1-y^2-z^2-v^2+2yzv\right)^{\alpha-\frac{1}{2}}dv.$$

Changing the order of integration and using the formula (see [9], p. 298)

$$\int_a^b(b-x)^{\mu-1}(x-a)^{\nu-1}dx=(b-a)^{\mu+\nu-1}\frac{\Gamma(\mu)\,\Gamma(\nu)}{\Gamma(\mu+\nu)},$$

we obtain

$$\|\tau_t f\|_{p,\,\alpha}\le 2^{\alpha+\frac{3}{2}}c_1(\alpha)\,c_2(\alpha)\left(1-y^2\right)^{-\alpha}\int_0^1\left|f\left(2v^2-1\right)\right|^p dv\times$$

$$\times\int_{yv-\sqrt{1-y^2}\sqrt{1-v^2}}^{yv+\sqrt{1-y^2}\sqrt{1-v^2}}\left(1-y^2-z^2-v^2+2yzv\right)^{\alpha-\frac{1}{2}}dz=$$

$$= 2^{3\alpha+\frac{3}{2}} c_1(\alpha)\, c_2(\alpha) \frac{\Gamma^2\left(\alpha+\frac{1}{2}\right)}{\Gamma(2\alpha+1)} \int\limits_0^1 \left(1-v^2\right)^{\alpha} \left|f\left(2v^2-1\right)\right|^p dv.$$

Substituting $v=\sqrt{\frac{1+u}{2}}$ and the equality (see [12], p. 760)

$$\frac{2^{2\alpha}\Gamma^2\left(\alpha+\frac{1}{2}\right)}{\Gamma(2\alpha+1)} = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\alpha+\frac{1}{2}\right)}{\Gamma(\alpha+1)} = \frac{1}{c_2(\alpha)},$$

we obtain

$$\|\tau_t f\|_{p,\,\alpha} \le \left(\int\limits_{-1}^1 |f(u)|^p\, dm_\alpha(u)\right)^{\frac{1}{p}} = \|f\|_{p,\,\alpha},$$

from which follows (2.10).

On the other hand,

$$\|\tau_t 1\|_{p,\,\alpha} = \left(\int\limits_{-1}^1 \left|\int\limits_{-1}^1 dm_\alpha(r)\right|^p d\mu_\alpha(x)\right)^{\frac{1}{p}} = \|1\|_{p,\,\alpha} = 1.$$

From this and (2.10) it follows that in the case $X=L_{p,\,\alpha}$, $p\ge 1$,

$$\|\tau_t 1\|_{p,\,\alpha} = 1, \quad (t\in[-1,\,1]). \tag{2.11}$$

But the case $X=L'$ is elementary.

Really, assuming

$$z = xt + r\sqrt{1-x^2}\sqrt{1-t^2} - \frac{1}{2}\left(1-r^2\right)(1-x)(1-t) \le$$
$$\le xt + r\sqrt{1-x^2}\sqrt{1-t^2} \le xt + \sqrt{1-x^2}\sqrt{1-t^2}$$

and taking $x=\cos u$, $t=\cos v$, we obtain

$$z \le \cos x\ \cos t + \sin x\ \sin t = \cos\,(x-t) \le 1.$$

On the other hand,

$$z = xt + r\sqrt{1-x^2}\sqrt{1-t^2} - \frac{1}{2}\left(1-r^2\right)(1-x)(1-t) \ge -1 \Leftrightarrow$$
$$\Leftrightarrow 2xt + 2r\sqrt{(1-x^2)(1-t^2)} + r^2(1-x)(1-t) - 1 - xt + x + t \ge -2 \Leftrightarrow$$
$$\Leftrightarrow r^2(1-x)(1-t) + 2r\sqrt{(1-x^2)(1-t^2)} + xt + x + t + 1 \ge 0 \Leftrightarrow$$
$$\Leftrightarrow r^2(1-x)(1-t) + 2r\sqrt{(1-x^2)(1-t^2)} + (1+x)(1+t) \ge 0 \Leftrightarrow$$
$$\Leftrightarrow \left(r\sqrt{(1-x)(1-t)} + \sqrt{(1+x)(1+t)}\right)^2 \ge 0.$$

Thus $-1\le z\le 1$, then by (2.8),

$$\|\tau_t f\|_C \le \sup_{-1\le z\le 1} |f(z)| = \|f\|_C\,.$$

This implies that

$$\|\tau_t 1\|_C = 1. \tag{2.12}$$

Property (a) follows from (2.11) and (2.12).

We prove (b). Let $f \in L_{p,\,\alpha}$ $(p \geq 1)$. Then by density of C in $L_{p,\,\alpha}$ for an arbitrary number $\varepsilon > 0$ there exists $\psi \in C\,[-1,\,1]$ such that

$$\|f - \psi\|_{p,\,\alpha} < \frac{\varepsilon}{3}. \tag{2.13}$$

If $\psi \in C\,[-1,\,1]$, then $\forall \varepsilon > 0$, $\exists \delta\,(\varepsilon) > 0$.

$$|\psi\,(x,\,t,\,r) - \psi\,(x)| < \frac{\varepsilon}{3} \tag{2.14}$$

for all $t \in (1-\delta;\ 1)$. Then for any $t \in (1-\delta;\ 1)$ one has

$$|(\tau_t \psi)\,(x) - \psi\,(x)| < \frac{\varepsilon}{3}, \tag{2.15}$$

from which follows

$$\|\tau_t \psi - \psi\|_{p,\,\alpha} = \left(\int\limits_{-1}^{1} |(\tau_t \psi\,(x) - \psi\,(x))|^p \, d\mu_\alpha\,(x) \right)^{\frac{1}{p}} < \frac{\varepsilon}{3}. \tag{2.16}$$

Now taking into account (2.10), (2.13) and (2.16), for any $t \in (1-\delta,\,1)$, we obtain

$$\|\tau_t f - f\|_{p,\,\alpha} \leq \|\tau_t \psi - f\|_{p,\,\alpha} + \|\psi - \tau_t \psi\|_{p,\,\alpha} + \|\tau_t \psi - \tau_t f\|_{p,\,\alpha} < \varepsilon,$$

equivalent to

$$\lim_{t \to 1-0} \|\tau_t f - f\|_{p,\,\alpha} = 0, \quad p \geq 1. \tag{2.17}$$

Now, from (2.14) and (2.8) follows

$$\lim_{t \to 1-0} \|\tau_t f - f\|_C = 0. \tag{2.18}$$

The validity of assertion (b) follows from (2.17) and (2.18).

We prove (c). Doing as in proving property (a), we obtain

$$(\tau_t f)^\wedge(n) = \int\limits_{-1}^{1} R_n^{\left(\alpha,\,-\frac{1}{2}\right)}(x)\,(\tau_t f)\,(x)\,d\mu_\alpha\,(x) =$$

$$= 2^{\alpha+\frac{1}{2}} c_1\,(\alpha)\,c_2\,(\alpha)\left(1-y^2\right)^{-\alpha} \int\limits_{0}^{1} R_n^{\left(\alpha,\,-\frac{1}{2}\right)}\left(2z^2-1\right) \times$$

$$\times \int\limits_{yz-\sqrt{1-y^2}\sqrt{1-z^2}}^{yz+\sqrt{1-y^2}\sqrt{1-z^2}} f\left(2v^2-1\right)\left(1-y^2-z^2-v^2+2yzv\right)^{\alpha-\frac{1}{2}} dz dv.$$

It is known (see [5], p. 71) that

$$P_n^{\left(\alpha,\,-\frac{1}{2}\right)}\left(2z^2-1\right)=\frac{\Gamma\left(n+\alpha+1\right)\Gamma\left(2n+1\right)}{\Gamma\left(2n+\alpha+1\right)\Gamma\left(n+1\right)}P_{2n}^{(\alpha,\,\alpha)}\left(z\right), \tag{2.19}$$

where $P_n^{(\alpha,\,\alpha)}(z)$ are ultraspherical polynomials which form the orthogonal system of functions on the segment $[-1,\,1]$ with weight $\left(1-z^2\right)^{\alpha}$. Then according (1.2), we have

$$R_n^{\left(\alpha,\,-\frac{1}{2}\right)}\left(2z^2-1\right)=\frac{\Gamma\left(\alpha+1\right)\Gamma\left(2n+1\right)}{\Gamma\left(2n+\alpha+1\right)}P_{2n}^{(\alpha,\,\alpha)}\left(z\right). \tag{2.20}$$

Taking into account (2.20) and changing the order of integration, we obtain

$$(\tau_t f)\wedge(n)=2^{\alpha+\frac{1}{2}}c_1(\alpha)\,c_2(\alpha)\,\frac{\Gamma(\alpha+1)\,\Gamma(2n+1)}{\Gamma(2n+\alpha+1)}\left(1-y^2\right)^{-\alpha}\int_0^1 f\left(2v^2-1\right)\times$$

$$\times\int_{yv-\sqrt{1-y^2}\sqrt{1-v^2}}^{yv+\sqrt{1-y^2}\sqrt{1-v^2}} P_{2n}^{(\alpha,\,\alpha)}(z)\left(1-y^2-z^2-v^2+2yzv\right)^{\alpha-\frac{1}{2}}dzdv.$$

Substituting the inner integral

$$z=yv+r\sqrt{1-y^2}\sqrt{1-v^2},$$

we obtain

$$(\tau_t f)\wedge(n)=2^{\alpha+\frac{1}{2}}c_1(\alpha)\,c_2(\alpha)\,\frac{\Gamma(\alpha+1)\,\Gamma(2n+1)}{\Gamma(2n+\alpha+1)}\int_0^1\left(1-v^2\right)^{\alpha}f\left(2v^2-1\right)\times$$

$$\times\int_{-1}^{1}P_{2n}^{(\alpha,\,\alpha)}\left(yv+r\sqrt{1-y^2}\sqrt{1-v^2}\right)\left(1-r^2\right)^{\alpha-\frac{1}{2}}drdv.$$

By the "multiplication theorem", for ultraspherical polynomials

$$P_{2n}^{(\alpha,\,\alpha)}(y)\,P_{2n}^{(\alpha,\,\alpha)}(v)=$$

$$=\frac{\Gamma\left(2n+\alpha+1\right)}{\Gamma\left(2n+1\right)\Gamma\left(\alpha+1\right)}\int_{-1}^{1}P_{2n}^{(\alpha,\,\alpha)}\left(yv+r\sqrt{1-y^2}\sqrt{1-v^2}\right)dm_{\alpha}(r),$$

we have

$$(\tau_t f)\wedge(n)=\frac{\Gamma\left(\alpha+\frac{3}{2}\right)\Gamma^2\left(2n+1\right)\Gamma\left(\alpha+1\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma^2\left(2n+\alpha+1\right)}P_{2n}^{(\alpha,\,\alpha)}(y)\times$$

$$\times\int_0^1\left(1-v^2\right)^{\alpha}f\left(2v^2-1\right)P_{2n}^{(\alpha,\,\alpha)}(v)\,dv.$$

And by formula (2.20),

$$(\tau_t f)\wedge(n) = \frac{\Gamma\left(\alpha+\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(\alpha+1)} R_n^{\left(\alpha,-\frac{1}{2}\right)}\left(2y^2-1\right)\times$$
$$\times\int_0^1 \left(1-v^2\right)^{\alpha} f\left(2v^2-1\right) R_n^{\left(\alpha,-\frac{1}{2}\right)}\left(2v^2-1\right) dv.$$

Since $2y^2-1=t$, making the change of variables $v=\sqrt{\frac{1+u}{2}}$, we obtain

$$(\tau_t f)\wedge(n) = R_n^{\left(\alpha,-\frac{1}{2}\right)}(t)\int_{-1}^{1} f(u) R_n^{\left(\alpha,-\frac{1}{2}\right)}(u)\, d\mu_\alpha(u) = R_n^{\left(\alpha,-\frac{1}{2}\right)}(t)\, f\overset{\wedge}{(n)}.$$

Property (d) is obvious by the definition.

It remains to prove (e). Let x_n be the greater root of $P_n^{\left(\alpha,-\frac{1}{2}\right)}(x)$. From Lemma 2.1 $(a,\ b)$ and Lemma 2.2 (c) for $n\in N$ one can deduce that

$$\left| f\overset{\wedge}{(n)}\right| = \left|(f-\tau_{x_n f})\wedge(n)\right| \le \|f-f_{x_n}\|_X. \tag{2.21}$$

By Stieltjes inequality (see [5], p. 131, formula (6.21.5)),

$$\frac{2n-1}{2n+1}\pi \le x_n \le \frac{2n}{2n+1}\pi \Rightarrow \lim_{n\to\infty} x_n = 1. \tag{2.22}$$

Assertion (e) follows from Lemma 2.2 (b), (2.21) and (2.22). Part (e) is a Riemann–Lebesgue type result.

For the functions f, g defined on $[-1,\ 1]$, the Jacobi convolution is given by

$$(f*g)(x) = \int_{-1}^{1} g(u)(\tau_u f)(x)\, d\mu_\alpha(u),$$

whenever the integral exists. □

Lemma 2.3. *If $f\in X$, $g\in L_{1,\alpha}$, then $f*g$ exists (a.e.) and belongs to X. Furthermore, one has:*

(a) $(f*g)(x) = (g*f)(x)$

(b) $\|f*g\|_X \le \|f\|_X\, \|g\|_{1,\alpha}$ $(X=L_{p,\alpha},\ 1\le p<\infty)$,

(c) $(f*g)\wedge(n) = f\overset{\wedge}{(n)}\, g\overset{\wedge}{(n)},\quad n\in P.$

Proof. We prove (a). By definition (see the prove of Lemma 2.2 (a)),

$$(f*g)(x) = \int_{-1}^{1} g(u)\left(\int_{-1}^{1} f(x,t,r)\, dm_\alpha(r)\right) d\mu_\alpha(u) =$$

$$=2^{\alpha+\frac{1}{2}}c_1(\alpha)\,c_2(\alpha)\int_0^1\int_{-1}^1 g\left(2u^2-1\right) f\left[2\left(uy+r\sqrt{1-u^2}\sqrt{1-y^2}\right)^2-1\right]\times$$

$$\times\left(1-r^2\right)^{\alpha-\frac{1}{2}}\left(1-u^2\right)^{\alpha} drdu=$$

$$=2^{\alpha+\frac{1}{2}}c_1(\alpha)\,c_2(\alpha)\left(1-y^2\right)^{-\alpha}\int_0^1 g\left(2u^2-1\right)\int_{uy=\sqrt{1-u^2}\sqrt{1-y^2}}^{uy+\sqrt{1-u^2}\sqrt{1-y^2}} f\left(2v^2-1\right)\times$$

$$\times\left(1-u^2-y^2-v^2+2uyv\right)^{\alpha-\frac{1}{2}} dvdu=$$

$$=2^{\alpha+\frac{1}{2}}c_1(\alpha)\,c_2(\alpha)\left(1-y^2\right)^{-\alpha}\int_0^1 f\left(2v^2-1\right)\times$$

$$\times\int_{vy-\sqrt{1-v^2}\sqrt{1-y^2}}^{vy+\sqrt{1-v^2}\sqrt{1-y^2}} g\left(2u^2-1\right)\left(1-u^2-y^2-v^2+2uyv\right)^{\alpha-\frac{1}{2}} dudv=$$

$$=2^{\alpha+\frac{1}{2}}c_1(\alpha)\,c_2(\alpha)\int_0^1\left(1-v^2\right) f\left(2v^2-1\right)\times$$

$$\times\int_{-1}^1 g\left[2\left(vy+r\sqrt{1-v^2}\sqrt{1-y^2}\right)^2-1\right]\left(1-r^2\right)^{\alpha-\frac{1}{2}} drdv=$$

$$=\int_{-1}^1 f(u)\,(\tau_u g)(x)\,d\mu_\alpha(u)=(g*f)(x)\,.$$

We prove (b). Let $X=L_{p,\alpha}$, $1\le p<\infty$.
By Minkovski's inequality (see [10], p. 179) and (2.10), we obtain

$$\|f*g\|_{p,\alpha}=\left(\int_{-1}^1\left|\int_{-1}^1 g(u)\,(\tau_x f)(u)\,d\mu_\alpha(u)\right|^p d\mu_\alpha(x)\right)^{\frac{1}{p}}\le$$

$$\le\int_{-1}^1|g(u)|\left(\int_{-1}^1|\tau_x f(u)|^p\,d\mu_\alpha(x)\right)^{\frac{1}{p}} d\mu_\alpha(u)=$$

$$=\int_{-1}^1\|\tau_u f\|_{p,\alpha}\,|g(u)|\,d\mu_\alpha(u)\le\|f\|_{p,\alpha}\,\|g\|_{1,\alpha}\,. \tag{2.23}$$

Let now $X = L'$. Then

$$\|f * g\|_{L'} = \sup_{|x|\le 1} \left| \int_{-1}^{1} g(u)(\tau_u f)(x)\, d\mu_\alpha(u) \right| \le$$

$$\le \int_{-1}^{1} \|\tau_u f\|_C \, |g(u)| \, d\mu_\alpha(u) \le \|f\|_C \, \|g\|_{1,\alpha} . \tag{2.24}$$

Property (b) follows from (2.23) and (2.24).

