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ON ONE NONLINEAR ANALOGUE OF THE DARBOUX PROBLEM

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Abstract. For one nonlinear oscillation equation, we consider a problem which is a nonlinear analogue of the Darboux problem and consists in the simultaneous determination of a regular solution and its domain of definition. The question of solvability of the formulated problem is solved by the method of characteristics.

რეზიუმე. არაწრფივი რხევების ერთი განტოლებისათვის განხილულია ამოცანა, რომელიც წარმოადგენს დარბუს ამოცანის არაწრფივ ანალოგს და მოითხოვს რეგულარული ამოხსნისა და მისი განსაზღვრის არის ერთდროულად დადგენას. განხილული ამოცანის ამოხსნადობის პრობლემა გადაჭრილია მახასიათებელთა მეთოდით.

As is known, the carrier of the initial characteristic Darboux problem for linear equations consists of two curves drawn from the common point of these curves [1]. One of these curves is characteristic, and the other has nowhere characteristic direction.

The characteristics of linear hyperbolic equations are completely defined by means of principal coefficients. In nonlinear cases these coefficients already depend on a sought solution and its lower derivatives. Since the characteristics, too, depend on them, the linear formulation of a Darboux characteristic problem cannot be automatically extended to the case of nonlinear equations which are of particular interest from the standpoint of application [2]. Therefore the formulations of Darboux problems for such equations should be revised with regard for general characteristic invariants [3]–[7].

In this paper, an attempt is made to formulate correctly a partially characteristic problem for a quasilinear equation, which arises when studying

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nonlinear oscillations

$$x^2(u_y^4 u_{xx} - u_{yy}) = cuu_y^4, \quad c = \text{const}\,. \tag{1}$$

The particular case of equation (1) for the purpose of outfitting warships by order of Pentagon has been investigated in [8], [9]. The general solution of the equation has been constructed for $c = 0$.

Equation (1) is interesting by the degeneracy of its order and, perhaps, by the hyperbolicity, too. The former is completely defined and occurs on the coordinate axis. The parabolic degeneracy [10] depends on the behavior of the derivative u_y of an unknown solution $u(x, y)$. Hence the set of points of this degeneracy is not a priori prescribed in this case and has to be defined simultaneously with a solution.

Since the set of points of parabolic degeneracy and the characteristics are not defined by the equation, they have to be defined by the conditions of the problem. For this we need all characteristic rules of equation (1).

The characteristic roots of equation (1)

$$\lambda_1 = u_y^{-2}, \quad \lambda_2 = -u_y^{-2}$$

provide differential relations of characteristic directions

$$u_y^2 dy - dx = 0, \quad u_y^2 dy + dx = 0. \tag{2}$$

If, taking into account (2), we consider equation (1), we come to the differential characteristic relations

$$x^2 u_y^4\, du_x - x^2 u_y^2\, du_y - cu_y^4 u\, dx = 0, \quad x^2 u_y^4\, du_x + x^2 u_y^2\, du_y - cu_y^4 u\, dx = 0.$$

The following theorem [11] is true.

Theorem 1. *Assuming $c > -\frac{1}{4}$, each of the characteristic systems of equation* (1) *admits exactly two first integrals, and they are represented explicitly as*

$$\begin{cases} \xi \equiv (u_y^{-1} + u_x)x^{\alpha} - \alpha u x^{\alpha-1}, \\ \xi_1 \equiv (u_y^{-1} + u_x)x^{1-\alpha} - (1-\alpha)u x^{-\alpha}, \end{cases} \tag{3}$$

for the family of the root λ_1, and as

$$\begin{cases} \eta \equiv (u_y^{-1} - u_x)x^{\alpha} + \alpha u x^{\alpha-1}, \\ \eta_1 \equiv (u_y^{-1} - u_x)x^{1-\alpha} + (1-\alpha)u x^{-\alpha}, \quad \alpha = \dfrac{1}{2}\left(1 + \sqrt{4c+1}\right), \end{cases} \tag{4}$$

for the family of the root λ_2.

By virtue of these two pairs of first integrals (ξ, ξ_1) and (η, η_1), which are actually characteristic invariants, it follows that in the class of hyperbolic solutions we can construct two intermediate integrals

$$\xi_1 = \varphi'(\xi), \quad \eta_1 = \psi'(\eta)$$

of equation (1) [12]. In these integrals, φ, ψ are arbitrary smooth functions such that they ensure the differentiability of the sought solution up to the second order.

