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ON SOLVING THE DIRICHLET GENERALIZED PROBLEM FOR A HARMONIC FUNCTION IN THE CASE OF AN INFINITE PLANE WITH A CRACK-TYPE CUT

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Abstract. An algorithm for the approximate solution of the Dirichlet generalized problem is proposed. The term "generalized" indicates that the boundary function has a finite number of first kind break points. The solution consists of the following stages: 1) the reduction of the Dirichlet generalized problem to an ordinary auxiliary problem for a harmonic function; 2) an approximate solution of the auxiliary problem by the modified version of the MFS (the method of fundamental solutions); 3) the construction of an approximate solution of the generalized problem from the solution of the auxiliary problem. An example is considered in which the break points are the cusp ones.

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1. Introduction

Let D be the infinite plane $z = x + iy \equiv (x, y)$ with a finite crack-type cut G, which is bounded by a closed piecewise smooth contour $S = \bigcup\limits_{j=1}^{m} S_j$, without multiple points. On each piece of the cut its width d is positive, and the length of the cut $l \gg d$. Moreover, we assume that the contour S has cusp points and the parametric equations of the smooth curves S_J are given. It is known that the classical statement of the Dirichlet ordinary boundary problem for the Laplace equation requires continuity of the boundary function. However, in practical problems (for example, for the determination of temperature of the thermal field or the potential of the electric field, and so on) there are the cases where the boundary condition is piecewise continuous and therefore it is necessary to consider the Dirichlet generalized problem (see [1,2,3,4]).

Problem A. *A function $g(\tau)$ is given on the boundary S of the domain D which is continuous everywhere, except at a finite number of points $\tau_1, \tau_2, \dots, \tau_n$ at which it has the first kind break points. It is required to find a function $u(z) \equiv u(x, y) \in C^2(D) \bigcap C(\overline{D} \setminus \{\tau_1, \tau_2, \dots, \tau_n\})$ satisfying the conditions*

$$\Delta u(z) = 0, \qquad z \in D \tag{1.1}$$

$$u(\tau) = g(\tau), \qquad \tau \in S, \quad \tau \neq \tau_k \ \ (k = 1, 2, \dots, n), \tag{1.2}$$

$$u(z) = c + O\left(\frac{1}{|z|}\right) \qquad for \quad z \to \infty, \tag{1.3}$$

where Δ is the Laplace operator and c is a real constant such that $|c| < \infty$.

It is known (see [1,2,5]) that Problem (1.1)–(1.3) has a unique solution depending continuously on the data, and for a generalized solution $u(z)$ the generalized extremum principle is valid:

$$\min_{z \in S} u(z) < \underset{z \in D}{u(z)} < \max_{z \in S} u(z), \tag{1.4}$$

where for $z \in S$ it is assumed that $z \neq \tau_k$ $(k = \overline{1, n})$.

It should be noted that condition (1.3) is important in the extremum principle (1.4) and, consequently, in the uniqueness theorem of the solution of Problem A (see [1,2]).

It can be easily shown that if we fix in advance the value of the constant c, this will be a rather strong restriction. Since under conditions (1.1), (1.2), (1.3) for $u(x, y)$ the minimax principle is satisfied, Problem A with c fixed a priori may have no solution. To avoid this the constant c should be defined from condition (1.2).

If $g^-(\tau_k)$ and $g^+(\tau_k)$ are the limiting values of the boundary function $g(\tau)$, as τ tends to the point τ_k along S, respectively, in the positive and negative directions, then the following theorem explains the behavior of the generalized solution in the neighborhood of the point τ_k (see [1,5]).

Theorem 1. *The limiting values of the solution $u(z)$ of the Dirichlet generalized problem, when the point $z \in D$ approaches the point τ_k, lie between $g^-(\tau_k)$ and $g^+(\tau_k)$.*

Remark 1. In general, the method of conformal mapping (MCM) (see e.g., [1,6,7]) may be applied to obtain an approximate solution of Problem A, but in the case under consideration it is ineffective. Indeed, a strong tension (or compression) of parts of the boundary (see [8,9]) take place under the conformal mapping of the domains of type D onto the unit disk and inversely.

