Proceedings of A. Razmadze
Mathematical Institute
Vol. **165** (2014), 13–30

TIME SCALE HARDY-TYPE INEQUALITIES WITH GENERAL KERNEL FOR SUPERQUADRATIC FUNCTIONS

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Abstract. In this paper we give the extension of Hardy-type inequalities with general kernel for superquadratic functions to arbitrary time scales.

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1. Introduction

1.1. **On time scale calculus.** The theory of time scales was introduced by Stefan Hilger in his Ph.D. thesis [15] in 1988, as a unification of the theory of difference equations with that of differential equations, unifying integral and differential calculus with the calculus of finite differences offering a formalism for studying hybrid discrete-continuous dynamical systems. It has applications in any field that requires simultaneous modelling of discrete and continuous time. Now, we briefly introduce the time scales calculus and refer to [5, 16, 17] and the books [11, 18] for further details.

By a time scale (or measure chain) $\mathbb{T}$ we mean any closed subset of $\mathbb{R}$ with the topology of subspace of $\mathbb{R}$. The two most popular examples of time scales are the real numbers $\mathbb{R}$ and the discrete time scale $\mathbb{Z}$. Since time scale $\mathbb{T}$ may or may not be connected, we need the concept of jump operators.

For $t \in \mathbb{T}$, we define the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$$

and the *backward jump operator* by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}.$$

In this definition, the convention is $\inf\ \emptyset = \sup\ \mathbb{T}$ (i.e., $\sigma(t) = t$ if $\mathbb{T}$ has a maximum t) and $\sup\ \emptyset = \inf\ \mathbb{T}$ (i.e., $\rho(t) = t$ if $\mathbb{T}$ has a minimum t).

2010 *Mathematics Subject Classification.* Primary 26D15, 26D20, 26D99,34N99.
Key words and phrases. Hardy-Type inequalities, superquadratic functions, time scales.

If $\sigma(t) > t$, we say that t is *right-scattered* and if $\rho(t) < t$ we say that t is *left-scattered*. The points that are right-scattered and left-scattered at the same time are called *isolated*. Also, if $\sigma(t) = t$ then t is said to be *right-dense*, and if $\rho(t) = t$ then t is said to be *left-dense*. The points that are simultaneously right-dense and left-dense are called *dense*. The mapping $\mu : \mathbb{T} \to [0, \infty)$ defined by

$$\mu(t) = \sigma(t) - t$$

is called the *graininess function*.

If $\mathbb{T}$ has a left-scattered maximum M, then define $\mathbb{T}^\kappa = \mathbb{T} \setminus \{M\}$; otherwise $\mathbb{T}^\kappa = \mathbb{T}$. If $f : \mathbb{T} \to \mathbb{R}$ is a function, then we define the function $f^\sigma : \mathbb{T} \to \mathbb{R}$ by

$$f^\sigma(t) = f(\sigma(t)) \quad \text{for all} \quad t \in \mathbb{T}.$$

In the following considerations $\mathbb{T}$ will denote a time scale, $I_\mathbb{T} = I \cap \mathbb{T}$ will denote a time scale interval, for any open or closed interval I in $\mathbb{R}$ and $[0, \infty)_\mathbb{T}$ will be used for the time scale interval $[0, \infty) \cap \mathbb{T}$.

Definition A. Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$. Then we define $f^\triangle(t)$ to be the number (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood U of t such that

$$\left| f(\sigma(t)) - f(s) - f^\triangle(t)(\sigma(t) - s) \right| \le \epsilon \left| \sigma(t) - s \right| \quad \text{for all} \quad s \in U_\mathbb{T}.$$

We call $f^\triangle(t)$ the delta derivative of f at t.

We say that f is *delta differentiable* on $\mathbb{T}^\kappa$ provided $f^\triangle(t)$ exists for all $t \in \mathbb{T}^\kappa$.

For all $t \in \mathbb{T}^\kappa$, we have the following properties:

(i) If f is delta differentiable at t, then f is continuous at t.

(ii) If f is continuous at t and t is right-scattered, then f is delta differentiable at t with $f^\triangle(t) = \dfrac{f(\sigma(t)) - f(t)}{\mu(t)}$.

(iii) If t is right-dense, then f is delta differentiable at t, iff the limit $\lim\limits_{s \to t} \dfrac{f(t) - f(s)}{t - s}$ exists as a finite number. In this case, $f^\triangle(t) = \lim\limits_{s \to t} \dfrac{f(t) - f(s)}{t - s}$.

(iv) If f is delta differentiable at t, then $f(\sigma(t)) = f(t) + \mu(t) f^\triangle(t)$.