Theorem 2. *If $\varphi, \psi \in C^3(R)$, then equation* (1) *is equivalent to the triple of the following relations* [11]

$$x = \Big(\frac{\varphi'(\xi) + \psi'(\eta)}{\xi + \eta}\Big)^{\frac{1}{1-2\alpha}}, \tag{5}$$

$$y = \frac{1}{4(1-2\alpha)}\Big[(\xi+\eta)\big(\psi'(\eta) - \varphi'(\xi)\big) + 2\big(\varphi(\xi) - \psi(\eta)\big)\Big], \tag{6}$$

$$u = \frac{1}{1-2\alpha}\Big[\xi\Big(\frac{\varphi'(\xi) + \psi'(\eta)}{\xi + \eta}\Big)^{\frac{1-\alpha}{1-2\alpha}} - \varphi'(\xi)\Big(\frac{\varphi'(\xi) + \psi'(\eta)}{\xi + \eta}\Big)^{\frac{\alpha}{1-2\alpha}}\Big]. \tag{7}$$

To relations (5)–(7) we come from equation (1) without any additional conditions. By removing arbitrary parameters φ, ψ, from these relations we return to equation (1). Hence this triple of relations can be taken as a general integral of equation (1), and the invariants ξ, η as characteristic variables.

However, the above-constructed general integral (5)–(7) does not define in any way at least one characteristic of either of the families in order to take it as a data carrier of a mixed characteristic problem. Hence we have to choose such a characteristic arbitrarily, at our discretion. Suppose it is some arc γ given in explicit form

$$\gamma: \ y = g(x), \quad 0 < a \le x \le b, \quad g \in C^3(R). \tag{8}$$

The function g is assumed to be strictly monotonic, and the arc γ to be ascending. Without loss of generality, it can be assumed that

$$g(a) = 0.$$

Let the function h given on some segment $[a, d]$ be twice continuously differentiable and contracting this segment to the segment $[a, b]$. It is assumed that h satisfies the conditions

$$h(a) = a, \quad h(d) = b, \ \ h' < 0.$$

The Problem. Find a regular hyperbolic solution $u(x, y)$ of equation (1) and define simultaneously its domain of definition if along this solution the curve γ is characteristic, the solution itself satisfies the conditions

$$u(a, 0) = \mu, \quad u_x(a, 0) = \theta, \tag{9}$$

and each pair of points $(x, 0)$, $(h(x), g(h(x)))$ connected with the mapping of h belongs to the respective general characteristic of the family of the root λ_2.

According to the formulation of the problem, the curve γ is actually attributed to the family of characteristics of the root λ_1. This is equivalent to the equality

$$g'(x) = u_y^{-2}(x, g(x)). \tag{10}$$

Thus we can define two variants of values of the derivative u_y along the curve γ:

$$u_y = \frac{1}{\pm\sqrt{g'(x)}}\,. \tag{11}$$

Of them we choose the arithmetic value of the root. The reasoning for the other root is analogous. To solve the problem, along with (11) we also need to define on the arc γ the values of a solution u and its derivative u_x. To this end, we have to use the characteristic invariants ξ, ξ_1 of the family λ_1. The values of u and u_x at the initial point $(a, 0)$ of the curve γ are known. Using (9), (11), we calculate the characteristic invariants ξ, ξ_1 at the point $(a, 0)$ for which we introduce the notation

$$\begin{aligned} \xi\big|_{(a,0)} &= \big(\sqrt{g'(a)} + \theta\big)a^{\alpha} - \alpha\mu a^{\alpha-1} \equiv [\xi]_a, \\ \xi_1\big|_{(a,0)} &= \big(\sqrt{g'(a)} + \theta\big)a^{1-\alpha} - (1-\alpha)\mu a^{-\alpha} \equiv [\xi_1]_a. \end{aligned} \tag{12}$$

Since the characteristic invariants ξ, ξ_1 take constant values along γ, we have

$$\begin{aligned} \big[(u_y^{-1} + u_x)x^{\alpha} - \alpha u x^{\alpha-1}\big]\Big|_{\gamma} &= [\xi]_a, \\ \big[(u_y^{-1} + u_x)x^{1-\alpha} - (1-\alpha)u x^{-\alpha}\big]\Big|_{\gamma} &= [\xi_1]_a. \end{aligned}$$

Considering these two relations as a system relative to u and u_x, we define their values on γ as follows:

$$\begin{aligned} u\big|_{\gamma} &= \frac{1}{2-\alpha}\Big[x^{1-\alpha}[\xi]_a - [\xi_1]_a x^{\alpha}\Big], \\ u_x\big|_{\gamma} &= \frac{1-\alpha}{2-\alpha}\,[\xi]_a x^{-\alpha} - \frac{\alpha}{1-2\alpha}\,[\xi_1]_a x^{\alpha-1} - \sqrt{g'(x)}\,. \end{aligned}$$

Thus we have succeeded in defining the values of the sought solution and its first order derivatives all over the characteristic γ. Using these values, we can define the solution and its first order derivatives outside γ and establish the limits of their propagation.

To define the values of $u(x, 0)$, $u_x(x, 0)$ and $u_y(x, 0)$, from an arbitrary point $P(x, 0)$, $a < x \le d$, we draw the characteristic Γ of the family of the root λ_2, which by the conditions of problem (1), (9) intersects the characteristic γ at the point $N(h(x), g(h(x)))$. The invariants η and η_1 must be constant along the characteristic Γ.