2. On the Application of the MFS for Generalized Problem

In general, it is known (see [3,6]) that the methods used to obtain an approximate solution of ordinary boundary value problems are less suitable (or not suitable at all) for solving problems with singularities. In particular, the convergence is very slow and, consequently, the accuracy is very low in the neighborhood of the boundary singularities. Similar phenomena take place in solving the generalized Dirichlet boundary problem with the MFS. For this reason, many researchers have tried to perform preliminary improvements on the boundary value problem. More precisely, they have to reduce, if possible, the posed problem by smoothing a boundary function to solving the ordinary problem (see, e.g.,[3,5,6]). For example, the plane harmonic and biharmonic problems for simply connected domains with certain singularities have been considered in [10,11,12]. To solve such problems, the authors have used modified versions of the MFS, which were based on the direct subtraction of the leading terms of the singular local solution (which have to be determined) from the original mathematical problem. The problems of the type A are studied in [13] for finite simply– and multiply-connected domains.

Theoretically, the MFS may be used for an approximate solution of generalized problems of the type A (see [14,15]), but since the fundamental solutions have a high degree of smoothness on the contour S, such functions are less suitable for the approximation of discontinuous functions. In order to get a desired result from the point of view of its accuracy, it is necessary to have a very large number of auxiliary points (see Sect. 4). In its turn, the above-mentioned situation produces technical difficulties in numerical implementation.

3. A Method of Reduction of the Dirichlet Generalized Problem to an Ordinary Problem

To reduce Problem A to an ordinary problem, it suffices to construct a function $u_0(z)$ which would be a solution of equation (1.1), bounded in $\overline{D}$, continuous everywhere in $\overline{D}$, except at the points $\tau = \tau_k$, and would have the same jumps at the points τ_k, as $g(\tau)$. Indeed, if such a function is constructed, we define a new (unknown) function

$$v(z) = u(z) - u_0(z). \tag{3.1}$$

To determine $v(z)$, we have to solve an ordinary Dirichlet problem.

Problem B.

$$\Delta v(z) = 0, \quad z \in D, \tag{3.2}$$

$$v(\tau) = f(\tau), \quad \tau \in S, \tag{3.3}$$

where $f(\tau) = g(\tau) - u_0(\tau)$ is a continuous function on the contour S (since the function $f(\tau)$ has removable break points at τ_k, i.e.,$f(\tau_k) = f^-(\tau_k) = f^+(\tau_k)$).

Since the domain D is infinite, for the uniqueness of the solution of Problems B and A (see [1,2]) we require, additionally, that

$$\lim v(z) = c_1, \quad \text{for } z \to \infty, \tag{3.4}$$

$$\lim u_0(z) = c_2, \quad \text{for } z \to \infty. \tag{3.5}$$

It is evident that in this case, since $c = c_1 + c_2$, therefore c_2 should be given in advance, and c_1 should be found while solving Problem (3.2),(3.3). Conditions (1.4), (3.4) and (3.5) are essential, respectively, for the uniqueness of the solution of Problems A and B in the case of an infinite domain.

After the function $v(z) = v(x, y)$ is constructed, from (3.1) we have

$$u(z) = v(z) + u_0(z), \quad z \in \overline{D}, \quad z \neq \tau_k. \tag{3.6}$$

In [5], it is shown that for smoothing the function $g(\tau)$ on the contour S we can take the function

$$u_0(z) = \sum_{k=1}^{n} u_k(z), \tag{3.7}$$

$$u_k(z) = \frac{h_k}{\delta_k} w_k(z),$$

$$w_k(z) = \arg\left(\overline{\frac{z - \tau_k}{(z - z_0)(z_0 - \tau_k)}}\right),$$

where h_k and δ_k are the jumps of the functions $g(\tau)$ and $w_k(\tau)$ at the point τ_k along S, respectively; in particular,

$$h_k = g^+(\tau_k) - g^-(\tau_k), \quad \delta_k = \varphi_k^+ - \varphi_k^-,$$
$$\varphi_k^+ = \lim_{\tau\to\tau_k+} w_k(\tau), \quad \varphi_k^- = \lim_{\tau\to\tau_k-} w_k(\tau), \quad \tau \in S.$$

In the expression for (3.7), z_0 is the inner point of the cut (to avoid difficulties in calculations, it is better to take z_0 as the "center" of G).