It remains to prove (c). By Fubini's theorem (see [11], p. 379) and Lemma 2.2 (c), we obtain

$$(f * g)^\wedge(n) = \int_{-1}^{1} R_n^{\left(\alpha,-\frac{1}{2}\right)}(x) \left(\int_{-1}^{1} g(u)(\tau_u f)(x)\, d\mu_\alpha(u) \right) d\mu_\alpha(x) =$$

$$= \int_{-1}^{1} R_n^{\left(\alpha,-\frac{1}{2}\right)}(x)(\tau_u f)(x)\, d\mu_\alpha(x) \int_{-1}^{1} g(u)\, d\mu_\alpha(u) =$$

$$= \int_{-1}^{1} g(u)(\tau_u f)^\wedge(n)\, d\mu_\alpha(u) =$$

$$= f\overset{\wedge}{(n)} \int_{-1}^{1} g(u)\, R_n^{\left(\alpha,-\frac{1}{2}\right)}(u)\, d\mu_\alpha(u) = f\overset{\wedge}{(n)}\, g\overset{\wedge}{(n)} .$$

Thus the lemma is proved. □

3. The Jacobi Derivative and Integral

We start with the definition of a strong (or norm) derivative.

Definition 3.1. If for $f \in X$ there exists $g \in X$ such that

$$\lim_{t\to 1-0} \left\| \frac{f - \tau_t f}{1-t} - g \right\|_X = 0,$$

then g is called a strong Jacobi derivative of f which we denote by Df. For any $r \in N$, the r-th strong derivative of f is defined with $D^0 f = f$ by $D^r f = D\left(D^{r-1} f\right)$, whenever this is meaningful. The set of all $f \in X$ for which $D^r f$ exists as an element of X, we denote by W_X^r.

Lemma 3.1. *If $f \in W_X^r$, $r \in N$, then*

$$(D^r f)^\wedge(n) = \left(\frac{n\left(n + \alpha + \frac{1}{2}\right)}{2(\alpha+1)} \right)^r f\overset{\wedge}{(n)}, \quad n \in P. \tag{3.1}$$

Proof. Let $r=1$. Using Lemma 2.2 (c) and Lemma 2.1 (a, b), we obtain

$$\left|\frac{1-R_n^{\left(\alpha,\,-\frac{1}{2}\right)}(t)}{1-t}\,\hat{f}(n)-(D\hat{f})(n)\right|=$$

$$=\left|\left(\frac{f-\tau_t f}{1-t}-Df\right)^{\wedge}(n)\right|\le\left\|\frac{f-\tau_t f}{1-t}-Df\right\|_X.$$

Since the right-hand side tends to zero as $t\to 1-0$, it follows that

$$\lim_{t\to 1-0}\frac{1-R_n^{\left(\alpha,\,-\frac{1}{2}\right)}(t)}{1-t}\,\hat{f}(n)=(Df)^{\wedge}(n).$$

Taking into account (2.2) and (2.3), we obtain

$$\lim_{t\to 1-0}\frac{1-R_n^{\left(\alpha,\,-\frac{1}{2}\right)}(t)}{1-t}=\left(R_n^{\left(\alpha,\,-\frac{1}{2}\right)}(t)\right)'_{t=1}=\frac{\left(n+\alpha+\frac{1}{2}\right)P_{n-1}^{\left(\alpha+1,\,\frac{1}{2}\right)}(1)}{2P_n^{\left(\alpha,\,-\frac{1}{2}\right)}(1)}=$$

$$=\frac{\left(n+\alpha+\frac{1}{2}\right)\Gamma(\alpha+1)\,\Gamma(n+1)}{2\Gamma(\alpha+2)\,\Gamma(n)}=\frac{n\left(n+\alpha+\frac{1}{2}\right)}{2(\alpha+1)},\qquad(3.2)$$

and then for $r=1$,

$$(Df)^{\wedge}(n)=\frac{n\left(n+\alpha+\frac{1}{2}\right)}{2(\alpha+1)}\,\hat{f}(n).$$

The result for $r\ge 2$ follows by induction.

Lemma is proved. □

A simple consequence of this results is

Corollary 3.1. *$f\in W_X^r$ and $D^r f=0$ (a.e.) for some $r\in N$ holds if and only if $f=\mathrm{const}$ (a.e.).*

Proof. Direct assertion follows from Corollary 2.1 of Lemma 2.1. The converse follows from the definition of $D^r f$, since $(\tau_t f)(x)=f(x)$ (a.e.) if $f=\mathrm{const}$ (a.e.).

In order to define an inverse operator D^r, one has to look for a function $\psi_r\in L_{1,\,\alpha}(-1,\,1)$ whose Jacobi transform is given by

$$\psi_r{}^{\wedge}(n)=\left(\frac{2(\alpha+1)}{n\left(n+\alpha+\frac{1}{2}\right)}\right)^r,\quad r\in N.\qquad(3.3)$$

Thus Corollary is proved. □

Proposition 3.1. *The functions*

$$\psi_1(u)=2(\alpha+1)\int_{-1}^{u}\left[(1-x)^{-1-\alpha}(1+x)^{-\frac{1}{2}}\int_{-1}^{x}(1-t)^{\alpha}(1+t)^{-\frac{1}{2}}\,dt\right]dx,$$

$$u\in(-1,\,1)\quad \psi_r(u)=(\psi_1*\psi_{r-1})(u),\quad r=2,\,3,\ldots$$

belong to $L_{1,\alpha}(-1,1)$ *for each* $r\in N$*, and their Jacobi coefficients are given by* (3.3)*.*

Proof. First we'll show that $\psi_1\in L_{1,\alpha}(-1,1)$.
Really

$$\int_{-1}^{1}\psi_1(u)\,d\mu_\alpha(2\alpha+2)\times$$

$$\times\int_{-1}^{1}\left\{\int_{-1}^{u}\left[(1-x)^{-1-\alpha}(1+x)^{-\frac{1}{2}}\int_{-1}^{x}\frac{(1-t)^{\alpha+\frac{1}{2}}\,dt}{(1-t)^{\frac{1}{2}}(1+t)^{\frac{1}{2}}}\right]dx\right\}d\mu_\alpha(u)\le$$

$$\le 2^{\alpha+\frac{1}{2}}(\alpha+1)\int_{-1}^{1}\left\{\int_{-1}^{u}\left[\frac{(1-x)^{-\alpha-\frac{3}{2}}}{(1+x)^{\frac{1}{2}}}\int_{-1}^{x}(1+t)^{-\frac{1}{2}}\,dt\right]dx\right\}d\mu_\alpha(u)=$$

$$=2^{\alpha+\frac{5}{2}}(\alpha+1)\int_{-1}^{1}\left[\int_{-1}^{u}(1-x)^{-\alpha-\frac{3}{2}}(1+x)^{-\frac{1}{2}}(1+t)^{\frac{1}{2}}\Big|_{-1}^{x}dx\right]d\mu_\alpha(u)=$$

$$=2^{\alpha+\frac{5}{2}}\frac{\alpha+1}{\alpha+\frac{1}{2}}\int_{-1}^{1}(1-u)^{\alpha}(1+u)^{-\frac{1}{2}}\left((1-u)^{-\alpha-\frac{1}{2}}-2^{-\alpha-\frac{1}{2}}\right)du=$$

$$=2^{\alpha+\frac{7}{2}}\frac{\alpha+1}{2\alpha+1}\left(\int_{-1}^{1}(1-u^2)^{-\frac{1}{2}}\,du-2^{-\alpha-\frac{1}{2}}\int_{-1}^{1}(1-u)^{\alpha}(1+u)^{-\frac{1}{2}}\,du\right)=$$

$$=2^{\alpha+\frac{7}{2}}\frac{\alpha+1}{2\alpha+1}\arcsin u|_{-1}^{1}-\frac{8(\alpha+1)}{2\alpha+1}\frac{2^{\alpha+\frac{1}{2}}\Gamma\left(\frac{1}{2}\right)\Gamma(\alpha+1)}{\Gamma\left(\alpha+\frac{3}{2}\right)}=$$

$$=2^{\alpha+\frac{7}{2}}\frac{(\alpha+1)\pi}{2\alpha+1}-\frac{8(\alpha+1)}{2\alpha+1}\frac{2^{\alpha+\frac{1}{2}}\Gamma\left(\frac{1}{2}\right)\Gamma(\alpha+1)}{\Gamma\left(\alpha+\frac{3}{2}\right)}=$$

$$=2^{\alpha+\frac{7}{2}}\frac{\alpha+1}{2\alpha+1}\Gamma\left(\frac{1}{2}\right)\left(\Gamma\left(\frac{1}{2}\right)-\frac{\Gamma(\alpha+1)}{\Gamma\left(\alpha+\frac{3}{2}\right)}\right)=c_\alpha.$$

This implies that

$$\|\psi_1\|_{1,\alpha}\le c_\alpha.$$

Now we'll show that

$$\psi_r\in L_{1,\alpha}(-1,1),\quad r=2,3,\dots.$$

Assume $r=2$. Using Lemma 2.3 (b), we can write

$$\|\psi_2\|_{1,\alpha}=\|\psi_1*\psi_1\|_{1,\alpha}\le\|\psi_1\|_{1,\alpha}^2\le c_\alpha^2.$$

The result for $r \geq 3$ follows by induction

$$\|\psi_r\|_{1,\,\alpha} = \|\psi_1 * \psi_{r-1}\|_{1,\,\alpha} \leq \|\psi_1\|_{1,\,\alpha}\,\|\psi_{r-1}\|_{1,\,\alpha} \leq c_\alpha^2.$$

Now we show that for $\psi_r\,(u)$, $r \in N$ the equality (3.3) holds. According to Lemma 2.3, it suffices to have the differential equation (see [5], p. 73)

$$P_n^{\left(\alpha,-\frac{1}{2}\right)}(u) = -\frac{(1-u)^{-\alpha}\,(1+u)^{\frac{1}{2}}}{n\left(n+\alpha+\frac{1}{2}\right)}\frac{d}{du}\left[(1-u)^{\alpha+1}\,(1+u)^{\frac{1}{2}}\,\frac{d}{du}P_n^{\left(\alpha,-\frac{1}{2}\right)}(u)\right].$$

Integrating by parts, we find

$$\overset{\wedge}{\psi_1}(n) = 2\,(\alpha+1)\int_{-1}^{1}\left\{\int_{-1}^{u}\left[(1-x)^{-\alpha-1}\,(1+x)^{-\frac{1}{2}}\int_{-1}^{x}d\mu_\alpha\,(t)\right]dx\right\}\times$$
$$\times R_n^{\left(\alpha,\,-\frac{1}{2}\right)}(u)\,d\mu_\alpha\,(u) =$$
$$= -\frac{2\,(\alpha+1)}{n\left(n+\alpha+\frac{1}{2}\right)}\int_{-1}^{1}\left\{\int_{-1}^{u}\left[(1-x)^{-\alpha-1}\,(1+x)^{-\frac{1}{2}}\int_{-1}^{x}d\mu_\alpha\,(t)\right]dx\right\}\times$$
$$\times d\left[(1-u)^{\alpha+1}\,(1+u)^{\frac{1}{2}}\,\frac{d}{du}R_n^{\left(\alpha,\,-\frac{1}{2}\right)}(u)\right] =$$
$$= \frac{2\,(\alpha+1)}{n\left(n+\alpha+\frac{1}{2}\right)}\int_{-1}^{1}\frac{d}{du}R_n^{\left(\alpha,\,-\frac{1}{2}\right)}(u)\int_{-1}^{u}d\mu_\alpha\,(t) =$$
$$= \frac{2\,(\alpha+1)}{n\left(n+\alpha+\frac{1}{2}\right)}\int_{-1}^{1}\left(\int_{-1}^{u}d\mu_\alpha\,(t)\right)dR_n^{\left(\alpha,\,-\frac{1}{2}\right)}(u) =$$
$$= \frac{2\,(\alpha+1)}{n\left(n+\alpha+\frac{1}{2}\right)}\left[R_n^{\left(\alpha,\,-\frac{1}{2}\right)}(1)\int_{-1}^{1}d\mu_\alpha\,(t) - \int_{-1}^{1}R_n^{\left(\alpha,\,-\frac{1}{2}\right)}(u)\,d\mu_\alpha\,(u)\right] =$$
$$= \frac{2\,(\alpha+1)}{n\left(n+\alpha+\frac{1}{2}\right)},$$

where we have also used Lemma 2.1 (c) and (2.4).
Thus Proposition 3.1 is proved. □

For $r \in N$, the Jacobi integral I^r can now be defined as follows:

$$(I^r f)\,(x) := (f * \psi_r)\,(x)\,, \quad (x \in [-1,\,1]\,;\; f \in X)\,. \tag{3.4}$$

Proposition 3.2. *The integral I^r is the bounded linear operator from X into itself, which satisfies for each $r \in N$ and $f \in X$, $r, s \in N$:*
(a) $(I^r I^s f)\,(x) = (I^s I^r f)\,(x) = (I^{r+s} f)\,(x)$, (*a.e.*)

(b) $(I^r f)^\wedge(n) = \left(\frac{2(\alpha+1)}{n\left(n+\alpha+\frac{1}{2}\right)}\right)^r \hat{f}(n)\,, \qquad n \in N;$

(c) *for* $f \in W_X^r$, *one has*

$$(I^r D^r f)(x) = f(x) - c, \qquad (a.e.)\,; \tag{3.5}$$

(d) *for any* $f,\, g \in X$, *the equality*

$$\int_{-1}^{1} g(x)\,(I^r f)(x)\,d\mu_\alpha(x) = \int_{-1}^{1} f(x)\,(I^r g)(x)\,d\mu_\alpha(x)$$

is valid.