Since the values of the invariants η, η_1 at the point N

$$\eta\big|_N = 2\sqrt{g'(h(x))}\, h^{\alpha}(x) - [\xi]_a,$$
$$\eta_1\big|_N = 2\sqrt{g'(h(x))}\, h^{1-\alpha}(x) - [\xi_1]_a,$$

remain unchanged all over the characteristic Γ, the point $(x,0)$ inclusive, the equalities

$$\eta(x,0) = \eta\big|_N, \quad \eta_1(x,0) = \eta_1\big|_N$$

will be fulfilled. These invariants can be written in the explicit form

$$[\eta]_x \equiv \big(u_y^{-1}(x,0) - u_x(x,0)\big)x^{\alpha} + \alpha u(x,0)x^{\alpha-1} = = 2\sqrt{g'(h(x))}\, h^{\alpha}(x) - [\xi]_a, \tag{13}$$

$$[\eta_1]_x \equiv \big(u_y^{-1}(x,0) - u_x(x,0)\big)x^{1-\alpha} + (1-\alpha)u(x,0)x^{-\alpha} = = 2\sqrt{g'(h(x))}\, h^{1-\alpha}(x) - [\xi_1]_a. \tag{14}$$

We take these equalities as a linear algebraic system and define the sought solution at an arbitrary point $(x,0)$ of the segment $[a,d]$

$$u(x,0) = \frac{2}{2\alpha-1}\sqrt{g'(h(x))}\,\big(h^{\alpha}(x)x^{1-\alpha} - h^{1-\alpha}(x)x^{\alpha}\big) - - \frac{1}{2\alpha-1}\,[\xi]_a x^{1-\alpha} + \frac{1}{2\alpha-1}\,[\xi_1]_a x^{\alpha}. \tag{15}$$

This is quite sufficient in order to define at the same points the first order derivatives $u_x(x,0)$ and $u_y(x,0)$ of the sought solution u. The derivative u_x is obtained by direct differentiation of (15)

$$\begin{aligned} u_x(x,0) = {} & \frac{1}{2\alpha-1}\,\frac{g''(h(x))\cdot h'(x)}{\sqrt{g'(h(x))}}\,\big(h^{\alpha}(x)x^{1-\alpha} - h^{1-\alpha}(x)x^{\alpha}\big) + \\ & + \frac{2}{2\alpha-1}\sqrt{g'(h(x))}\,\Big(\alpha h^{\alpha-1}(x)h'(x)x^{1-\alpha} + (1-\alpha)h^{\alpha}(x)x^{-\alpha} - \\ & - (1-\alpha)h^{-\alpha}(x)h'(x)x^{\alpha} - \alpha h^{1-\alpha}(x)x^{\alpha-1}\Big) - \\ & - \frac{1-\alpha}{2\alpha-1}\,[\xi]_a x^{-\alpha} + \frac{\alpha}{2\alpha-1}\,[\xi_1]_a x^{\alpha-1}. \end{aligned} \tag{16}$$

The other derivative u_y is defined by substituting (15), (16) into (13) or (14)

$$u_y(x,0)=\Bigg\{\frac{2\sqrt{g'(h(x))}}{2\alpha-1}\Big[(\alpha-1)h^{\alpha}(x)x^{-\alpha}+(1-\alpha)h^{1-\alpha}(x)x^{\alpha-1}+ \\ +\alpha h^{\alpha-1}(x)h'(x)x^{1-\alpha}-(1-\alpha)h^{-\alpha}(x)h'(x)x^{\alpha}\Big]+\frac{1}{2\alpha-1}\times \\ \times\frac{g''(h(x))\cdot h'(x)}{\sqrt{g'(h(x))}}\big(h^{\alpha}(x)x^{1-\alpha}-h^{1-\alpha}(x)x^{\alpha}\big)+\frac{1-\alpha}{2\alpha-1}[\xi]_a x^{\alpha}\Bigg\}^{-1}. \quad (17)$$

Because of the nonlinearity of equation (1) and depending on the $u_x(x,0)$, $u_y(x,0)$, the segment $[a,d]$ may turn out to be the characteristic of either of the families. This is the cause for which the problem under consideration may be ill-posed or even unsolvable.

In order to avoid transformation of the segment $[a,d]$ to the characteristic, we should find the conditions ensuring an a priori estimate

$$0<|u_y(x,0)|<\infty.$$

It is understood that these conditions should be expressed in terms g, h.

The above estimate excludes for $y=0$ not only the characteristic direction of the carrier, but also the parabolic degeneracy of equation (1).