From (3.7), for the value of the constant c_2 (see (3.5)) we have

$$c_2 = \lim_{z\to\infty} u_0(z) = \sum_{k=1}^{n} \frac{h_k}{\delta_k} \arg(z_0 - \tau_k). \tag{3.8}$$

4. On the Application of the Modified Version of MFS for the Solution of the Dirichlet Ordinary External Problem

It is known (see [14,15,16]) that the method of fundamental solutions can be used in the general case to solve approximately both internal and external boundary value problems. The functions (see [14,15])

$$\{\varphi_k(z)\}_{k=1}^{\infty} = \{\ln|z - \widetilde{z}_k|\}_{k=1}^{\infty}, \quad z \in S, \tag{4.1}$$

are the fundamental solutions of the Laplace operator. In (4.1), $\{\widetilde{z}_k\}_{k=1}^{\infty}$ is a countable set of points lying everywhere densely on the auxiliary closed Liapunov contour $\widetilde{S}$, lying inside of the finite domain G and $\min \rho(S, \widetilde{S}) > 0$, where ρ is the distance between S and $\widetilde{S}$. It is known that system (4.1) is linearly independent and complete not only in the space $L_2(S)$, but also in $C(S)$. Theoretically, by means of system (4.1), the boundary function $g(z)$ can be approximated to within any accuracy. When using the MFS, an approximate solution is sought in the form

$$u_N(z) = \sum_{k=1}^{N} a_k^{(N)} \ln|z - \widetilde{z}_k|, \quad z \in \overline{D},$$

where the points $\widetilde{z}_k (k = 1, 2, \ldots, N)$ are situated "uniformly" on the auxiliary contour $\widetilde{S}$, and $a_k^{(N)}$ are the coefficients of the expansion of the function $g(z)$. It is obvious that $\lim u_N(z) = \infty$ as $z \to \infty$, which means that condition (3.4) for the solution to be unique is not satisfied.

Remark 2. Further, (see [16]), while solving approximately Problem B, the contour $\widetilde{S}$ is the Jordan contour which represents the boundary of the domain $\widetilde{G}(\widetilde{G} \subset G)$. The domain $\widetilde{G}$ is "similar" to G, oriented in the same way and they have the same "center" of gravity. As for the values $\rho(S, \widetilde{S})$ and N, they can be chosen during the numerical implementation of the algorithm, taking into account *a posteriori* estimates of the accuracy of the results.

Remark 3. If we seek a solution to Problem B in the form

$$u_N(z) = \sum_{k=1}^{N} a_k^{(N)} \ln|z - \widetilde{z}_k| + c_N,$$

under the condition $\sum\limits_{k=1}^{N} a_k^{(N)} = 0$, where c_N is a real constant and $|c_N| < \infty$, then it can be easily proved that $u_N(\infty) = c_N$. However, when finding the constants $a_k^{(N)}$ $(k = 1, 2, \dots, N)$ and c_N there arise considerable difficulties connected with the solvability of the system and conditioning its matrix.

To avoid the above situations, in [17] the modified version of the system of fundamental solutions (4.1) is constructed by conformal mapping of the form

$$\{\psi_k(z)\}_{k=1}^{\infty} \equiv \{\psi(z, \widetilde{z}_k)\}_{k=1}^{\infty} = \left\{ \ln\left|\frac{\widetilde{z}_k - z}{(z - z_0)(\widetilde{z}_k - z_0)}\right| \right\}_{k=1}^{\infty}, \tag{4.2}$$

For system (4.2) the following conditions are satisfied (see [17]):

1_0. $\Delta\psi_k(z) = 0$, $\forall z \in D$;

2_0. The system $\{\psi_k(z)\}_{k=1}^{\infty}$ is linearly independent and complete not only in the space $L_2(S)$, but also in $C(S)$.

3_0. $\lim \psi_k(z)$ is finite as $z \to \infty$.

Since in our case $f(z) \in C(S)$, on the basis of property 2_0 for arbitrary $\varepsilon > 0$ there exist natural numbers $N^0(\varepsilon)$ and a set of coefficients $a_k^{(N)}$ $(k = 1, 2, \dots, N)$, such that if $N \geq N^0$, then

$$\max_{z \in S} \left| f(z) - \sum_{k=1}^{N} a_k^{(N)} \psi_k(z) \right| < \varepsilon.$$

If we introduce the notation

$$v_N(z) \equiv v_N(x, y) = \sum_{k=1}^{N} a_k^{(N)} \psi_k(z),$$

then from the minimax principle we obtain $\max\limits_{z \in \overline{D}} |v(z) - v_N(z)| < \varepsilon$, where $v(z)$ is the exact solution of Problem B, i.e. $v_N(z)$ converges uniformly to $v(z)$ in $\overline{D}$ for $N \to \infty$.