Proof. The linearity of the operator is obvious, and the boundedness follows from the inequality (see Lemma 2.3 (b))

$$\|I^r f\|_X = \|f * \psi_r\|_X \le \|\psi_r\|_{1,\,\alpha}\,\|f\|_X \le c_\alpha^r\,\|f\|_X\,.$$

We prove (a). By definition,

$$\psi_{r+s} = \psi_1 * \psi_{r+s-1} = \psi_1 * (\psi_1 * \psi_{r+s-2}) =$$
$$= (\psi_1 * \psi_1) * \psi_{r+s-2} = \psi_2 * \psi_{r+s-2} = \cdots = \psi_r * \psi_s$$

from which we have

$$(I^r I^s f)(x) = (I^r f * \psi_s)(x) = ((f * \psi_r) * \psi_s)(x) =$$
$$= (f * (\psi_r * \psi_s))(x) = (f * \psi_{r+s})(x) = \left(I^{r+s} f\right)(x)\,.$$

We'll prove (b). By Proposition 3.1 and Lemma 2.3 (c), we obtain

$$(I^r f)^\wedge(n) = (f * \psi_r)^\wedge(n) = \hat{f}(n)\,\overset{\wedge}{\psi_r}(n) = \left(\frac{2(\alpha+1)}{n\left(n+\alpha+\frac{1}{2}\right)}\right)^r \hat{f}(n)\,.$$

We'll prove (c). By Proposition 3.2 (b) and (3.1), we have

$$(I^r D^r f)^\wedge(n) = \left(\frac{2(\alpha+1)}{n\left(n+\alpha+\frac{1}{2}\right)}\right)^r (D^r f)^\wedge(n) = \hat{f}(n), \quad n \in N. \tag{3.6}$$

Since $R_0^{\left(\alpha,\,-\frac{1}{2}\right)}(x) = 1$ (see [5], p.82), by the orthogonality of polynomials $R_0^{\left(\alpha,\,-\frac{1}{2}\right)}(x)$, we have

$$\hat{c}(n) = c\int_{-1}^{1} R_0^{\left(\alpha,\,-\frac{1}{2}\right)}(x)\,R_n^{\left(\alpha,\,-\frac{1}{2}\right)}(x)\,d\mu_\alpha(x) = \begin{cases} 0, & n \in N, \\ c, & n = 0. \end{cases}$$

Then from (3.6) we find that

$$(I^r D^r f)^\wedge(n) = \hat{f}(n) - \hat{c}(n) = (f(x) - c)^\wedge(n)\,, \quad n \in N,$$

from which it follows that

$$(I^r D^r f)(n) = f(x) - c, \quad (a.e.)\,.$$

We'll prove (d).

$$\int_{-1}^{1} g(x)\,(I^r f)(x)\,d\mu_\alpha(x) = \int_{-1}^{1} g(x)\,(f*\psi_r)(x)\,d\mu_\alpha(x) =$$

$$= \int_{-1}^{1} g(x)\left\{\int_{-1}^{1} f(t)\left[\int_{-1}^{1}\psi_r(x,\,t,\,r)\,dm_\alpha(r)\right]d\mu_\alpha(t)\right\}d\mu_\alpha(x) =$$

$$= \int_{-1}^{1} f(t)\left\{\int_{-1}^{1} g(x)\left[\int_{-1}^{1}\psi_r(x,\,t,\,r)\,dm_\alpha(r)\right]d\mu_\alpha(x)\right\}d\mu_\alpha(t) =$$

$$= \int_{-1}^{1} f(t)\left[\int_{-1}^{1} g(x)\,(\tau_t\psi_r)(x)\,d\mu_\alpha(x)\right]d\mu_\alpha(t) =$$

$$= \int_{-1}^{1} f(t)\,(g*\psi_r)(t)\,d\mu_\alpha(t) = \int_{-1}^{1} f(t)\,(I^r g)(t)\,d\mu_\alpha(t)\,.$$

The proposition is proved. □

The analogue of (3.5) for $r = 1$ in the classical analysis is

$$\int_{-1}^{x} f'(t)dt = f(x) - f(-1)\,, \tag{3.7}$$

which holds for each continuous function f integrable derivative. Making interchange of order of integration and performing differentiation in (3.7), one can get an equation which will be valid for each continuous function f, namely,

$$\frac{d}{dx}\int_{-1}^{x} f(t)\,dt = f(x)\,.$$

Therefore there arises the question whether the equation

$$(D^r I^r f)(x) = f(x) - c \qquad (a.e.) \tag{3.8}$$

is likewise true for each $f \in X$.

Thus we consider the function

$$(x;\,t)=\begin{cases}\frac{\alpha+1}{c_1(\alpha)}\left(\log\frac{2}{1+t}\right)^{-1}\int_t^x (1-u)^{-\alpha-1}(1+u)^{-\frac{1}{2}}\,du, & -1<t\le x<1;\\ 0, & \text{otherwise}\ ,\end{cases}$$

and put $A_t f = f * (\cdot\,;\,t)$, $t\in(-1,\,1)$.

Lemma 3.2. *For each $t \in (-1,\,1)$ and $x \in [-1,\,1]$, the function $(\cdot\,;\,t)$ belongs to $L_{1,\,\alpha}\,(-1,\,1)$, it is nonnegative and satisfies:*

(a) $$((\cdot\,;\,t))\wedge(n) = \frac{(\alpha+1)\left(1-R_n^{\left(\alpha,\,-\frac{1}{2}\right)}(t)\right)}{n\left(n+\alpha+\frac{1}{2}\right)}\left(\log\frac{2}{1+t}\right)^{-1},\quad n \in N;$$

(b) *For each $t \in (-1,\,1)$, the A_t is the positive linear bounde operator from X into itself and*

$$\lim_{t\to 1-0}\|A_t f - f\|_X = 0 \quad (f \in X)\,. \tag{3.9}$$

Let $n \in N$. Then by the partial integration (see the proof of Proposition 3.1.), we have

$$((\cdot\,;\,t))\wedge(n) = \frac{\alpha+1}{c_1(\alpha)}\left(\log\frac{2}{1+t}\right)^{-1}\int\limits_t^1\left[\int\limits_t^x (1-u)^{-\alpha-1}(1+u)^{-\frac{1}{2}}\,du\right]\times$$

$$\times(1-x)^{\alpha}(1+x)^{-\frac{1}{2}}R_n^{\left(\alpha,\,-\frac{1}{2}\right)}(x)\,dx = -\frac{(\alpha+1)\left(\log\frac{2}{1+t}\right)^{-1}}{c_1(\alpha)\,n\left(n+\alpha+\frac{1}{2}\right)}\times$$

$$\times\int\limits_t^1\left[\int\limits_t^x (1-u)^{-\alpha-1}(1+u)^{-\frac{1}{2}}\,du\right]d\left[\frac{(1-u)^{\alpha+1}}{(1+u)^{-\frac{1}{2}}}\frac{d}{du}R_n^{\left(\alpha,\,-\frac{1}{2}\right)}(u)\right] =$$

$$= \frac{(\alpha+1)\left(\log\frac{2}{1+t}\right)^{-1}}{n\left(n+\alpha+\frac{1}{2}\right)}\int\limits_t^1\frac{d}{dx}R_n^{\left(\alpha,-\frac{1}{2}\right)}(x)\,dx = \frac{\alpha+1}{n\left(n+\alpha+\frac{1}{2}\right)}\frac{1-R_n^{\left(\alpha,-\frac{1}{2}\right)}(t)}{\log\frac{2}{1+t}}.$$

We can show that $X\,(x;\,t)$ belongs to $L_{1,\,\alpha}\,(-1,\,1)$.

$$\|(\cdot\,;\,t)\|_{1,\,\alpha} = \frac{\alpha+1}{\log\frac{2}{1+t}}\int\limits_t^1 (1-x)^{\alpha}(1+x)^{-\frac{1}{2}}\int\limits_t^x (1-u)^{-\alpha-1}(1+u)^{-\frac{1}{2}}\,du dx =$$

$$= -\left(\log\frac{2}{1+t}\right)^{-1}\int\limits_t^1 (1+x)^{-\frac{1}{2}}\left[\int\limits_t^x (1-u)^{-\alpha-1}(1+u)^{-\frac{1}{2}}\,du\right]d\,(1-x)^{\alpha+1}\,.$$

By the partial integration, we obtain

$$\|(\cdot\,;\,t)\|_{1,\,\alpha} = \left(\log\frac{2}{1+t}\right)^{-1}\times$$

$$\times\int\limits_t^1\left[\frac{(1-x)^{-\alpha-1}}{1+x} - \frac{1}{2}(1+x)^{-\frac{3}{2}}\int\limits_t^x (1-u)^{-\alpha-1}(1+u)^{-\frac{1}{2}}\,du\right](1-x)^{\alpha+1}\,dx =$$

$$= \left(\log\frac{2}{1+t}\right)^{-1}\times$$

$$\times\int_t^1\left[\frac{1}{1+x}-\frac{1}{2}(1-x)^{\alpha+1}(1+x)^{-\frac{3}{2}}\int_t^x(1-u)^{-\alpha-1}(1+u)^{-\frac{1}{2}}\,du\right]dx=$$

$$=1-\frac{1}{2\log\frac{2}{1+t}}\int_t^1(1-x)^{\alpha+1}(1+x)^{-\frac{3}{2}}\int_t^x(1-u)^{-\alpha-1}(1+u)^{-\frac{1}{2}}\,dudx.$$

Then we have

$$\|(\cdot\,;t)\|_{1,\alpha}\le 1.$$

From this, taking into account Lemma 2.3 (b), we obtain

$$\|A_t f\|_X\le\|(\cdot\,;;t)\|_{1,\alpha}\|f\|_X\le\|f\|_X\,. \tag{3.10}$$

Further, by Lemma 2.3 (c), we have

$$\left(A_tR_n^{(\alpha,-\frac{1}{2})}\right)^\wedge(n)=\left(R_n^{(\alpha,-\frac{1}{2})}*(\cdot\,;t)\right)^\wedge(n)=$$

$$=\left(R_n^{(\alpha,-\frac{1}{2})}\right)^\wedge(n)\left((\cdot\,;t)\right)^\wedge(n)=$$

$$=\frac{(\alpha+1)\left(1-R_n^{\alpha,-\frac{1}{2}}(t)\right)}{n\left(n+\alpha+\frac{1}{2}\right)\log\frac{2}{1+t}}\left(R_n^{(\alpha,-\frac{1}{2})}\right)^\wedge(n)\,.$$

This implies that

$$A_tR_n^{(\alpha,-\frac{1}{2})}(x)=\frac{2(\alpha+1)\left(1-R_n^{(\alpha,-\frac{1}{2})}(t)\right)}{n\left(n+\alpha+\frac{1}{2}\right)2\log\frac{2}{1+t}}R_n^{(\alpha,-\frac{1}{2})}(x)\ \ (a.e.) \tag{3.11}$$

Using (3.2) in (3.11), by de L'Hospital's rule, we obtain

$$\lim_{t\to1-0}A_tR_n^{(\alpha,-\frac{1}{2})}(x)=\frac{2(\alpha+1)}{n\left(n+\alpha+\frac{1}{2}\right)}R_n^{(\alpha,-\frac{1}{2})}(x)\lim_{t\to1-0}\frac{1-R_n^{(\alpha,-\frac{1}{2})}(t)}{1-t}\times$$

$$\times\lim_{t\to-0}\frac{1-t}{2\log\frac{2}{1+t}}=R_n^{(\alpha,-\frac{1}{2})}(x)\,,$$

from which it follows that

$$\lim_{t\to1-0}\left\|A_tR_n^{(\alpha,-\frac{1}{2})}-R_n^{(\alpha,-\frac{1}{2})}\right\|_X=0. \tag{3.12}$$

Using the density of the space of polynomials in the space X for any $f\in X$, there exists the polynomial $Q_n(x)$ such that for every $\varepsilon>0$ and for sufficient n,

$$\|f-Q_n\|_X<\frac{\varepsilon}{3}.$$

On the other hand (see [12], p. 334), there can appear $Q_n(x)$ in the form

$$Q_n(x) = \sum_{k=0}^{n} \alpha_k R_k^{\left(\alpha,\, -\frac{1}{2}\right)}(x)\,,$$

where α_k is same number.

But then by (3.12) for the chosen ε, there exists δ $(0 < \delta < 1)$ such that for $t \in (1-\delta;\, 1)$, we have

$$\|A_t Q_n - Q_n\|_X \le \sum_{k=0}^{n} \alpha_k \left\| A_t R_k^{\left(\alpha,\, -\frac{1}{2}\right)} - R_k^{\left(\alpha,\, -\frac{1}{2}\right)} \right\|_X < \frac{\varepsilon}{3} \qquad (3.13)$$

Taking now into account (3.10) and (3.13), from the inequality

$$\|A_t f - f\|_X \le \|A_t(f - Q_n)\|_X + \|A_t Q_n - Q_n\|_X + \\ + \|f - Q_n\|_X \le 2\,\|f - Q_n\| + \|A_t Q_n - Q_n\|_X < \varepsilon$$

we obtain the approval (3.9).

The following theorem is the analogue, suitable for Theorem 1 from [4], obtained for the Legendre transform.

Theorem 3.3. *The following statements are equivalent to $f \in X$, $r \in N$:*
(a) $f \in W_X^r = \{f \in X;\ D^r f \in X\}$
(b) *there exists $g_1 \in X$ such that*

$$\overset{\wedge}{g_1}(n) = \left(\frac{n\left(n + \alpha + \frac{1}{2}\right)}{2(\alpha+1)}\right)^r \hat{f}(n),$$

for any $n \in N$.
(c) *there exists $g_2 \in X$ such that*

$$f(x) = (I^r g_2)(x) + \text{const} \qquad (a.e.). \qquad (3.14)$$

The functions g_1, g_2 are uniquely determined $(a.e.)$ a part form additive constant and one has

$$(D^r f)(x) = g_1(x) - \overset{\wedge}{g_1}(0) = g_2(x) - \overset{\wedge}{g_2}(0) \qquad (a.e.) \qquad (3.15)$$

Proof. Let $f \in W_X^r$. By Lemma 3.1, one has

$$(D^r f)^\wedge(n) = \left(\frac{n\left(n + \alpha + \frac{1}{2}\right)}{2(\alpha+1)}\right)^r \hat{f}(n)$$

i.e., (b) is valid with $g_1 = D^r f$.