The assumptions $h\in C^1(\overline{J})$, $g\in C^2(\overline{I})$, $J\equiv(a,d)$, $I\equiv(a,b)$ ensure the fulfillment of the condition

$$u_y(x,0)\neq 0,\quad x\in\overline{J}$$

and the existence of minimal and maximal values of the functions g', $|g''|$ on $\overline{I}$ and of $|h|$ on $\overline{J}$. We denote by n the smallest of minimal values and by N the largest of maximal values. We obtain the estimate

$$|u_y(x,0)|<+\infty,\quad x\in\overline{J}$$

if one of the following conditions

$$2\sqrt{N}\bigg[(\alpha-1)\Big(\frac{a}{d}\Big)^{\alpha}-(\alpha-1)\Big(\frac{d}{a}\Big)^{\alpha-1}-\alpha\Big(\frac{b}{a}\Big)^{\alpha-1}N-(\alpha-1)\Big(\frac{d}{a}\Big)^{\alpha}N\bigg]- \\ -\frac{N^2}{\sqrt{n}}\big(d^{\alpha}a^{1-\alpha}-d^{1-\alpha}a^{\alpha}\big)-(\alpha-1)[\xi]_a a^{\alpha}>0, \quad (18)$$

is fulfilled, where the value $[\xi]_a$ is given by formula (12), and $\alpha>1$;

$$\sqrt{N}\,\eta_*(1+\operatorname{sgn}\eta_*)+\sqrt{n}\,\eta_*(1-\operatorname{sgn}\eta_*)+\frac{N^2}{\sqrt{n}}\big(d^{\alpha}a^{1-\alpha}-d^{1-\alpha}a^{\alpha}\big)- \\ -(\alpha-1)\xi^{[a]}\Big(a^{\alpha}\,\frac{1+\operatorname{sgn}[\xi]_a}{2}+d^{\alpha}\,\frac{1-\operatorname{sgn}[\xi]_a}{2}\Big)<0, \quad (19)$$

$$\eta_* \equiv (\alpha-1)\Big(\frac{b}{a}\Big)^{\alpha} - (\alpha-1)\Big(\frac{a}{b}\Big)^{\alpha-1} - \alpha\Big(\frac{a}{d}\Big)^{\alpha-1} n - (\alpha-1)\Big(\frac{a}{b}\Big)^{\alpha} n, \quad \alpha>1,$$

$$g''(x) \le 0, \quad x \in \overline{I}, \quad \alpha = 1; \tag{20}$$

$$g''(x) > 0, \quad -2n^2 + N^2(d-a) < 0, \quad x \in \overline{I}, \quad \alpha = 1; \tag{21}$$

$$\sqrt{N}\,\eta^*(1-\operatorname{sgn}\eta^*) + \sqrt{n}\,\eta^*(1+\operatorname{sgn}\eta^*) - \frac{N^2}{\sqrt{n}}(d-a) +$$
$$+(1-\alpha)\xi^{[a]}\Big[a^{\alpha}\,\frac{1+\operatorname{sgn}[\xi]_a}{2} + d^{\alpha}\,\frac{1-\operatorname{sgn}[\xi]_a}{2}\Big] > 0, \tag{22}$$

$$\eta^* \equiv (\alpha-1)\Big(\frac{b}{a}\Big)^{\alpha} + (1-\alpha)\Big(\frac{a}{d}\Big)^{1-\alpha} - \alpha\Big(\frac{d}{a}\Big)^{1-\alpha} N +$$
$$+ (1-\alpha)\Big(\frac{a}{d}\Big)^{\alpha} n, \quad \frac{1}{2} < \alpha < 1;$$

$$\eta_0\sqrt{n}\,(1-\operatorname{sgn}\eta_0) + \eta_0\sqrt{N}\,(1+\operatorname{sgn}\eta_0) + \frac{n^2}{\sqrt{n}}(d-a) +$$
$$+ (1-\alpha)[\xi]_a\Big(d^{\alpha}\,\frac{1+\operatorname{sgn}[\xi]_a}{2} + a^{\alpha}\,\frac{1-\operatorname{sgn}[\xi]_a}{2}\Big) < 0, \tag{23}$$

$$\eta_0 \equiv (\alpha-1)\Big(\frac{a}{d}\Big)^{\alpha} + (1-\alpha)\Big(\frac{b}{a}\Big)^{1-\alpha} - \alpha\Big(\frac{a}{b}\Big)^{1-\alpha} n +$$
$$+(1-\alpha)\Big(\frac{d}{a}\Big)^{\alpha} N, \quad \frac{1}{2} < \alpha < 1.$$

Let $(\rho,0)$ and $(\sigma,0)$ be arbitrarily chosen points from the segment $[a,d]$.