Thus, the approximate solution $v_N(z)$ of Problem B by the modified version of MFS has the form

$$v_N(z) \equiv v_N(x, y) = \sum_{k=1}^{N} a_k^{(N)} \ln\left|\frac{\widetilde{z}_k - z}{(z - z_0)(\widetilde{z}_k - z_0)}\right|, \tag{4.3}$$

where the auxiliary points (simulation sources) $\widetilde{z}_k (k = 1, 2, \dots, N)$ are situated "uniformly" on the contours $\widetilde{S}$.

As for the coefficients $a_k^{(N)}$, they can be found (see [14,15,16]) from the system of linear algebraic equations of the form

$$\sum_{k=1}^{N} a_k^{(N)} \psi(z_j, \widetilde{z}_k) = f(z_j) \ , \tag{4.4}$$

where the collocation points z_j $(j = 1, 2, \ldots, N)$ are situated "uniformly" on the contour S. The matrix of system (4.4) has the same properties as the matrix obtained when solving internal problems by virtue of system (4.1) (see[17]).

From (4.3), for an approximate value of the constant c_1 we have

$$c_1^N = \lim_{z\to\infty} v_N(z) = v_N(\infty) = -\sum_{k=1}^{N} a_k^{(N)} \ln \left| \widetilde{z}_k - z_0 \right| \ ,$$

or $|c_1^N| < \infty$.

5. Numerical Example

In the example considered below, the coefficients $a_k^{(N)}$ of the expansion (4.3) are found from system (4.4).

In Table 1, N is a number of auxiliary and collocation points on the contours $\widetilde{S}$ and S, respectively; ε is an *a posteriori* error estimate of the solution of Problem B or Problem A:

$$\varepsilon = \max\{|f(z_j) - v_N(z_j)|\},$$

where $f(z_j) = g(z_j) - u_0(z_j)(z_j \neq \tau_k)$ and τ_k is the point of discontinuity. The points $z_j(j = 1, 2, \ldots, M)$ are situated "uniformly" on the contour S. If $z_j = \tau_k$, then $f(z_j) = f^+(\tau_k) \equiv f^-(\tau_k)$.

Example. The domain D is the exterior of the crack-type cut G with the boundary $S = S_1 \bigcup S_2$, where the equations of the curves S_1 and S_2 have the following form:

$$\begin{aligned} S_1 : y &= \frac{b}{1+x^2} - \frac{b}{1+a^2}, \qquad x \in [-a, a]; \\ S_2 : y &= -\frac{b}{1+x^2} + \frac{b}{1+a^2}, \qquad x \in [-a, a]. \end{aligned} \tag{5.1}$$

For illustration, the form of the contour S is given in Figure 1 for $a = 3$ and $b = 1$.

In the solution of Problem B, $\tilde{S} = \tilde{S_1} \bigcup \tilde{S_2}$ was taken as the auxiliary contour, where

$$\begin{aligned} \tilde{S}_1 : \tilde{y} &= \frac{b_1}{1+\tilde{x}^2} - \frac{b_1}{1+a_1^2}, \qquad \tilde{x} \in [-a_1, a_1]; \\ \tilde{S}_2 : \tilde{y} &= -\frac{b_1}{1+\tilde{x}^2} + \frac{b_1}{1+a_1^2}, \qquad \tilde{x} \in [-a_1, a_1]. \end{aligned} \tag{5.2}$$