Let now $g_1 \in X$ and

$$g_1{}^\wedge(n) = \left(\frac{n\left(n + \alpha + \frac{1}{2}\right)}{2(\alpha+1)}\right)^r \hat{f}(n), \quad (n \in N)\,,$$

then by Lemma 3.1, $\overset{\wedge}{g_1}(n) = (D^r f)^\wedge(n)$, from which it follows that $g_1(x) = (D^r f)(x)$ (a.e.), i.e., $D^r f \in X$. On the other hand, if (b) holds, then by Proposition 3.2 (b),

$$\hat{f}(n) = \left(\frac{2(\alpha+1)}{n\left(n+\alpha+\frac{1}{2}\right)}\right)^r g_1{}^\wedge(n) = (I^r g_1)^\wedge(n), \quad (n \in N).$$

This implies that $f(x) = (I^r g_1)(x)$ (a.e.). But since $g_1 \in X$ and $I^r : X \to X$, therefore $f \in X$.

We show that (a) is equivalent to (c). If $f \in W_X^r$, then by Proposition 3.2 (c),

$$f(x) = (I^r D^r f)(x) + c \quad (a.e.)$$

and it suffices to put $g_2 = D^r f$.

Now let (c) be satisfied with $r = 1$. We show that

$$\frac{f(x) - (\tau_t f)(x)}{1-t} = \frac{2\log\frac{2}{1+t}}{1-t}\left[(A_t g_2)(x) - \overset{\wedge}{g_2}(0)\right] \quad (a.e.). \tag{3.16}$$

On the one hand,

$$f(x) - (\tau_t f)(x) = (I g_2)(x) - (\tau_t I g_2)(x) =$$
$$= (g_2 * \psi_1)(x) - \tau_t (g_2 * \psi_1)(x).$$

By Lemmas 2.2 (c), 2.3 (c) and (3.3), we have

$$((g_2 * \psi_1) - \tau_t (g_2 * \psi)(x))^\wedge(n) = \overset{\wedge}{g_2}(n)\,\overset{\wedge}{\psi_1}(n)\left(1 - R_n^{\left(\alpha,-\frac{1}{2}\right)}(t)\right)$$
$$\frac{2(\alpha+1)}{n\left(n+\alpha+\frac{1}{2}\right)}\left(1 - R_n^{\left(\alpha,-\frac{1}{2}\right)}(t)\right)\overset{\wedge}{g_2}(n). \tag{3.17}$$

On the other hand, by Lemmas 2.3 (c), 3.2 (b) and (3.3),

$$\left(A_t g_2 - \overset{\wedge}{g_2}(0)\right)^\wedge(n) = (g_2 * X(\cdot;t))^\wedge(n) - \left(\overset{\wedge}{g_2}(0)\right)^\wedge(n) =$$

$$= \overset{\wedge}{g_2}(n)\,(X(\cdot\,;t))^\wedge(n) = \frac{\alpha+1}{n\left(n+\alpha+\frac{1}{2}\right)}\,\overset{\wedge}{g_2}(n)\,\frac{1 - R_n^{\left(\alpha,-\frac{1}{2}\right)}(t)}{\log\frac{2}{1+t}}. \tag{3.18}$$

Now (3.16) follows from (3.17) and (3.18).
Taking into account (3.9) and (3.16), we obtain

$$\left\|\frac{f - \tau_t f}{1-t} - g_2 + \overset{\wedge}{g_2}(0)\right\|_X = \left\|\frac{2\log\frac{2}{1+t}}{1-t}\left(A_t g_2 - \overset{\wedge}{g_2}(0)\right) - g_2 + \overset{\wedge}{g_2}(0)\right\|_X \le$$

$$\le \left|1 - \frac{2\log\frac{2}{1+t}}{1-t}\right|\left(\|A_t g_2\|_X + \left|\overset{\wedge}{g_2}(0)\right|\right) + \|A_t g_2 - g_2\|_X = 0(1), \ \ t \to 1-0,$$

since by de L'Hospital's rule,

$$\lim_{t\to 1-0}\frac{2\log\frac{2}{1+t}}{1-t}=\lim_{t\to 1-0}\frac{2}{1+t}=1,$$

from which it follows that (3.16) holds for $r=1$, i.e., $Df\in X$. The general case follows by induction.

We show that the presentation (3.15) is unique. We suppose that there exists $g_1\in X$ such that

$$f(x)=(I^r g_1)(x)+\text{const},$$

then

$$(D^r f)(x)=g_1(x)-\overset{\wedge}{g_1}(0)\,. \tag{3.19}$$

From (3.15) and (3.19) it follows that for $n\in N$,

$\overset{\wedge}{g_2}(n)=\overset{\wedge}{g_1}(n)\Rightarrow g_2(x)=g_1(x)$, (a.e.)

i.e., the presentation (3.15) is unique.

Thus the theorem is proved. □

Corollary 3.2. *If $f\in W_X^r$, $r\in N$ and $g\in L_{1,\,\alpha}$, then $f*g\in W_X^r$ and*

(a) $$(D^r(f*g))(x)=((D^r f)*g)(x)\quad (a.e.), \tag{3.20}$$

(b) $$\left(D^r R_n^{(\alpha,\,-\frac{1}{2})}\right)(x)=\left(\frac{n\left(n+\alpha+\frac{1}{2}\right)}{2(\alpha+1)}\right)^r R_n^{(\alpha,\,-\frac{1}{2})}(x)\,,$$

$$(x\in[-1,1]\,,\; r\in N,\; n\in P)\,.$$

(c) *The Jacobi integral $I^r f$ of $f\in X$ belongs to W_X^r for each $r\in N$, and one has*

$$(D^r I^r f)(x)=f(x)-\hat{f}(0)\quad (a.e.) \tag{3.21}$$

(d) *$A_t f\in W_X^r$ of $f\in X$ for each $t\in(-1,\,1)$ and*

$$(DA_t f)(x)=\frac{f(x)-(\tau_t f)(x)}{2\log\frac{2}{1+t}}\quad (a.e.),$$

moreover,

$$(DA_t f)(x)=(A_t Df)(x)\quad (a.e.).$$

(e) *The operator $D^r: W_X^r\to X$ is closed, i.e., if*

$$\lim_{n\to\infty}\|f_n-f\|_X=\lim_{n\to\infty}\|D^r f_n-g\|_X=0$$

for a sequence $\{f_n\}_{n=1}^{\infty}\in W_X^r$ and $f,\,g\in X$, then $f\in W_X^r$ and $D^r f=g$ (a.e.).

We prove (a). Let $f \in W_X^r$ and $g \in L_{1,\alpha}(-1, 1)$, then $D^r f \in X$, and by Lemma 2.3, (a) $(D^r f) * g \in X$. But by Lemma 2.3 (c) and Lemma 3.1, we have

$$((D^r f) * g) \wedge (n) = (D^r f) \wedge (n)\, \hat{g}(n) = \left(\frac{n\left(n+\alpha+\frac{1}{2}\right)}{2(\alpha+1)}\right)^r \hat{f}(n)\, \hat{g}(n)$$

$$\left(\frac{n\left(n+\alpha+\frac{1}{2}\right)}{2(\alpha+1)}\right)^r (f * g) \wedge (n) = (D^r (f * g)) \wedge (n),$$

whence follows (3.20), and $f * g \in W_X^r$.

We prove (b). First, let $r = 1$. By (2.9) and (3.2), we have

$$\lim_{t \to 1-0} \frac{R_n^{\left(\alpha, -\frac{1}{2}\right)}(x) - \left(\tau_t R_n^{\left(\alpha, -\frac{1}{2}\right)}\right)(x)}{1-t} =$$

$$= \lim_{t \to 1-0} \frac{1 - R_n^{\left(\alpha, -\frac{1}{2}\right)}}{1-t} R_n^{\left(\alpha, -\frac{1}{2}\right)}(x) = \frac{n\left(n+\alpha+\frac{1}{2}\right)}{2(\alpha+1)} R_n^{\left(\alpha, -\frac{1}{2}\right)}(x).$$

The last means that $\forall \varepsilon > 0$ $\exists \delta(\varepsilon) > 0$ such that $\forall t \in (1-\delta, 1)$ and $\forall x \in [-1, 1]$

$$\left| \frac{R_n^{\left(\alpha, -\frac{1}{2}\right)}(x) - \left(\tau_t R_n^{\left(\alpha, -\frac{1}{2}\right)}\right)(x)}{1-t} - \frac{n\left(n+\alpha+\frac{1}{2}\right)}{2(\alpha+1)} R_n^{\left(\alpha, -\frac{1}{2}\right)}(x) \right| < \varepsilon,$$

whence it follows that for $\forall t \in (1-\delta, 1)$,

$$\left\| \frac{R_n^{\left(\alpha, -\frac{1}{2}\right)} - \left(\tau_t R_n^{\left(\alpha, -\frac{1}{2}\right)}\right)}{1-t} - \frac{n\left(n+\alpha+\frac{1}{2}\right)}{2(\alpha+1)} R_n^{\left(\alpha, -\frac{1}{2}\right)} \right\| < \varepsilon,$$

i.e., $\left(D R_n^{\left(\alpha, -\frac{1}{2}\right)}\right)(x) = \frac{n\left(n+\alpha+\frac{1}{2}\right)}{2(\alpha+1)} R_n^{\left(\alpha, -\frac{1}{2}\right)}(x)$.

The general case for $r \geq 2$ follows by induction. If in Theorem 3.1 (c) we put $g_2 = f$, then we obtain (3.21) and the assertion $I^r f \in W_X^r$.

We prove (d). By Lemmas 3.1 and 2.3, we have

$$(D A_t f) \wedge (n) = \frac{n\left(n+\alpha+\frac{1}{2}\right)}{2(\alpha+1)} (A_t f) \wedge (n)$$

$$\frac{n\left(n+\alpha+\frac{1}{2}\right)}{2(\alpha+1)} (f * (\cdot\,; t)) \wedge (n) = \frac{n\left(n+\alpha+\frac{1}{2}\right)}{2(\alpha+1)} \hat{f}(n)\, \overset{\wedge}{(n)}. \tag{3.22}$$

From (3.22), according to Lemma 3.2 (a), we obtain

$$(D A_t f) \wedge (n) = \frac{1 - R_n^{\left(\alpha, -\frac{1}{2}\right)}(t)}{2 \log \frac{2}{1+t}} \hat{f}(n) = \left(\frac{f - \tau_t f}{2 \log \frac{2}{1+t}}\right) \wedge (n),$$

from which it follows that

$$(DA_t f)(x) = \frac{f(x) - (\tau_t f)(x)}{2\log\frac{2}{1+t}} \quad (a.e.).$$

Further, if $f \in W_X^r$, then

$$(A_t Df)^\wedge(n) = (Df * (\cdot;\, t))^\wedge(n) = (Df)^\wedge(n)\,((\cdot;\, t))^\wedge(n) =$$
$$= \frac{n\left(n+\alpha+\frac{1}{2}\right)}{2(\alpha+1)} f^\wedge(n)\ \overset{\wedge}{(n)}\,. \tag{3.23}$$

From (3.22) and (3.23) follows

$$(DA_t f)(x) = (A_t Df)(x) \quad (a.e.).$$

Finally, we prove (e). From (3.15), we have

$$(D^r f)(x) = g_1(x) - g_1{}^\wedge(0)\,,\ \ g_1 \in X$$

and

$$(D^r f_n)(x) = g_{1n}(x) - g_{1n}{}^\wedge(0)\,,\ \ g_{1n} \in X,\ n = 1, 2, \ldots\,.$$

By supposition

$$\lim_{n\to\infty} \|g_1 - g_{1n}\|_X = 0.$$

But then along with Lemma 2.1 (a), we have

$$|\,(D^r f - g)^\wedge(n)\,| \le \|D^r f - g\|_X \le$$
$$\le \|D^r f - D^r f_n\|_X + \|D^r f_n - g\|_X \le$$
$$\le \|D^r f_n - g\|_X + \|g_1 - g_{1n}\|_X + |g_1{}^\wedge(0) - g_{1n}{}^\wedge(0)\,| \le \|D^r f_n - g\|_X +$$
$$+ 2\|g_1 - g_{1n}\|_X \to 0 \quad \text{as} \quad n \to \infty,$$

which implies that $(D^r f)(x) = g(x)$ (a.e.), but $g \in X \Rightarrow f \in W_X^r$.

Thus Corollary is proved.

4. Auxiliary Assertions

In this section we prove some facts that we'll need later on, though they are of independent interest.

In what follows, by M we denote positive constants, independent of x and n.

Lemma 4.1. *Let* $-\frac{1}{2} < \alpha < \frac{1}{2}$. *For* $n = 2, 3, \ldots$, *the estimations*

$$\left|\int_{-1}^{t} (1-x)^{\alpha}(1+x)^{-\frac{1}{2}} P_n^{(\alpha,-\frac{1}{2})}(x)dx\right| \le$$
$$\le M(1-t)^{\frac{\alpha}{2}-\frac{1}{4}}(1+t)^{-\frac{1}{2}} n^{-\frac{3}{2}},\, t \in (-1, 1)\,, \tag{4.1}$$

$$\left| \int_{-1}^{t} (1-x)^{\alpha}(1+x)^{-\frac{1}{2}} P_n^{(\alpha,-\frac{1}{2})}(x)dx \right| \leq M n^{\frac{2}{2\alpha+3}-\frac{3}{2}}, t \in [-1,1] \tag{4.2}$$

are valid.

Here and in the sequel, M will denote different constants.