Using the values of $u(\sigma,0)$, $u_x(\sigma,0)$, $u_y(\sigma,0)$, we define the constants $[\xi]_\sigma$, $[\xi_1]_\sigma$, whose values must coincide with the invariants ξ, ξ_1 on the unknown yet characteristic Γ_1 of the family of the root λ_1 drawn from the point $A(\sigma,0)$. Assume that this characteristic is given by the formula $y = m(x)$, where the function m is to be defined. Then on this curve, we have

$$\xi\big|_{\Gamma_1} = \big(u_y^{-1}(x,m(x)) + u_x(x,m(x))\big)x^{\alpha} - \alpha u(x,m(x))x^{\alpha-1} = [\xi]_\sigma, \tag{24}$$

$$\xi_1\big|_{\Gamma_1} = \big(u_y^{-1}(x,m(x)) + u_x(x,m(x))\big)x^{1-\alpha} -$$
$$-(1-\alpha)u(x,m(x))x^{-\alpha} = [\xi_1]_\sigma, \tag{25}$$

where

$$[\xi]_\sigma = \xi\big|_A = \frac{4\sqrt{g'(h(\sigma))}}{2\alpha-1}\Big[(\alpha-1)h^{\alpha}(\sigma) + (1-\alpha)h^{1-\alpha}(\sigma)\sigma^{2\alpha-1} -$$
$$- \alpha h^{\alpha-1}(\sigma)h'(\sigma)\sigma^{1-\alpha} - (1-\alpha)h^{-\alpha}(\sigma)h'(\sigma)\sigma^{2\alpha}\Big] +$$
$$+ \frac{2}{2\alpha-1}\cdot\frac{g''(h(\sigma))h'(\sigma)}{\sqrt{g'(h(\sigma))}}\big(h^{\alpha}(\sigma)\sigma - h^{1-\alpha}\sigma^{2\alpha}\big) +$$

$$+\frac{2-2\alpha}{2\alpha-1}\,[\xi]_a\sigma^{2\alpha}-2\sqrt{g'(h(\sigma))}\,h^{\alpha}(\sigma)+[\xi]_a,$$

$$[\xi_1]_\sigma=\xi_1\big|_A=\frac{4\sqrt{g'(h(\sigma))}}{2\alpha-1}\Big[(\alpha-1)h^{\alpha}(\sigma)\sigma^{-1}+(1-\alpha)h^{1-\alpha}(\sigma)\sigma^{2\alpha-2}+$$
$$+\alpha h^{\alpha-1}(\sigma)h'(\sigma)-(1-\alpha)h^{-\alpha}(\sigma)h'(\sigma)\sigma^{2\alpha-1}\Big]+$$
$$+\frac{2}{2\alpha-1}\cdot\frac{g''(h(\sigma))h'(\sigma)}{\sqrt{g'(h(\sigma))}}\big(h^{\alpha}(\sigma)-h^{1-\alpha}(\sigma)\sigma^{2\alpha-1}\big)+$$
$$+\frac{2-2\alpha}{2\alpha-1}\,[\xi]_a\sigma^{2\alpha-1}-2\sqrt{g(h(\sigma))}+[\xi_1]_a.$$

In an absolutely analogous manner we define the invariants η, η_1 on the characteristic Γ_2 of the other family drawn from the point $B(\rho,0)$. Assume that this characteristic is given by the equation $y=\ell(x)$, where ℓ is an unknown yet function. Thus we have

$$\eta\big|_{\Gamma_2}=\big(u_y^{-1}(x,\ell(x))-u_x(x,\ell(x))\big)x^{\alpha}+\alpha u(x,\ell(x))x^{\alpha-1}=[\eta]_\rho, \tag{26}$$

$$\eta_1\big|_{\Gamma_2}=\big(u_y^{-1}(x,\ell(x))-u_x(x,\ell(x))\big)x^{1-\alpha}+$$
$$+(1-\alpha)u(x,\ell(x))x^{-\alpha}=[\eta_1]_\rho, \tag{27}$$

where

$$[\eta]_\rho=2\sqrt{g'(h(\rho))}\,h^{\alpha}(\rho)-[\xi]_a,$$
$$[\eta_1]_\rho=2\sqrt{g'(h(\rho))}\,h^{1-\alpha}(\rho)-[\xi_1]_a.$$

At the intersection point (x_1,y_1) of these characteristics, if such a point exists, conditions (24)–(27) and $\ell(x_1)=m(x_1)$ must be fulfilled simultaneously. Therefore in the left-hand parts of (24), (25) we can replace $m(x_1)$ by $\ell(x_1)$. As a result, we obtain the following system for defining the values of x, u, u_x, u_y at the point $C(x_1,\ell(x_1))$

$$\big(u_y^{-1}(x_1,\ell(x_1))+u_x(x_1,\ell(x_1))\big)x_1^{\alpha}-\alpha u(x_1,\ell(x_1))x_1^{\alpha-1}=[\xi]_\sigma, \tag{28}$$
$$\big(u_y^{-1}(x_1,\ell(x_1))+u_x(x_1,\ell(x_1))\big)x_1^{1-\alpha}-(1-\alpha)u(x_1,\ell(x_1))x_1^{-\alpha}=[\xi_1]_\sigma, \tag{29}$$
$$\big(u_y^{-1}(x_1,\ell(x_1))-u_x(x_1,\ell(x_1))\big)x_1^{\alpha}+\alpha u(x_1,\ell(x_1))x_1^{\alpha-1}=[\eta]_\rho, \tag{30}$$
$$\big(u_y^{-1}(x_1,\ell(x_1))-u_x(x_1,\ell(x_1))\big)x_1^{1-\alpha}+(1-\alpha)u(x_1,\ell(x_1))x_1^{-\alpha}=[\eta_1]_\rho. \tag{31}$$

Taking these equalities as a linear algebraic system, we define the values of the abscissa x_1 of the intersection point C of the characteristics Γ_1 and Γ_2, and also of the sought solution $u(x_1,\ell(x_1))$ together with its first order derivatives $u_x(x_1,\ell(x_1))$ and $u_y(x_1,\ell(x_1))$.