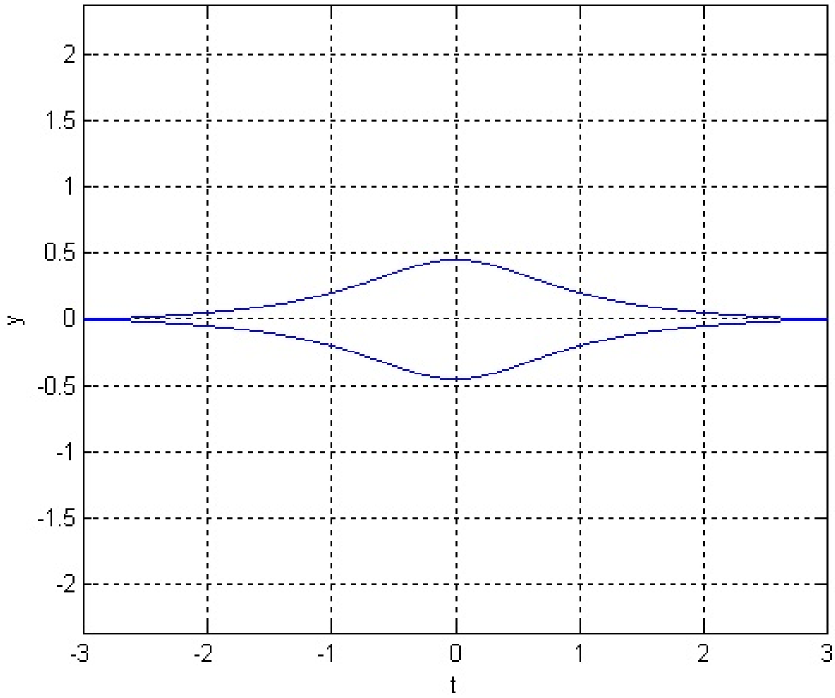

Figure 1.

In the above equations a, b, a_1 and b_1 are the real constants.

The function $g(\tau)$ was taken to be a function with two break points $\tau_1 = (a; 0)$, $\tau_2 = (-a; 0)$. In particular, we have taken

$$g(\tau) = \begin{cases} 1, & \tau \in \tau_1\tau_2, \\ -1, & \tau \in \tau_2\tau_1, \end{cases} \tag{5.3}$$

where $\tau_1\tau_2$ and $\tau_2\tau_1$ are the open curves S_1 and S_2, respectively.

It is evident that the jumps of the function $g(\tau)$ at the break points τ_1 and τ_2 are equal to $h_1 = 2$ and $h_2 = -2$. In the implementation: $z_0 = (0; 0)$ (see (3.7), (4.3)); For $f(\tau_1)$ and $f(\tau_2)$ we took $f(\tau_1^*)$ and $f(\tau_2^*)$, where $\tau_1^* = (a - \varepsilon_1; y) \in S_1$, $\tau_2^* = (-a + \varepsilon_1; y) \in S_1$ and $\varepsilon_1 = 0.0001$; The auxiliary points,the collocation points and the points for the calculation of ε in solving Problem B by the modified version of the MFS were situated uniformly on $\tilde{S}$ and S with respect to the abscissa x, and ordinates of these points were found from (5.1) and (5.2).

In Table 1: $N = N_1 + N_2$ $(N_1 = N_2)$; $M = M_1 + M_2$ $(M_1 = M_2)$; $u_N(z_k)$ is the value of an approximate solution to Problem A at the point $z_k \in D$, calculated by virtue of (3.6). The results of numerical experiments for the various values of a, b, a_1, b_1 and N are given. Computations were realized by the MATLAB system.

Table 1

$M = 5000$; $\varepsilon_2 = 10^{-5}$				
$z_1 = a + \varepsilon_2$; $z_2 = -a - \varepsilon_2$; $z_3 = \infty$				
$a = 20$; $b = 1$; $a_1 = a - 0.00001$; $b_1 = b - 0.1$				
N	ε	$u_N(z_1)$	$u_N(z_2)$	$u_N(z_3)$
400	0.03	5.835(−5)	−5.844(−5)	−8.436(−6)
800	0.001	5.834(−5)	−5.842(−5)	−8.400(−6)
1000	0.0003	5.835(−5)	−5.839(−5)	−1.381(−6)
$a = 20$; $b = 0.1$; $a_1 = a - 0.00001$; $b_1 = b - 0.01$				
N	ε	$u_N(z_1)$	$u_N(z_2)$	$u_N(z_3)$
500	0.07	7.066(−6)	−4.600(−6)	2.136(−4)
1000	0.02	5.999(−6)	−5.666(−6)	3.915(−5)
2000	0.002	5.845(−6)	−5.820(−6)	3.697(−6)
2200	0.0009	5.840(−6)	−5.827(−6)	2.444(−6)

6. Concluding Remarks

From Table 1 it is clear that for an approximate solution $u_N(z)$ of Problem A at the above-considered points of the domain D, the conditions of the generalized extremum principle and Theorem 1 are satisfied.

The results of numerical experiments indicate the effectiveness of the proposed algorithm for an approximate solution of Problem A in the case of an infinite plane with a crack-type cut. In particular, the algorithm is rather simple for numerical implementation and characterized by an accuracy which is sufficient practically for many problems.

Finally, it should be noted that we can apply the proposed algorithm to the solution of generalized Dirichlet three-dimensional problems, which can be reduced to problems of type A.

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