Proof. Using the formula ([5], p. 73)

$$P_n^{(\alpha,-\frac{1}{2})}(x) = -\frac{1}{n(n+\alpha+\frac{1}{2})}\Big\{(1-x^2)\frac{d^2}{dx^2}P_n^{(\alpha,-\frac{1}{2})}(x) - \\ - \left[\frac{1}{2}+\alpha+\Big(\alpha+\frac{3}{2}\Big)x\right]\frac{d}{dx}P_n^{(\alpha,-\frac{1}{2})}(x)\Big\},$$

we obtain

$$\int_{-1}^{t} (1-x)^{\alpha}(1+x)^{-\frac{1}{2}} P_n^{(\alpha,-\frac{1}{2})}(x)dx =$$

$$= -\frac{1}{n(n+\alpha+\frac{1}{2})}\left\{ \int_{-1}^{t} (1-x)^{\alpha+1}(1+x)^{\frac{1}{2}} \frac{d^2}{dx^2} P_n^{(\alpha,-\frac{1}{2})}(x) - \right.$$

$$\left. - \int_{-1}^{t} (1-x)^{\alpha}(1+x)^{-\frac{1}{2}} \left[\frac{1}{2}+\alpha+\left(\alpha+\frac{3}{2}\right)x\right] \frac{d}{dx}P_n^{(\alpha,-\frac{1}{2})}(x)dx \right\} =$$

$$= -\frac{1}{n(n+\alpha+1/2)}\left\{ (1-x)^{\alpha+1}(1+x)^{\frac{1}{2}} \frac{d}{dx}P_n^{(\alpha,-\frac{1}{2})}(x)\Big|_{-1}^{t} + \right.$$

$$+ \int_{-1}^{t} (1-x)^{\alpha}(1+x)^{-\frac{1}{2}} \left[\frac{1}{2}+\alpha+\left(\alpha+\frac{3}{2}\right)x\right] \frac{d}{dx}P_n^{(\alpha,-\frac{1}{2})}(x)dx -$$

$$\left. - \int_{-1}^{t} (1-x)^{\alpha}(1+x)^{-\frac{1}{2}} \left[\frac{1}{2}+\alpha+\left(\alpha+\frac{3}{2}\right)x\right] \frac{d}{dx}P_n^{(\alpha,-\frac{1}{2})}(x)dx \right\} =$$

$$= -\frac{1}{n(n+\alpha+1/2)}(1-t)^{\alpha+1}(1+t)^{\frac{1}{2}} \frac{d}{dt}P_n^{(\alpha,-\frac{1}{2})}(t). \tag{4.3}$$

Since ([5], pp. 84, 82)

$$(1-x^2)\frac{d}{dx}P_n^{(\alpha,-\frac{1}{2})}(x) = \frac{(n+\alpha+\frac{1}{2})[(2n+\alpha+\frac{3}{2})x+\alpha+\frac{1}{2}]P_n^{(\alpha,-\frac{1}{2})}(x)}{2n+\alpha+\frac{3}{2}} - \\ -\frac{2(n+1)(n+\alpha+\frac{1}{2})}{2n+\alpha+\frac{3}{2}}P_{n+1}^{(\alpha,-\frac{1}{2})}(x) = \left(n+\alpha+\frac{1}{2}\right)xP_{n+1}^{(\alpha,-\frac{1}{2})}(x)+$$

$$+\frac{(\alpha+\frac{1}{2})(n+a+\frac{1}{2})}{2n+\alpha+\frac{3}{2}}P_n^{(\alpha,-\frac{1}{2})}(x)-\frac{2(n+1)(n+\alpha+\frac{1}{2})}{2n+\alpha+\frac{3}{2}}P_{n+1}^{(\alpha,-\frac{1}{2})}(x) \quad (4.4)$$

and

$$2(n+1)\left(n+\alpha+\frac{1}{2}\right)\left(2n+\alpha-\frac{1}{2}\right)P_{n+1}^{(\alpha,-\frac{1}{2})}(x)=$$
$$=\left(2n+\alpha+\frac{1}{2}\right)\left[\left(2n+\alpha+\frac{3}{2}\right)\left(2n+\alpha-\frac{1}{2}\right)x+\alpha^2-\frac{1}{4}\right]P_n^{(\alpha,-\frac{1}{2})}(x)-$$
$$-2(n+\alpha)\left(n-\frac{1}{2}\right)\left(2n+\alpha+\frac{3}{2}\right)P_{n+1}^{(\alpha,-\frac{1}{2})}(x),$$

therefore

$$\left(2n+\alpha+\frac{1}{2}\right)\left(2n+\alpha+\frac{3}{2}\right)\left(2n+\alpha-\frac{1}{2}\right)xP_n^{(\alpha,-\frac{1}{2})}(x)=$$
$$=2(n+1)\left(n+\alpha+\frac{1}{2}\right)\left(2n+\alpha-\frac{1}{2}\right)P_{n+1}^{(\alpha,-\frac{1}{2})}(x)+$$
$$+2(n+\alpha)\left(n-\frac{1}{2}\right)\left(2n+\alpha+\frac{3}{2}\right)P_{n+1}^{(\alpha,-\frac{1}{2})}(x)-$$
$$-\left(2n+\alpha+\frac{1}{2}\right)\left(\alpha^2-\frac{1}{4}\right)P_n^{(\alpha-\frac{1}{2})}(x).$$

Further,

$$xP_n^{(\alpha,-\frac{1}{2})}(x)=\frac{2(n+1)(n+\alpha+\frac{1}{2})}{(2n+\alpha+\frac{1}{2})(2n+\alpha+\frac{3}{2})}P_{n+1}^{(\alpha,-\frac{1}{2})}(x)+$$
$$+\frac{2(n+\alpha)(n-\frac{1}{2})}{(2n+\alpha+\frac{1}{2})(2n+\alpha-\frac{1}{2})}P_{n+1}^{(\alpha,-\frac{1}{2})}(x)+$$
$$+\frac{\frac{1}{4}-\alpha^2}{(2n+\alpha+\frac{3}{2})(2n+\alpha-\frac{1}{2})}P_n^{(\alpha,-\frac{1}{2})}(x). \quad (4.5)$$

Using (4.5) in (4.4) and (4.3), we obtain

$$\int_{-1}^{\mathrm{t}}(1-x)^\alpha(1+x)^{-\frac{1}{2}}P_n^{(\alpha,-\frac{1}{2})}(x)dx=\frac{(1-t)^\alpha(1+t)^{-\frac{1}{2}}}{n(n+\alpha+\frac{1}{2})}\times$$
$$\times\Bigg\{\frac{2(n+1)(n+\alpha+\frac{1}{2})}{2n+\alpha+\frac{3}{2}}P_{n+1}^{(\alpha,-\frac{1}{2})}(x)-\frac{(\alpha+\frac{1}{2})(n+\alpha+\frac{1}{2})}{2n+\alpha+\frac{3}{2}}P_n^{(\alpha,-\frac{1}{2})}(x)-$$
$$-\frac{2(n+1)(n+\alpha+\frac{1}{2})^2}{(2n+\alpha+\frac{1}{2})(2n+\alpha+\frac{3}{2})}P_{n+1}^{(\alpha,-\frac{1}{2})}(x)-$$
$$-\frac{2(n+\alpha)(n-\frac{1}{2})(n+\alpha+\frac{1}{2})}{(2n+\alpha+\frac{3}{2})(2n+\alpha-\frac{1}{2})}P_{n+1}^{(\alpha,-\frac{1}{2})}(x)-$$

$$-\frac{(\frac{1}{4}-\alpha^2)(n+\alpha+\frac{1}{2})}{(2n+\alpha+\frac{3}{2})(2n+\alpha-\frac{1}{2})}P_{n+1}^{(\alpha,-\frac{1}{2})}(x)\Big\},$$

which implies that

$$\int\limits_{-1}^{t}(1-x)^{\alpha}(1+x)^{-\frac{1}{2}}P_n^{(\alpha,-\frac{1}{2})}(x)dx = O(\frac{1}{n})(1-t)^{\alpha}(1+t)^{-\frac{1}{2}}\left|P_n^{(\alpha,-\frac{1}{2})}(t)\right|.$$

Using estimation of the latter correlation ([6], p. 265)

$$(1-x)^{\frac{\alpha}{2}+\frac{1}{4}}\left|P_n^{(\alpha,-\frac{1}{2})}(x)\right| \le Mn^{-\frac{1}{2}},\ \ \alpha \ge -\frac{1}{2},\ \ -1 \le x \le 1, \tag{4.6}$$

we obtain (4.1).

Now we prove inequality (4.2). Let $-1 \le t \le -1+n^{-4/(3+2\alpha)}$.

Then using (4.6), we obtain

$$\left|\int\limits_{-1}^{\mathrm{t}}(1-x)^{\alpha}(1+x)^{-\frac{1}{2}}P_n^{(\alpha,-\frac{1}{2})}(x)dx\right| \le$$

$$\le \int\limits_{-1}^{-1+n^{-4/(3+2\alpha)}}(1-x)^{\alpha}(1+x)^{-\frac{1}{2}}\left|P_n^{(\alpha,-\frac{1}{2})}(x)\right|dx \le$$

$$\le Mn^{-\frac{1}{2}}\int\limits_{-1}^{-1+n^{-4/(3+2\alpha)}}(1-x)^{\frac{\alpha}{2}-\frac{1}{4}}(1+x)^{-\frac{1}{2}}dx \le$$

$$\le Mn^{-\frac{1}{2}}\int\limits_{-1}^{-1+n^{-4/(3+2\alpha)}}(1+x)^{-\frac{1}{2}}dx \le Mn^{-\frac{1}{2}}n^{-\frac{2}{3+2a}} \le Mn^{\frac{2}{2\alpha+3}-\frac{3}{2}}. \tag{4.7}$$

Let now $-1+n^{-4/(3+2\alpha)} \le t \le \delta < 1$.

Then from (4.1) we obtain

$$\left|\int\limits_{-1}^{t}(1-x)^{\alpha}(1+x)^{-\frac{1}{2}}P_n^{(\alpha,-\frac{1}{2})}(x)dx\right| \le Mn^{\frac{2}{2\alpha+3}-\frac{3}{2}}. \tag{4.8}$$

Let $\delta \le t \le 1-n^{-\frac{4}{3+2\alpha}}$. Again, from (4.1), we have

$$\left|\int\limits_{-1}^{t}(1-x)^{\alpha}(1+x)^{-\frac{1}{2}}P_n^{(\alpha,-\frac{1}{2})}(x)dx\right| \le Mn^{(\frac{1}{4}-\frac{\alpha}{2})\frac{4}{2\alpha+3}-\frac{3}{2}} =$$

$$= Mn^{\frac{1-2\alpha}{3+2\alpha}-\frac{3}{2}} \le M\cdot n^{\frac{2}{2\alpha+3}-\frac{3}{2}}. \tag{4.9}$$

At last, let

$$1-n^{-\frac{4}{3+2\alpha}} \le t \le 1.$$

Then we have

$$\int_{-1}^{t}(1-x)^{\alpha}(1+x)^{-\frac{1}{2}}P_n^{(\alpha,-\frac{1}{2})}(x)dx =$$

$$= \int_{-1}^{1-n^{-\frac{4}{3+2\alpha}}}(1-x)^{\alpha}(1+x)^{-\frac{1}{2}}P_n^{(\alpha,-\frac{1}{2})}(x)dx+$$

$$+\int_{1-n^{-\frac{4}{3+2\alpha}}}^{t}(1-x)^{\alpha}(1+x)^{-\frac{1}{2}}P_n^{(\alpha,-\frac{1}{2})}(x)dx = J_1+J_2. \tag{4.10}$$

Applying the estimates (4.7)–(4.9) for J_1, we obtain the estimate (4.11)

$$J_1 \le Mn^{\frac{2}{2\alpha+3}-\frac{3}{2}}. \tag{4.11}$$

It remains to estimate J_2. Using inequality (4.6), we obtain

$$J_2 \le \int_{1-n^{-\frac{4}{3+2\alpha}}}^{t}(1-x)^{\alpha}(1+x)^{-\frac{1}{2}}\left|P_n^{(\alpha,-\frac{1}{2})}(x)\right|dx \le$$

$$\le Mn^{-\frac{1}{2}}\int_{1-n^{-\frac{4}{3+2\alpha}}}^{t}(1-x)^{\frac{\alpha}{2}-\frac{1}{4}}(1+x)^{-\frac{1}{2}}dx \le$$

$$\le Mn^{-\frac{1}{2}}\int_{1-n^{-\frac{4}{3+2\alpha}}}^{t}(1-x)^{\frac{\alpha}{2}-\frac{1}{4}}dx \le Mn^{-\frac{1}{2}}n^{-\left(\frac{\alpha}{2}+\frac{3}{4}\right)\frac{4}{3+2\alpha}} = Mn^{-\frac{3}{2}}. \tag{4.12}$$

Using (4.11) and (4.12) in (4.10), we obtain for which $1-n^{-4/(3+2a)} \le t \le 1$ the estimate

$$\left|\int_{-1}^{t}(1-x)^{\alpha}(1+x)^{-\frac{1}{2}}P_n^{(\alpha,-\frac{1}{2})}(x)dx\right| \le Mn^{\frac{2}{2\alpha+3}-\frac{3}{2}}. \tag{4.13}$$

is valid. Now, using (4.7), (4.8), (4.9) and (4.13), we obtain inequality (4.2). Thus Lemma 4.1 is proved. □

Since ([5], p. 70 and [11], p. 131)

$$P_n^{(\alpha,-\frac{1}{2})}(1) = \frac{\Gamma(n+1+\alpha)}{\Gamma(n+1)\Gamma(\alpha+1)} = \frac{n^{\alpha}}{\Gamma(\alpha+1)}\left\{1+O\Big(\frac{1}{n}\Big)\right\}, \tag{4.14}$$

for Lemma 4.1 we get the following

Corollary 4.1. *For $n = 2, 3, \ldots$; $-\frac{1}{2} < \alpha < \frac{1}{2}$, the estimates*

$$\left|\int\limits_{-1}^{t}(1-x)^{\alpha}(1+x)^{-\frac{1}{2}}R_n^{(a,-\frac{1}{2})}(x)dx\right| \leq$$

$$\leq M\begin{cases}(1-t)^{\frac{\alpha}{2}-\frac{1}{4}}(1+t)^{-\frac{1}{2}}n^{-\alpha-\frac{3}{2}}, & t\in(-1,1)\,,\\ n^{\frac{2}{2\alpha+3}-\frac{3}{2}-\alpha}, & t\in[-1,1]\end{cases}$$

are true.

Lemma 4.2. *Let $-\frac{1}{2} < \alpha < \frac{1}{2}$. Then for $n = 2, 3, \ldots$, the estimates*

$$\left|\int\limits_{-1}^{t}(1-x)^{\alpha}(1+x)^{-\frac{1}{2}}(I^rR_n^{(\alpha,-\frac{1}{2})})(x)dx\right| \leq$$

$$\leq M\begin{cases}(1-t)^{\frac{\alpha}{2}-\frac{1}{4}}(1+t)^{-\frac{1}{2}}n^{-2r-\alpha-\frac{3}{2}}, & t\in(-1,1)\,,\\ n^{\frac{2}{2\alpha+3}-\frac{3+2\alpha}{2}-2r}, & t\in[-1,1]\end{cases}$$

are true.