So far ρ, σ have been chosen arbitrarily on the segment $[a,d]$ and it has been through them that we have defined the coordinates (x_1,y_1) of the intersection point of the characteristics. Now, if we assume that they run through this segment, we obtain the set of intersection points of the characteristics drawn from all possible pairs of points $(\rho, 0)$, $(\sigma, 0)$. That is why in the notations of solutions of the algebraic system (28)–(31) we omit the indexes

$$x = X(\rho,\sigma), \tag{32}$$

$$u = U(\rho,\sigma), \tag{33}$$

$$u_x = P(\rho,\sigma), \tag{34}$$

$$u_y = Q(\rho,\sigma), \tag{35}$$

where

$$X(\rho,\sigma) = \Big(\frac{[\xi]_\sigma + [\eta]_\rho}{[\xi_1]_\sigma + [\eta_1]_\rho}\Big)^{\frac{1}{2\alpha-1}},$$

$$U(\rho,\sigma) = \frac{1}{1-2\alpha}\left[\Big(\frac{[\xi]_\sigma + [\eta]_\rho}{[\xi_1]_\sigma + [\eta_1]_\rho}\Big)^{\frac{1-\alpha}{2\alpha-1}}[\xi]_\sigma - [\xi_1]_\sigma\Big(\frac{[\xi]_\sigma + [\eta]_\rho}{[\xi_1]_\sigma + [\eta_1]_\rho}\Big)^{\frac{\alpha}{2\alpha-1}}\right],$$

$$P(\rho,\sigma) = \Big(\frac{[\xi]_\sigma + [\eta]_\rho}{[\xi_1]_\sigma + [\eta_1]_\rho}\Big)^{-\frac{\alpha}{2\alpha-1}}\Big(\frac{1}{2-4\alpha}[\xi]_\sigma - - \frac{1}{2}[\eta]_\rho\Big) - \frac{\alpha}{1-2\alpha}[\xi_1]_\sigma\Big(\frac{[\xi]_\sigma + [\eta]_\rho}{[\xi_1]_\sigma + [\eta_1]_\rho}\Big)^{\frac{\alpha-1}{2\alpha-1}},$$

$$Q(\rho,\sigma) = 2\Big(\frac{[\xi]_\sigma + [\eta]_\rho}{[\xi_1]_\sigma + [\eta_1]_\rho}\Big)^{\frac{\alpha}{2\alpha-1}}\big([\xi]_\sigma + [\eta]_\rho\big)^{-1}.$$

To describe the structure of this set of points, we must express the ordinate y as a function of arguments ρ, σ, in the same way as all other were represented by formulas (32)–(35). To construct the function $y = Y(\rho,\sigma)$, we need the explicit representations of X and Q in the form

$$X(\rho,\sigma) = \left\{\frac{4\sqrt{g'(h(\sigma))}}{2\alpha-1}\Big[(\alpha-1)h^\alpha(\sigma) + (1-\alpha)h^{1-\alpha}(\sigma)\sigma^{2\alpha-1} - -\alpha h^{\alpha-1}(\sigma)h'(\sigma)\sigma^{1-\alpha} - (1-\alpha)h^{-\alpha}(\sigma)h'(\sigma)\sigma^{2\alpha}\Big] + + \frac{2}{2\alpha-1}\cdot\frac{g''(h(\sigma))h'(\sigma)}{\sqrt{g'(h(\sigma))}}\big(h^\alpha(\sigma)\sigma - h^{1-\alpha}(\sigma)\sigma^{2\alpha}\big) + \frac{2-2\alpha}{2\alpha-1}[\xi]_a\sigma^{2\alpha} - -2\sqrt{g'(h(\sigma))}\,h^\alpha(\sigma) + 2\sqrt{g'(h(\rho))}\,h^\alpha(\rho)\right\}^{\frac{1}{2\alpha-1}} \times$$

$$\times\left\{\frac{4\sqrt{g'(h(\sigma))}}{2\alpha-1}\Big[(\alpha-1)h^\alpha(\sigma) + (1-\alpha)h^{1-\alpha}(\sigma)\sigma^{2\alpha-1} - -\alpha h^{\alpha-1}(\sigma)h'(\sigma)\sigma^{1-\alpha} - (1-\alpha)h^{-\alpha}(\sigma)h'(\sigma)\sigma^{2\alpha}\Big] + + \frac{2}{2\alpha-1}\cdot\frac{g''(h(\sigma))h'(\sigma)}{\sqrt{g'(h(\sigma))}}\big(h^\alpha(\sigma)\sigma - h^{1-\alpha}(\sigma)\sigma^{2\alpha}\big) + \frac{2-2\alpha}{2\alpha-1}[\xi]_a\sigma^{2\alpha} -\right.$$

$$-2\sqrt{g'(h(\sigma))}\,h^{1-\alpha}(\sigma)+2\sqrt{g'(h(\rho))}\,h^{1-\alpha}(\rho)\Big\}^{\frac{1}{1-2\alpha}}, \tag{36}$$

$$Q(\rho,\sigma)=2\bigg\{\frac{4\sqrt{g'(h(\sigma))}}{2\alpha-1}\Big[(\alpha-1)h^{\alpha}(\sigma)+(1-\alpha)h^{1-\alpha}(\sigma)\sigma^{2\alpha-1}-$$
$$-\alpha h^{\alpha-1}(\sigma)h'(\sigma)\sigma^{1-\alpha}-(1-\alpha)h^{-\alpha}(\sigma)h'(\sigma)\sigma^{2\alpha}\Big]+$$
$$+\frac{2}{2\alpha-1}\cdot\frac{g''(h(\sigma))h'(\sigma)}{\sqrt{g'(h(\sigma))}}\left(h^{\alpha}(\sigma)\sigma-h^{1-\alpha}(\sigma)\sigma^{2\alpha}\right)+\frac{2-2\alpha}{2\alpha-1}\,[\xi]_a\sigma^{2\alpha}-$$
$$-2\sqrt{g'(h(\sigma))}\,h^{\alpha}(\sigma)+2\sqrt{g'(h(\rho))}\,h^{\alpha}(\rho)\bigg\}^{\frac{\alpha}{2\alpha-1}}\times$$
$$\times\bigg\{\frac{4\sqrt{g'(h(\sigma))}}{2\alpha-1}\Big[(\alpha-1)h^{\alpha}(\sigma)+(1-\alpha)h^{1-\alpha}(\sigma)\sigma^{2\alpha-1}-$$
$$-\alpha h^{\alpha-1}(\sigma)h'(\sigma)\sigma^{1-\alpha}-(1-\alpha)h^{-\alpha}(\sigma)h'(\sigma)\sigma^{2\alpha}\Big]+$$
$$+\frac{2}{2\alpha-1}\cdot\frac{g''(h(\sigma))h'(\sigma)}{\sqrt{g'(h(\sigma))}}\left(h^{\alpha}(\sigma)\sigma-h^{1-\alpha}(\sigma)\sigma^{2\alpha}\right)+\frac{2-2\alpha}{2\alpha-1}\,[\xi]_a\sigma^{2\alpha}-$$
$$-2\sqrt{g'(h(\sigma))}\,h^{1-\alpha}(\sigma)+2\sqrt{g'(h(\rho))}\,h^{1-\alpha}(\rho)\bigg\}^{\frac{\alpha}{1-2\alpha}}\times$$
$$\times\bigg\{\frac{4\sqrt{g'(h(\sigma))}}{2\alpha-1}\Big[(\alpha-1)h^{\alpha}(\sigma)\sigma^{-1}+(1-\alpha)h^{1-\alpha}(\sigma)\sigma^{2\alpha-2}+$$
$$+\alpha h^{\alpha-1}(\sigma)h'(\sigma)-(1-\alpha)h^{-\alpha}(\sigma)h'(\sigma)\sigma^{2\alpha-1}\Big]+$$
$$+\frac{2}{2\alpha-1}\cdot\frac{g''(h(\sigma))h'(\sigma)}{\sqrt{g'(h(\sigma))}}\left(h^{\alpha}(\sigma)-h^{1-\alpha}(\sigma)\sigma^{2\alpha-1}\right)+\frac{2-2\alpha}{2\alpha-1}\,[\xi]_a\sigma^{2\alpha-1}-$$
$$-2\sqrt{g'(h(\sigma))}\,h^{1-\alpha}(\sigma)+2\sqrt{g'(h(\rho))}\,h^{\alpha}(\rho)\bigg\}^{-1}. \tag{37}$$

To define the function m, the equation of the characteristic Γ_1 is formally written in the form

$$y=m(x)=m[X(\rho,\overline{\sigma})]\equiv M(\rho,\overline{\sigma}),$$

where the little line over the letter means this value is constant. The direction of this characteristic is defined by the root λ_1 or, in other words, by the values of the derivative $u_y=Q(\overline{\sigma},\rho)$. Therefore we have

$$\frac{dm(X(\rho,\overline{\sigma}))}{dX(\rho,\overline{\sigma})}=\frac{dm(X(\rho,\overline{\sigma}))}{X'_{\rho}(\rho,\overline{\sigma})\,d\rho}=\frac{1}{Q^2(\rho,\overline{\sigma})}$$

or, which is the same,

$$\frac{dM(\rho,\overline{\sigma})}{d\rho}=\frac{X'_\rho(\rho,\overline{\sigma})}{Q^2(\rho,\overline{\sigma})}\,. \tag{38}$$