Proof. From (2.4) and (1.9), we have

$$\int\limits_{-1}^{t}\left(I^rR_n^{(\alpha,-\frac{1}{2})}\right)(x)d\mu_\alpha(x) = \int\limits_{-1}^{t}\left(R_n^{(\alpha,-\frac{1}{2})}*\psi_r\right)(x)d\mu_\alpha(x) =$$

$$= \int\limits_{-1}^{t}\int\limits_{-1}^{1}\left(\tau_uR_n^{(\alpha,-\frac{1}{2})}\right)(x)\psi_r(u)\,d\mu_\alpha(u)\,d\mu_\alpha(x) =$$

$$= \int\limits_{-1}^{1}\psi_r(u)R_n^{(\alpha,-\frac{1}{2})}(u)d\mu_\alpha(u)\int\limits_{-1}^{t}R_n^{(\alpha,-\frac{1}{2})}(x)d\mu_x =$$

$$= \overset{\wedge}{\psi_r}(n)\int\limits_{-1}^{t}R_n^{(\alpha,-\frac{1}{2})}(x)d\mu_\alpha(x) =$$

$$= \overset{\wedge}{\psi_r}(n)\int\limits_{-1}^{t}R_n^{(\alpha,\,-\frac{1}{2})}(x)d\mu_\alpha(x). \tag{4.15}$$

Using (2.3), Corollary 4.1 in (4.15), we obtain assertions of Lemma 4.2. □

Lemma 4.3. *For $n = 1, 2, \ldots$, the estimates*

$$\left\|R_n^{(\alpha,-\frac{1}{2})}\right\|_{q,\alpha} \le M \begin{cases} n^{-\frac{1}{q}}, & (\alpha+\frac{1}{2})q > 1, \\ n^{-(\alpha+\frac{1}{2})}\sqrt{\log n}, & (\alpha+\frac{1}{2})q = 1, \\ n^{-(\alpha+\frac{1}{2})}, & (\alpha+\frac{1}{2})q < 1. \end{cases}$$

are true.

$$\left\|R_n^{(\alpha,-\frac{1}{2})}\right\|_{q,\alpha} = C_1(\alpha)\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{-\frac{1}{2}}\left|R_n^{(\alpha,-\frac{1}{2})}(x)\right|dx =$$

$$= 2^{a+\frac{3}{2}}C_1(\alpha)\int_{0}^{1}(1-x^2)^{\alpha}\left|R_n^{(\alpha,-\frac{1}{2})}(2x^2-1)\right|dx. \tag{4.16}$$

Using the equality ([5], p. 71)

$$P_n^{(\alpha,-\frac{1}{2})}(2x^2-1) = \frac{\Gamma(n+1+\alpha)\Gamma(2n+1)}{\Gamma(n+1)\Gamma(2n+\alpha+1)}P_{2n}^{(\alpha,\alpha)}(x)$$

and (4.14), we find

$$R_n^{(\alpha,-\frac{1}{2})}(2x^2-1) = \frac{\Gamma(1+\alpha)\Gamma(2n+1)}{\Gamma(2n+\alpha+1)}P_{2n}^{(\alpha,\alpha)}(x). \tag{4.17}$$

Using (4.17) in (4.16), we obtain

$$\left\|R_n^{(\alpha,-\frac{1}{2})}\right\|_{q,\mu}^{q} =$$

$$= 2^{\alpha+\frac{3}{2}}C_1(\alpha)\left\{\frac{\Gamma(1+\alpha)\Gamma(2n+1)}{\Gamma(2n+\alpha+1)}\right\}^q\int_{0}^{1}(1-x^2)^{\alpha}\left|P_{2n}^{(\alpha,\alpha)}(x)\right|^q dx.$$

by the Cauchy-Buniakowsky's inequality

$$\left\|R_n^{(\alpha,-\frac{1}{2})}\right\|_{q,\alpha}^{q} \le$$

$$\le 2^{\alpha+\frac{3}{2}}C_1(\alpha)\left\{\frac{\Gamma(\alpha+1)\Gamma(2n+1)}{\Gamma(2n+\alpha+1)}\right\}^q\left(\int_{0}^{1}(1-x^2)^{2\alpha}dx\right)^{\frac{1}{2}} \le$$

$$\le \left(\int_{0}^{1}|P_{2n}^{(\alpha,\alpha)}(x)|^{2q}dx\right)^{\frac{1}{2}} \le$$

$$\le C_3(\alpha)\left(\int_{0}^{1}|P_{2n}^{(\alpha,\alpha)}(x)|^{2q}dx\right)^{\frac{1}{2}} \tag{4.18}$$

where

$$C_3(\alpha)=2^{\alpha+1}C_1(\alpha)\left\{\frac{\Gamma(2\alpha+1)\Gamma(\frac{1}{2})}{\Gamma(2\alpha+\frac{3}{2})}\right\}^{\frac{1}{2}}\left\{\frac{\Gamma(\alpha+1)\Gamma(2n+1)}{\Gamma(2n+\alpha+1)}\right\}^{q}\leq Mn^{-\alpha q}$$

From the inequality ([5], p. 177)

$$\left|P_{2n}^{(\alpha,\alpha)}(x)\right| \leq Mn^{-\frac{1}{2}}(1-x+n^{-2})^{-\frac{\alpha}{2}-\frac{1}{4}},\ 0\leq x\leq 1$$

it follows that

$$\left(\int_0^1 \left|P_{2n}^{(\alpha,\alpha)}(x)\right|^{2q} dx\right)^{\frac{1}{2}} \leq Mn^{-\frac{q}{2}}\begin{cases} n^{\left(\alpha+\frac{1}{2}\right)q-1}, & (\alpha+\frac{1}{2})q>1,\\ \sqrt{\log n}, & (\alpha+\frac{1}{2})q=1,\\ 1, & (\alpha+\frac{1}{2})q<1,\end{cases} \leq$$

$$\leq M\begin{cases} n^{q\alpha-1}, & (\alpha+\frac{1}{2})q>1,\\ n^{-q/2}\sqrt{\log n}, & (\alpha+\frac{1}{2})q=1,\\ n^{-q/2}, & (\alpha+\frac{1}{2})q<1.\end{cases} \tag{4.19}$$

are true.

Using (4.14) and (4.19) in (4.15), we obtain the assertion of Lemma 4.3.

Lemma 4.4. *For $n=1,2,\ldots$ the estimates*

$$\left\|I^r R_n^{(\alpha,-\frac{1}{2})}\right\|_{q,\mu} \leq M\begin{cases} n^{-2r-\alpha-1/2}, & (\alpha+\frac{1}{2})q<1,\\ n^{-2r-\alpha-1/2}\sqrt{\log n}\frac{1}{2q}, & (\alpha+\frac{1}{2})\sqrt{\log n},\\ n^{-2r-1/q}, & (\alpha+\frac{1}{2})q>1\end{cases}$$

are true.

Proof. By (2.4) and (1.9), we obtain

$$\left\|I^r R_n^{(\alpha,-\frac{1}{2})}\right\|_{q,\alpha} = \left\|R_n^{(\alpha,-\frac{1}{2})} * \psi_r\right\|_{q,\alpha} =$$

$$= \left\|\int_{-1}^{1}\psi_r(u)\left(\tau_u R_n^{(\alpha,-\frac{1}{2})}(\cdot)\,d\mu_\alpha(u)\right)\right\|_{q,\alpha} =$$

$$= \left\|\int_{-1}^{1}\psi_r(u)R_n^{(\alpha,-\frac{1}{2})}(u)R_n^{(\alpha,-\frac{1}{2})}(\cdot)\,d\mu_\alpha(u)\right\|_{q,\alpha} =$$

$$= \left\|\overset{\wedge}{\psi_r}(n)\,R_n^{(\alpha,-\frac{1}{2})}\right\|_{q,\alpha} = \overset{\wedge}{\psi_r}(n)\left\|R_n^{(\alpha,-\frac{1}{2})}\right\|_{q,\alpha},$$

from which using (3.3) and Lemma 4.3, we obtain the statement of Lemma 4.4. $\square$

Lemma 4.5. *Let* $-\dfrac{1}{2} < \alpha < \dfrac{1}{2}$. *For* $r = 0, 1, \ldots$ *and* $n = 1, 2, \ldots$ *the equality*

$$\left\| I^r R_n^{(\alpha,-\frac{1}{2})} \right\|_{C[-1,1]} = \overset{\wedge}{\psi_r}(n) = \left\{ \frac{2(\alpha+1)}{n(n+\alpha+\frac{1}{2})} \right\}^r$$

is valid.

Proof. By (2.7) and (3.3), we obtain

$$\left\| I^r R_n^{(\alpha,-\frac{1}{2})} \right\|_{C[-1,1]} = \left\| R_n^{(\alpha,-\frac{1}{2})} * \psi \right\|_{C[-1,1]} =$$

$$= \left\| \int_{-1}^{1} \psi_r(u) \left(\tau_u R_n^{(\alpha,-\frac{1}{2})} \right) (\cdot) d\mu_\alpha(u) \right\|_{C[-1,1]} =$$

$$= \left\| \int_{-1}^{1} \psi_r(u) R_n^{(\alpha,-\frac{1}{2})}(u) d\mu_\alpha(u) R_n^{(\alpha,-\frac{1}{2})}(\cdot) \right\|_{C[-1,1]} =$$

$$= \overset{\wedge}{\psi_r}(n) \left\| R_n^{(\alpha,-\frac{1}{2})}(\cdot) \right\|_{C[-1,1]} = \overset{\wedge}{\psi_r}(n) = \left\{ \frac{2(\alpha+1)}{n(n+\alpha+\frac{1}{2})} \right\}^r .$$

Thus Lemma 4.5 is proved. □

Lemma 4.6. *Let* $-\dfrac{1}{2} < \alpha < \dfrac{1}{2}$. *For* $n = 2, 3, \ldots$ *and* $r = 0, 1, \ldots$, *the estimate*

$$\left\| I^r R_n^{(\alpha,-\frac{1}{2})} \right\|_{1,\alpha} \leq M n^{-2r-\alpha-\frac{1}{2}}$$

is valid.

Proof. Also, as when proving Lemma 4.4, we find

$$\left\| \left(I^r R_n^{(\alpha,-\frac{1}{2})} \right) \right\|_{1,\alpha} = \overset{\wedge}{\psi_r}(n) \left\| R_n^{(\alpha,-\frac{1}{2})} \right\|_{1,\alpha} . \tag{4.20}$$

Since (see proof of Lemma 4.3)

$$\| R_n^{(\alpha,-\frac{1}{2})} \|_{1,\alpha} = 2^{\alpha+\frac{3}{2}} C_1(\alpha) \frac{\Gamma(\alpha+1)\Gamma(2n+1)}{\Gamma(2n+\alpha+1)} \int_0^1 (1-x^2)^\alpha |P_{2n}^{(\alpha,\alpha)}(x)| dx,$$

using the estimate (4.6) and the relation (4.14), we obtain

$$\| R_n^{(\alpha,-\frac{1}{2})} \|_{1\alpha} \leq M n^{-\alpha-1/2} \int_0^1 (1-x^2)^{\frac{\alpha}{2}-\frac{1}{4}} dx \leq$$

$$\leq M n^{-\alpha-1/2} \int_0^1 (1-x)^{\frac{\alpha}{2}-\frac{1}{4}} x^{-\frac{1}{2}} dx = M \frac{\Gamma\left(\frac{2\alpha+3}{4}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{2\alpha+5}{4}\right)} n^{-\alpha-\frac{1}{2}}. \tag{4.21}$$

Using Lemma 4.5 and (4.12) in (4.20), we obtain the assertion of Lemma 4.6. □

5. On Estimations of Coefficients of Fourier–Jacobi Functions from W_X^r

In this section we give applications of general theorems from Section 1. We formulate them in conformity with our case.

Theorem A. Let $f \in W_X^r$ $(X = L_{p,\alpha},\ 1 < p < \infty)$, $\frac{1}{p} + \frac{1}{q} = 1$. If

1. $\|I^r \varphi_n\|_{q,\alpha} \leq M -$ const., $q > 1$, $r = 0, 1, \ldots$;

2. $\lim_{n\to\infty} \int_{-1}^{t} (I^r \varphi_n)(x) d\mu_\alpha(x) = 0$, $t \in [-1, 1]$,

then

$$\lim_{n\to\infty} \int_{-1}^{1} f(x)\varphi_n(x) d\mu_\alpha(x) = 0,$$

where $\varphi_n(x)$, $n = 0, 1, \ldots$ is the system of orthogonal functions with the weight $\mu_\alpha(x) = C_1(\alpha)(1-x)^\alpha(1+x)^{-\frac{1}{2}}$ on the segment $[-1, 1]$.

Theorem B. Let $f \in W_X^r$ $(X = L_{1,\alpha})$. If

1. $|(I^r \varphi_n(x))| \leq M -$ const, $r = 0, 1, \ldots$, $x \in [-1, 1]$;

2. $\lim_{n\to\infty} \int_{-1}^{t} (I^r \varphi_n)(x) d\mu_\alpha(x) = 0$, $t \in [-1, 1]$,

then

$$\lim_{n\to\infty} \int_{-1}^{1} f(x)\varphi_n(x) d\mu_\alpha(x) = 0.$$

Theorem C. Let $f \in W_X^r\big(X = L'\big)$. If

1. $\left|\int_{-1}^{x} (I^r \varphi_n)(t) d\mu_\alpha(t)\right| \leq M$ – const, $r = 0, 1, \ldots$, $x \in [-1, 1]$,

2. $\lim_{n\to\infty} \int_{-1}^{t} (I^r \varphi_n)(x) d\mu_\alpha(x) = 0$, $t \in [-1, 1]$,

then

$$\lim_{n\to\infty} \int_{-1}^{1} f(x)\varphi_n(x) d\mu_\alpha(x) = 0.$$

To each $f \in X$ let us now associate its Fourier–Jacobi series

$$f(x) \sim \sum_{n=0}^{\infty} \frac{(2n+\alpha+\frac{1}{2})\Gamma(n+\alpha+\frac{1}{2})\Gamma(n+\alpha+1)}{2^{\alpha+\frac{1}{2}}\Gamma^2(\alpha+1)\Gamma(n+\frac{1}{2})\Gamma(n+1)} \hat{f}(n)\, R_n^{(\alpha,-\frac{1}{2})}(x),$$

where

$$\hat{f}(n) = \int_{-1}^{1} f(t) R_n^{(\alpha,-\frac{1}{2})}(t) d\mu_\alpha(t).$$

Theorem 5.1. *Let* $f \in W_X^r$ $(X = L_{p,\alpha}, 1 < p < \infty)$, *then*

$$\lim_{n\to\infty} n^{2r+\alpha+\frac{1}{2}} \hat{f}(n) = 0, \;\; 0 < \Big(\alpha + \frac{1}{2}\Big) q < 1, \;\; r = 0, 1, \dots .$$

Proof. Proof of the theorem is reduced to the verification of the condition of Theorem A. According to (5.2),

$$n^{2r+\alpha+\frac{1}{2}} \hat{f}(n) = \int_{-1}^{1} \varphi_n(x) d\mu_\alpha(x),$$

where

$$\varphi_n^{(\alpha,-\frac{1}{2})} \equiv \varphi_n(x) = n^{2r+\alpha+\frac{1}{2}}(x) R_n^{(\alpha,-\frac{1}{2})}(x).$$

The first condition of Theorem A follows easily from Lemma 4.4.

$$\|I^r \varphi_n\|_{q,\alpha} = n^{2r+\alpha+\frac{1}{2}} \|R_n^{(\alpha,-\frac{1}{2})}\|_{q,\alpha} \le M,$$

and by Lemma 4.2, we have

$$\left| \int_{-1}^{t} (I^r \varphi_n)(x)\, d\mu_\alpha(x) \right| = n^{2r+\alpha+\frac{1}{2}} \left| \int_{-1}^{t} \Big(I^r R_n^{\alpha,-\frac{1}{2}} \Big)(x)\, d\mu_\alpha(x) \right| \le$$

$$\le M \begin{cases} (1-t)^{\frac{\alpha}{2}-\frac{1}{4}} (1+t)^{\frac{-1}{2}} \frac{1}{n}, & t \in (-1,\, 1), \\ n^{-\frac{2\alpha+1}{2\alpha+3}}, & t \in [-1,\, 1]. \end{cases}$$

From this we obtain condition 2 of Theorem A.