Hence by integration we obtain

$$M(\rho,\overline{\sigma})=\int\limits_a^\rho\frac{X'_t(t,\overline{\sigma})}{Q^2(t,\overline{\sigma})}\,dt+M(a,\overline{\sigma}),\quad \rho\in[a,d],$$

where the value $M(a,\overline{\sigma})$ is unknown yet.

By an analogous reasoning, using the notation $y=\ell(x)=\ell\big[X(\overline{\rho},\sigma)\big]\equiv L(\overline{\rho},\sigma)$ and taking into account the direction of the characteristic Γ_2 defined by the root λ_2, we obtain

$$\frac{dL(\overline{\rho},\sigma)}{d\sigma}=-\frac{X'_\sigma(\overline{\rho},\sigma)}{Q^2(\overline{\rho},\sigma)}$$

and

$$L(\overline{\rho},\sigma)=-\int\limits_a^\sigma\frac{X_z(\overline{\rho},z)}{Q^2(\overline{\rho},z)}\,dz+L(\overline{\rho},a),\quad \sigma\in[a,d],$$

where $L(\overline{\rho},a)$ is not known either and has to be defined.

To define the unknown values, note that $L(\overline{\rho},a)$ is the value of L at the intersection point of the characteristics γ and Γ_2. Therefore

$$L(\overline{\rho},a)=g(h(\overline{\rho})),$$

and

$$M(a,\overline{\sigma})=L(a,\overline{\sigma})=-\int\limits_a^{\overline{\sigma}}\frac{X'_z(a,z)}{Q^2(a,z)}\,dz+g(h(a)),$$

where $g(h(a))=g(a)=0$.

In defining the characteristics of the families of the roots λ_1 and λ_2, the functions $M(\rho,\overline{\sigma})$ and $L(\overline{\rho},\sigma)$ are given by the equalities

$$M(\rho,\overline{\sigma})=\int\limits_a^\rho\frac{X'_t(t,\overline{\sigma})}{Q^2(t,\overline{\sigma})}\,dt-\int\limits_a^{\overline{\sigma}}\frac{X'_z(a,z)}{Q^2(a,z)}\,dz \tag{39}$$

with an argument $\rho\in[a,d]$ and a parameter $\overline{\sigma}\in[a,d]$, and

$$L(\overline{\rho},\sigma)=-\int\limits_a^\sigma\frac{X'_z(\overline{\rho},z)}{Q^2(\overline{\rho},z)}\,dz+g(h(\overline{\rho})), \tag{40}$$

with a variable $\sigma\in[a,d]$ and a parameter $\overline{\rho}\in[a,d]$.

Thus the integral of problem (1), (9) is given by formulas (32), (33) and

$$y=Y(\rho,\sigma), \tag{41}$$

where

$$Y(\rho,\sigma) = -\int\limits_a^\sigma \frac{X_z'(\rho,z)}{Q^2(\rho,z)}\,dz + g(h(\rho)),$$

and the variables $\rho, \sigma \in [a, d]$.

The domain of definition D of the solution of problem (1), (9) is completely defined by relations (32), (41), where expressions of x, y depend on ρ, σ. The values of these functions are treated as the current coordinates describing the domain D.

The domain of definition of the solution of the problem under consideration is bounded by four characteristics. The first of them which is an arch of the curve γ is given by the condition of the problem. The other characteristics are represented parametrically. In our representations we take as parameters the values ρ, σ of the abscissa of the intersection points through which these characteristics pass:

$$\Gamma_3: \quad x = X(d,\sigma), \quad y = L(d,\sigma), \tag{42}$$

$$\Gamma_4: \quad x = X(\rho,d), \quad y = M(\rho,d), \tag{43}$$

$$\Gamma_5: \quad x = X(a,\sigma), \quad y = L(a,\sigma), \tag{44}$$

where the functions X, M, L are given by (36), (39), (40).

Such is the structure of the domain of definition of the solution of the problem when the values of the derivative u_y on the arc γ in formula (11) are defined by the positive root. The domain has the same kind of structure when the root in (11) is negative. The latter case is investigated by analogy with the preceding case.

Thus the following theorem is valid.

Theorem 3. *If along the curve γ it occurs that $u_y > 0$, then under the conditions* (18)–(23) *there exists the solution of problem* (1), (9) *given by the formulas* (32), (33), (41). *The domain D of the solution is bounded by the arcs of characteristic curves* (8), (42), (43) *and* (44).

The case $u_y > 0$ along the curve γ can be studied in a similar way.

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