Thus Theorem 5.1 is proved. □

Remark. If $f \in W_X^r$ $(X = L_{p,\alpha})$, then for each sequence of real numbers, γ_n tends to infinity

$$\lim_{n\to\infty} n^{2r+\alpha+\frac{1}{2}} \hat{f}(n)\, \gamma_n,$$

not approaching zero.

Proof. Just as in proving Lemma 4.2, we obtain

$$\|\gamma_n (I^r \varphi_n)\|_{2,\mu} = n^{2r+\alpha+\frac{1}{2}} \overset{\wedge}{\psi_r}(n) \left\| R_n^{(\alpha,-\frac{1}{2})} \right\|_{2,\mu} \gamma_n.$$

Using (1.1) and (1.2), we get

$$\|R_n^{(\alpha,-1/2)}\|_{2,\mu} = \left\{ \int_{-1}^{1} \Big[R_n^{(\alpha,-\frac{1}{2})}(x) \Big]^2 d\mu_\alpha(x) \right\}^{\frac{1}{2}} =$$

$$= \left\{ C_1(\alpha) \frac{2^{\alpha+\frac{1}{2}} \Gamma(n+\alpha+1) \Gamma(n+\frac{1}{2})}{(2n+\alpha+\frac{1}{2}) \Gamma(n+1) \Gamma(n+\alpha+\frac{1}{2})} \left(\frac{\Gamma(n+1)\Gamma(\alpha+1)}{\Gamma(n+\alpha+1)} \right)^2 \right\}^{\frac{1}{2}} =$$

$$= \left\{ C_1(\alpha) \frac{2^{\alpha+\frac{1}{2}}\Gamma(n+\frac{1}{2})\Gamma(n+1)\Gamma^2(\alpha+1)}{(2n+\alpha+\frac{1}{2})\Gamma(n+\alpha+1)\Gamma(n+\alpha+\frac{1}{2})} \right\}^{\frac{1}{2}} \sim$$

$$\sim \left(2^{\alpha+\frac{1}{2}}\Gamma^2(\alpha+1)C_1(\alpha)\frac{1}{n^{2\alpha+1}} \right)^{\frac{1}{2}} = 2^{\frac{\alpha}{2}+\frac{1}{4}}\Gamma(\alpha+1)\sqrt{C_1(\alpha)}n^{-\alpha-\frac{1}{2}}.$$

Then

$$\|\gamma_n(I^r\varphi_n)(\cdot)\|_{2,\mu} \sim$$

$$\sim 2^{\frac{\alpha}{2}+\frac{1}{4}}\Gamma(\alpha+1)C_1^{\frac{3}{2}}(\alpha)(2(\alpha+1))^r\gamma_n \frac{n^{2r+\alpha+\frac{1}{2}}n^{-\alpha-\frac{1}{2}}}{\left(n(n+\alpha+\frac{1}{2})\right)^r} \sim$$

$$\sim 2^{r+\frac{\alpha}{2}+\frac{1}{4}}(\alpha+1)^r(C_1(\alpha))^{3/2}\gamma_n,$$

since $\frac{\Gamma(\alpha+\lambda)}{\Gamma(\lambda)} \sim \alpha^\lambda, \alpha \to \infty$, ([9], p. 951).

Hence we find that

$$\lim_{n\to\infty} \|\gamma_n(I^r\varphi_n)\|_{2,\mu} = \infty,$$

i.e., not fluttering the first condition of Theorem A. Consequently, if $f \in W_X^r(L_{2,\alpha})$, then the order is final in Theorem 5.1. □

Theorem 5.2. *Let* $f \in W_X^r\,(X = L_{1,\alpha})$. *Then for* $-\dfrac{1}{2} < \alpha < \dfrac{1}{2}$ *and* $r = 0, 1, \dots,$

$$\lim_{\gamma\mapsto\infty} n^{2r}\hat{f}(n) = 0, \quad r = 0, 1, \dots,$$

but

$$\lim_{n\to\infty} \gamma_n n^{2r}\hat{f}(n)$$

not approaching zero, as γ_n*-tends to infinity.*

Proof. According to (5.2),

$$n^{2r}\hat{f}(n) = \int_{-1}^{1} f(x)\varphi_n(x)d\mu_\alpha(x),$$

where

$$\varphi_n(x) = n^{2r}R_n^{(\alpha,-\frac{1}{2})}(x).$$

The first condition of Theorem B follows by Lemma 4.5.

Using Lemma 4.2, we have

$$\left|\int_{-1}^{t}(I^r\varphi_n)\,d\mu_\alpha(x)\right| = n^{2r}\left|\int_{-1}^{t}\left(I^r R_n^{(\alpha,-\frac{1}{2})}\right)(x)d\mu_\alpha(x)\right| \le$$

$$\leq M \begin{cases} (1-t)^{\frac{\alpha}{2}-\frac{1}{4}}(1+t)^{\frac{-1}{2}} n^{-n-\frac{3}{2}}, & t \in (-1,\,1), \\ n^{\frac{2}{2\alpha+3}-\frac{2\alpha+3}{2}}, & t \in [-1,\,1], \end{cases}$$

from which we get the second condition of Theorem B. Thus the first assertion of Theorem 5.2 is proved.

We prove the second assertion of the theorem.

From the proof of Lemma 4.5, it follows that

$$\|\gamma_n\,(I^r\varphi_n)\|_{C[-1,1]} = \gamma_n n^{2r}\,\overset{\wedge}{\psi_r}(n)\,\||R_n^{(\alpha,-\frac{1}{2})}|\| =$$

$$= \gamma_n n^{2r}\,\overset{\wedge}{\psi_r}(n) = (2(\alpha+1))^r \frac{n^{2r}\gamma_n}{(n(n+\alpha+\frac{1}{2}))^r},$$

whence it follows that

$$\lim_{n\to\infty} \|\gamma_n\,(J^r\varphi_n)\|_{C[-1,1]} = \infty,$$

i.e., the first condition of Theorem B is not fulfill.

Thus Theorem 5.2 is proved. □

Theorem 5.3. *Let* $f \in W_X^r\left(X = L'\right)$. *Then for* $-\frac{1}{2} < \alpha < \frac{1}{2}$, *the equality*

$$\lim_{n\to\infty} n^{2r+\frac{2\alpha+3}{2}-\frac{2}{2\alpha+3}}\,\hat{f}(n) = 0, \quad (r = 0, 1, \dots)$$

is valid.

Proof. By (5.2), we have

$$n^{2r+\frac{2\alpha+3}{2}-\frac{2}{2\alpha+3}}\,\hat{f}(n) = \int_{-1}^{1} f(x)\varphi_n(x)d\mu_\alpha(x),$$

where

$$\varphi_n(x) = n^{2r+\frac{2\alpha+3}{2}-\frac{2}{2\alpha+3}} R_n^{(\alpha,-\frac{1}{2})}(x). \tag{5.1}$$

The first condition of Theorem C for (5.4) follows from Lemma 4.2. Further, just as in proving Lemma 4.5, we have

$$\int_{-1}^{1} (I^r\varphi_n)(x)d\mu_\alpha(x) = n^{2r+\frac{2\alpha+3}{2}-\frac{2}{2\alpha+3}} \int_{-1}^{t} (I^r\varphi_n)(x)d\mu_\alpha(x) =$$

$$= n^{2r+\frac{2\alpha+3}{2}-\frac{2}{2\alpha+3}} \left\{\frac{2(\alpha+1)}{n(n+\alpha+\frac{1}{2})}\right\}^r \int_{-1}^{t} R_n^{(\alpha,-\frac{1}{2})}(x)d\mu_\alpha(x) \leq$$

$$\leq (2(\alpha+1))^r \begin{cases} 0, & n = 1, 2, \dots; \qquad t = \pm 1, \\ n^{\frac{2\alpha+3}{2}-\frac{2}{2\alpha+3}} \int_{-1}^{t} R_n^{(\alpha,-\frac{1}{2})}(x)d\mu_\alpha(x), & t \neq \pm 1. \end{cases}$$

Taking into account Corollary 4.1, we obtain

$$\left|\int_{-1}^{t} (I^r\varphi_n)(x)d\mu_\alpha(x)\right| \le M(1-t)^{\frac{\alpha}{2}-\frac{1}{4}}(1+t)^{-\frac{1}{2}}n^{-\frac{2}{2\alpha+3}}.$$

This implies that the second condition of Theorem C for (5.4) is correct.

Thus Theorem 5.3 is proved. □

6. On the Convergence of Fourier–Jacobi Series

In this section, using the results of Section 5 dealt with the convergence of Fourier–Jacobi series,

$$\sum_{n=0}^{\infty} \frac{\left(2n+\alpha+\frac{1}{2}\right)\Gamma(n+1)\Gamma\left(n+\alpha+\frac{1}{2}\right)}{2^{\alpha+\frac{1}{2}}\Gamma(n+\alpha+1)\Gamma\left(n+\frac{1}{2}\right)}\times$$
$$\times\left(\int_{-1}^{1} P_n^{(\alpha,-\frac{1}{2})}(t)d\mu_\alpha(t)\right)P_n^{(\alpha,-\frac{1}{2})}(x). \tag{6.1}$$

Let us consider the $n-$th partial sum of series (6.1)

$$S_n(f;x) = \int_{-1}^{1} f(t)K_n(t,x)d\mu_\alpha(t), \tag{6.2}$$

where

$$K_n(t,x) = \sum_{k=0}^{n} \frac{\left(2k+\alpha+\frac{1}{2}\right)\Gamma(k+1)\Gamma\left(k+\alpha+\frac{1}{2}\right)}{2^{\alpha+\frac{1}{2}}\Gamma(k+\alpha+1)\Gamma\left(k+\frac{1}{2}\right)}P_k^{(\alpha,-\frac{1}{2})}(t)P_k^{(\alpha,-\frac{1}{2})}(x).$$

Applying multiplication theorem to the Jacobi polynomials [8]

$$\frac{\Gamma(\alpha+1)\Gamma(n+1)}{C_2(\alpha)\Gamma(n+\alpha+1)}P_n^{(\alpha,-\frac{1}{2})}(t)P_n^{(\alpha,-\frac{1}{2})}(x) =$$
$$= \int_{-1}^{1} P_n^{(\alpha,-\frac{1}{2})}\left(xt + r\sqrt{1-x^2}\sqrt{1-t^2}-\right.$$
$$\left.-\frac{1}{2}(1-r^2)(1-x)(1-t)\right)(1-r^2)^{\alpha-\frac{1}{2}}dr$$

in (6.2), we obtain

$$S_n(f;x) = \frac{C_2(\alpha)}{2^{\alpha+\frac{1}{2}}\Gamma(\alpha+1)}\int_{-1}^{1}(1-t)^{\alpha}(1+t)^{-\frac{1}{2}}f(t)\times$$

$$\times\left\{\sum_{k=0}^{n}\left(2k+\alpha+\frac{1}{2}\right)\frac{\Gamma(k+\alpha+\frac{1}{2})}{\Gamma(k+\frac{1}{2})}\int_{-1}^{1}(1-r^2)^{\alpha-\frac{1}{2}}P_n^{(\alpha,-\frac{1}{2})}\left(xt+\right.\right.$$

$$\left.\left.+r\sqrt{1-x^2}\sqrt{1-t^2}-\frac{1}{2}(1-r^2)(1-x)(1-t)\right)dr\right\}dt.$$

Substituting $t=\cos u$ and $y=\cos\frac{u}{2}$, we obtain

$$S_n(f;x)=\frac{C_2(\alpha)}{2^{\alpha+\frac{1}{2}}\Gamma(\alpha+1)}\int_{0}^{\pi}(1-\cos u)^{\alpha}(1+\cos u)^{-\frac{1}{2}}f(\cos u)\times$$

$$\times\left\{\sum_{k=0}^{n}\left(2k+\alpha+\frac{1}{2}\right)\frac{\Gamma(k+\alpha+\frac{1}{2})}{\Gamma(k+\frac{1}{2})}\int_{-1}^{1}(1-r^2)^{\alpha-\frac{1}{2}}P_n^{(\alpha,-\frac{1}{2})}\left(x\cos u+\right.\right.$$

$$\left.\left.+r\sqrt{1-x^2}\sin u-(1-r^2)(1-x)\sin^2\frac{u}{2}\right)dr\right\}\sin u du=$$

$$=\frac{C_2(\alpha)}{2^{\alpha+\frac{1}{2}}\Gamma(\alpha+1)}\int_{0}^{\pi}\left(2\sin^2\frac{u}{2}\right)^{\alpha}\left(2\cos^2\frac{u}{2}\right)^{-\frac{1}{2}}f\left(2\cos^2\frac{u}{2}-1\right)\times$$

$$\times\left\{\sum_{k=0}^{n}\left(2k+\alpha+\frac{1}{2}\right)\frac{\Gamma(k+\alpha+\frac{1}{2})}{\Gamma(k+\frac{1}{2})}\int_{-1}^{1}(1-r^2)^{\alpha-\frac{1}{2}}P_n^{(\alpha,-\frac{1}{2})}\left[x\left(2\cos^2\frac{u}{2}-1\right)+\right.\right.$$

$$\left.\left.+2r\sqrt{1-x^2}\sqrt{1-\cos^2\frac{u}{2}}\cos\frac{u}{2}-(1-r^2)(1-x)sin^2\frac{u}{2}\right]dr\right\}\times$$

$$\times 2\sqrt{1-\cos^2\frac{u}{2}}\cos\frac{u}{2}du=\frac{2C_2(\alpha)}{\Gamma(\alpha+1)}\int_{0}^{1}(1-y^2)^{\alpha}f(2y^2-1)\times$$

$$\times\left\{\sum_{k=0}^{n}\left(2k+\alpha+\frac{1}{2}\right)\frac{\Gamma(k+\alpha+\frac{1}{2})}{\Gamma(k+\frac{1}{2})}\int_{-1}^{1}(1-r^2)^{\alpha-\frac{1}{2}}P_n^{(\alpha,-\frac{1}{2})}\left[x\left(2y^2-1\right)+\right.\right.$$

$$\left.\left.+2ry\sqrt{1-x^2}\sqrt{1-y^2}-\frac{1}{2}(1-r^2)(1-x)(1-y^2)\right]dr\right\}dy.$$

Assuming that in the inside integral $x=\cos\theta$ and $z=\cos\frac{\theta}{2}$, we obtain

$$S_n(f;x)=\frac{2C_2(\alpha)}{\Gamma(\alpha+1)}\sum_{k=0}^{n}\left(2k+\alpha+\frac{1}{2}\right)\frac{\Gamma(k+\alpha+\frac{1}{2})}{\Gamma(k+\frac{1}{2})}\times$$

$$\times\int_{-1}^{1}(1-y^2)^{\alpha}f(2y^2-1)\left\{\int_{-1}^{1}(1-r^2)^{\alpha-\frac{1}{2}}P_n^{(\alpha,-\frac{1}{2})}\left[(2z^2-1)(2y^2-1)+\right.\right.$$

$$+4ryz\sqrt{1-z^2}\sqrt{1-y^2}-2(1-r^2)(1-z^2)(1-y^2)\Big]dr\Big\}dy=$$

$$=\frac{2C_2(\alpha)}{\Gamma(\alpha+1)}\sum_{k=0}^{n}\left(2k+\alpha+\frac{1}{2}\right)\frac{\Gamma(k+\alpha+\frac{1}{2})}{\Gamma(k+\frac{1}{2})}\int\limits_0^1(1-y^2)^{\alpha}f(2y^2-1)\times$$

$$\times\Bigg\{\int\limits_{-1}^{1}(1-r^2)^{\alpha-\frac{1}{2}}P_n^{(\alpha,-\frac{1}{2})}\left[2\left(zy+r\sqrt{1-z^2}\sqrt{1-y^2}\right)^2-1\right]dr\Bigg\}dy.$$

Making substitution in the inside integral

$$v=zy+r\sqrt{1-z^2}\sqrt{1-y^2}$$

and taking into account that

$$r=(v-zy)(1-y^2)^{-\frac{1}{2}}(1-z^2)^{-\frac{1}{2}},\ \ dr=(1-y^2)^{-\frac{1}{2}}(1-z^2)^{-\frac{1}{2}}dv,$$

we obtain

$$S_n(f;x)=\frac{2C_2(\alpha)}{\Gamma(\alpha+1)}\sum_{k=0}^{n}\left(2k+\alpha+\frac{1}{2}\right)\frac{\Gamma\left(k+\alpha+\frac{1}{2}\right)}{\Gamma\left(k+\frac{1}{2}\right)}\times$$

$$\times\int\limits_0^1(1-y^2)^{\alpha}f(2y^2-1)(1-y^2)^{-\frac{1}{2}}(1-z^2)^{-\frac{1}{2}}\times$$

$$\times\int\limits_{zy-\sqrt{1-z^2}\sqrt{1-y^2}}^{zy+\sqrt{1-z^2}\sqrt{1-y^2}}\left[1-\left(\frac{v-zy}{\sqrt{1-z^2}\sqrt{1-y^2}}\right)^2\right]^{\alpha-\frac{1}{2}}P_n^{(\alpha,-\frac{1}{2})}(2v^2-1)dvdy=$$

$$=\frac{2C_2(\alpha)}{\Gamma(\alpha+1)}\sum_{k=0}^{n}\left(2k+\alpha+\frac{1}{2}\right)\frac{\Gamma\left(k+\alpha+\frac{1}{2}\right)}{\Gamma\left(k+\frac{1}{2}\right)}(1-z^2)^{-\alpha}\int\limits_0^1 f(2y^2-1)\times$$

$$\times\left\{\int\limits_{zy-\sqrt{1-z^2}\sqrt{1-y^2}}^{zy+\sqrt{1-z^2}\sqrt{1-y^2}}(1-z^2-y^2-v^2+2zyv)^{\alpha-\frac{1}{2}}P_n^{(\alpha,-\frac{1}{2})}(2v^2-1)dv\right\}dy.$$

By the change of order of integration, we obtain

$$S_n(f;x)=\frac{2C_2(\alpha)}{\Gamma(\alpha+1)}\times$$

$$\times\sum_{k=0}^{n}\left(2k+\alpha+\frac{1}{2}\right)\frac{\Gamma\left(k+\alpha+\frac{1}{2}\right)}{\Gamma\left(k+\frac{1}{2}\right)}(1-z^2)^{-\alpha}\int\limits_0^1 P_n^{(\alpha,-\frac{1}{2})}(2v^2-1)\times$$

$$\times \left\{ \int\limits_{vz-\sqrt{1-v^2}\sqrt{1-z^2}}^{vz+\sqrt{1-v^2}\sqrt{1-z^2}} (1-z^2-y^2-v^2+2zyv)^{\alpha-\frac{1}{2}} f(2y^2-1)dy \right\} dv.$$

Now, making substitution

$$y = vz + r\sqrt{1-v^2}\sqrt{1-z^2},$$

we obtain

$$S_n(f;x) = \frac{2C_2(\alpha)}{\Gamma(\alpha+1)}\times$$

$$\times \sum_{k=0}^{n} \left(2k+\alpha+\frac{1}{2}\right) \frac{\Gamma\left(k+\alpha+\frac{1}{2}\right)}{\Gamma\left(k+\frac{1}{2}\right)} \int\limits_0^1 (1-v^2) P_n^{(\alpha,-\frac{1}{2})}(2v^2-1)\times$$

$$\times\left\{ \int\limits_{-1}^{1} (1-r^2)^{\alpha-\frac{1}{2}} f\left[2\left(vz+r\sqrt{1-v^2}\sqrt{1-z^2}\right)^2-1\right] dr\right\}dv =$$

$$= \frac{2C_2(\alpha)}{\Gamma(\alpha+1)} \sum_{k=0}^{n} \left(2k+\alpha+\frac{1}{2}\right) \frac{\Gamma\left(k+\alpha+\frac{1}{2}\right)}{\Gamma\left(k+\frac{1}{2}\right)}\times$$

$$\times \int\limits_0^1 (1-v^2)^{\alpha} P_n^{(\alpha,-\frac{1}{2})}(2v^2-1)\left\{ \int\limits_{-1}^{1} (1-r^2)^{\alpha-\frac{1}{2}} f\left[2v^2z^2+\right.\right.$$

$$\left.\left.+4rzv\sqrt{1-v^2}\sqrt{1-z^2}+r^2(1-v^2)(1-z^2)-1\right] dr\right\}dv =$$

$$= \frac{2C_2(\alpha)}{\Gamma(\alpha+1)} \sum_{k=0}^{n} \left(2k+\alpha+\frac{1}{2}\right) \frac{\Gamma(k+\alpha+\frac{1}{2})}{\Gamma(k+\frac{1}{2})} \int\limits_0^1 \left(1-v^2\right)^{\alpha} P_n^{(\alpha,-\frac{1}{2})}(2v^2-1)\times$$

$$\times\left\{ \int\limits_{-1}^{1} (1-r^2)^{\alpha-\frac{1}{2}} f\left[(2z^2-1)(2v^2-1)+\right.\right.$$

$$+4rzv\sqrt{1-v^2}\sqrt{1-z^2}+r^{2(1-v^2)}(1-z^2)-$$

$$\left.\left.-2(1-r^2)(1-z^2)(1-v^2)\right] dr\right\}dv = \left|z=\cos\frac{\theta}{2}; x=\cos\theta\right| =$$

$$= \frac{2C_2(\alpha)}{\Gamma(\alpha+1)} \sum_{k=0}^{n} \left(2k+\alpha+\frac{1}{2}\right) \frac{\Gamma(k+\alpha+\frac{1}{2})}{\Gamma(k+\frac{1}{2})}\times$$

$$\times \int\limits_0^1 \left(1-v^2\right)^{\alpha} P_n^{(\alpha,-\frac{1}{2})}(2v^2-1)\left\{ \int\limits_{-1}^{1} (1-r^2)^{\alpha-\frac{1}{2}} f\left[x(2v^2-1)+\right.\right.$$

$$\left.\left.+2rv\sqrt{1-x^2}\sqrt{1-v^2}-(1-r^2)(1-x)(1-v^2)\right] dr\right\}dv =$$

$$= \left| 2v^2 - 1 = u; dv = 2^{-\frac{3}{2}}(1+u)^{-\frac{1}{2}} du \right| =$$

$$= \frac{C_2(\alpha)}{2^{\alpha+\frac{1}{2}}\Gamma(\alpha+1)} \sum_{k=0}^{n} \left(2k + \alpha + \frac{1}{2}\right) \frac{\Gamma(k+\alpha+\frac{1}{2})}{\Gamma(k+\frac{1}{2})} \times$$

$$\times \int_{-1}^{1} (1-u)^{\alpha}(1+u)^{-\frac{1}{2}} P_n^{(\alpha,-\frac{1}{2})}(u) \left\{ \int_{-1}^{1} (1-r^2)^{\alpha-\frac{1}{2}} f \left[xu + r\sqrt{1-u^2}\sqrt{1-x^2} - \right.\right.$$

$$\left.\left. - \frac{1}{2}(1-r^2)(1-x)(1-u) \right] dr \right\} = C_2(\alpha) \int_{-1}^{1} (1-u)^{\alpha}(1+u)^{-\frac{1}{2}} (\tau_u f)(x) K_n(u) du,$$

where

$$K_n(u) = \sum_{k=0}^{n} \frac{(2k+\alpha+\frac{1}{2})\Gamma(2k+\alpha+\frac{1}{2})}{2^{\alpha+\frac{1}{2}}\Gamma(\alpha+1)\Gamma(k+\frac{1}{2})} P_k^{(\alpha,-\frac{1}{2})}(u).$$

Applying Cristoffel-Darbu formula

$$\sum_{\nu=0}^{n} \frac{(2\nu+\alpha+\frac{1}{2})\Gamma(\nu+\alpha+\frac{1}{2})}{2^{\alpha+\frac{1}{2}}\Gamma(\alpha+1)\Gamma(\nu+\frac{1}{2})} P_{\nu}^{(\alpha,-\frac{1}{2})}(u) =$$

$$= \frac{2^{\frac{1}{2}-\alpha}\Gamma(n+\alpha+\frac{3}{2})}{(2n+\alpha+\frac{3}{2})\Gamma(\alpha+\frac{1}{2})\Gamma(n+\frac{1}{2})} \times$$

$$\times \frac{(n+\alpha+1)P_n^{(\alpha,-\frac{1}{2})}(x) - (n+1)P_{n+1}^{(\alpha,-\frac{1}{2})}(x)}{1-x}$$

and using $S_n(1;x) \equiv 1$, we obtain

$$f(x) - S_n(f;x) =$$

$$= \frac{C_2(\alpha)2^{\frac{1}{2}-\alpha}\Gamma(n+\alpha+\frac{3}{2})(n+1+\alpha)}{\Gamma(\alpha+1)(2n+\alpha+\frac{3}{2})\Gamma(n+\frac{1}{2})} \int_{-1}^{1} \mu(u)\varphi_u(x) P_n^{(\alpha,-\frac{1}{2})}(u) du$$

$$\frac{C_2(\alpha)2^{\frac{1}{2}-\alpha}\Gamma(n+\alpha+\frac{3}{2})(n+1)}{\Gamma(\alpha+1)(2n+\alpha+\frac{3}{2})\Gamma(n+\frac{1}{2})} \int_{-1}^{1} \mu(u)\varphi_u(x) P_{n+1}^{(\alpha,-\frac{1}{2})}(u) du,$$

where

$$\varphi_u(x) = \frac{f(x) - (\tau_u f)(x)}{1-u}.$$

And since

$$P_n^{(\alpha,-\frac{1}{2})}(x) = P_n^{(\alpha,-\frac{1}{2})}(1) R_n^{(\alpha,-\frac{1}{2})}(x) = \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+1)} R_n^{(\alpha,-\frac{1}{2})}(x),$$

therefore

$$f(x) - S_n(f;x) =$$

$$= \frac{C_2(\alpha)2^{\frac{1}{2}-\alpha}\Gamma(n+\alpha+\frac{3}{2})\Gamma(n+\alpha+2)}{\Gamma^2(\alpha+1)\Gamma(n+1)(2n+\alpha+\frac{3}{2})\Gamma(n+\frac{1}{2})}\int_{-1}^{1}\mu(u)\varphi_u(x)R_n^{(\alpha,-\frac{1}{2})}(u)du-$$

$$-\frac{C_2(\alpha)2^{\frac{1}{2}-\alpha}\Gamma(n+\alpha+\frac{3}{2})\Gamma(n+\alpha+2)}{\Gamma^2(\alpha+1)\Gamma(n+1)(2n+\alpha+\frac{3}{2})\Gamma(n+\frac{1}{2})}\int_{-1}^{1}\mu(u)\varphi_u(x)R_{n+1}^{(\alpha,-\frac{1}{2})}(u)du=$$

$$= \frac{C_2(\alpha)2^{\frac{1}{2}-\alpha}\Gamma(n+\alpha+\frac{3}{2})\Gamma(n+\alpha+2)((\varphi_x\overset{\wedge}{(n)})-(\varphi_x\,(\hat{n}+1)))}{\Gamma^2(\alpha+1)C_1(\alpha)\Gamma(n+1)(2n+\alpha+\frac{3}{2})\Gamma(n+\frac{1}{2})}\sim$$

$$\sim \frac{C_2(\alpha)2^{\frac{1}{2}-\alpha}}{C_1(\alpha)\Gamma^2(\alpha+1)}((\varphi_x\overset{\wedge}{(n)})-(\varphi_x\,(\hat{n}+1)))n^{2\alpha+1}=$$

$$= \frac{2}{\Gamma(\alpha+\frac{3}{2})}((\varphi_x\overset{\wedge}{(n)})-(\varphi_x\,(\hat{n}+1)))n^{2\alpha+1}.$$

For the theorems in Section 5 we have obtained the following theorems on the order of pointwise convergence of Fourier–Jacobi series.

Theorem 6.1. *For each point* $x \in [-1,1|$, $\varphi_{\mathrm{u}}(x) \in W_X^r$ $(X = L_{p,\alpha}$, $1 < p < \infty)$, *for* $r = 0, 1, \ldots,$ *and* $0 < (\alpha + \frac{1}{2})\ q < 1$, *the equality*

$$\lim_{n\to\infty} n^{2r-\alpha-\frac{1}{2}}\{f(x) - S_n(f;x)\} = 0$$

is valid, moreover, for $p = 2$, *the order here is final.*

Theorem 6.2. *For each point* $x \in [-1,1|$, $\varphi_u(x) \in W_X^r$ $(X = L_{1,\alpha})$, *for* $r = 0, 1, \ldots,$ *and* $-\frac{1}{2} < \alpha < \frac{1}{2}$, *the equality*

$$\lim_{n\to\infty} n^{2r-2\alpha-1}\{f(x) - S_n(f;x)\} = 0,$$

is valid, moreover, the order here is final.

Theorem 6.3. *For each point* $x \in [-1,1|$, $\varphi_u(x) \in W_X^r$ $(X = L'_{1,\alpha})$, *for* $r = 0, 1, \ldots,$ *and* $-\frac{1}{2} < \alpha < \frac{1}{2}$, *the equality*

$$\lim_{n\to\infty} n^{2r+\frac{2\alpha-1}{4\alpha+6}-\alpha}\{f(x) - S_n(f;x)\} = 0 \quad \textit{is valid.}$$

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