Definition B. A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous if it is continuous at all right-dense points in $\mathbb{T}$ and its left-sided limits are finite at all left-dense points in $\mathbb{T}$. We denote by C_{rd} the set of all rd-continuous functions.

We say that f is *rd-continuously delta differentiable* (and write $f \in C^1_{rd}$) if $f^\triangle(t)$ exists for all $t \in \mathbb{T}^\kappa$ and $f^\triangle \in C_{rd}$.

Definition C. A function $F : \mathbb{T} \to \mathbb{R}$ is called a delta antiderivative of $f : \mathbb{T} \to \mathbb{R}$ if $F^{\triangle}(t) = f(t)$ for all $t \in \mathbb{T}^{\kappa}$. Then, we define the delta integral by $\int_a^t f(s)\triangle s = F(t) - F(a)$.

The importance of rd-continuous function is revealed by the following result.

Theorem A. *Every rd-continuous function has a delta antiderivative.*

Theorem B. *If $f \in C_{rd}$ and $t \in \mathbb{T}^{\kappa}$, then*

$$\int_t^{\sigma(t)} f(s)\triangle s = \mu(t)f(t).$$

Theorem C. *If $f^{\triangle} \geq 0$, then f is nondecreasing.*

Theorem D. *If $a, b, c \in \mathbb{T}$, $\beta \in \mathbb{R}$ and $f, g \in C_{rd}$, then*

(i) $\int_a^b (f(t) + g(t))\triangle t = \int_a^b f(t)\triangle t + \int_a^b g(t)\triangle t;$

(ii) $\int_a^b \alpha f(t)\triangle t = \alpha \int_a^b f(t)\triangle t;$

(iii) $\int_a^b f(t)\triangle t = -\int_b^a f(t)\triangle t;$

(iv) $\int_a^b f(t)\triangle t = \int_a^c f(t)\triangle t + \int_c^b f(t)\triangle t;$

(v) $\int_a^a f(t)\triangle t = 0;$

(vi) *if $f(t) \geq 0$ for all t, then* $\int_a^b f(t)\triangle t \geq 0.$

The Lebesgue integration theory on time scales is given in [9] and [10]. For a Δ-measurable set $E \subset \Omega^n = \{a = \{a_1, \ldots, a_n\}; a_i \in \mathbb{T}_i, i \in \{1, \ldots, n\}\}$ and a Δ-measurable function $f : E \to \mathbb{R}$, the corresponding Δ-integral of f over E will be denoted by $\int_E f(t_1, t_2, \ldots, t_n)\Delta t_1 \Delta t_2 \cdots \Delta t_n$, $\int_E f(t)\Delta t$ or $\int_E f(t)\mu_\Delta(t)$.

Theorem E. (Fubini's theorem on time scales) If $f : \Omega \times \Lambda \to \mathbb{R}$ is a $\mu_\Delta \times \lambda_\Delta-$integrable function and if we define the function $\varphi(y) = \int_\Omega f(x, y)\Delta x$ for a.e. $y \in \Lambda$ and $\psi(x) = \int_\Lambda f(x, y)\Delta y$ for a.e. $x \in \Omega$, then φ is λ_Δ-integrable on Λ, ψ is μ_Δ-integrable on Ω and

$$\int\limits_{\Omega} \Delta x \int\limits_{\Lambda} f(x,y)\Delta y = \int\limits_{\Lambda} \Delta y \int\limits_{\Omega} f(x,y)\Delta x$$

holds.

1.2. **On superquadratic functions.** The concept of superquadratic functions in one variable, as a generalization of the class of convex functions, was recently introduced by S. Abramovich, G. Jameson and G. Sinnamon in [2] and [3]. Here we quote some definitions and theorems that we use in this paper. More examples and properties of superquadratic functions can be found in [1], [7], [6] and its references.

Definition D. A function $\varphi : [0,\infty) \to \mathbb{R}$ is superquadratic provided that for all $x \geq 0$ there exists a constant $C(x) \in \mathbb{R}$ such that

$$\varphi(y) - \varphi(x) - \varphi(|y-x|) \geq C(x)(y-x)$$

for all $y \geq 0$. We say that f is subquadratic if $-f$ is a superquadratic function.

For example, function $\varphi(x) = x^p$ is superquadratic for $p \geq 2$ and subquadratic for $p \in (0,2]$.

The following lemma shows that positive superquadratic functions are also convex functions.

Lemma A. *Let φ be a superquadratic function with $C(x)$ as in Definition D. Then*

(i) $\varphi(0) \leq 0$.

(ii) *If $\varphi(0) = \varphi'(0) = 0$, then $C(x) = \varphi'(x)$ wherever φ is differentiable at $x > 0$.*

(iii) *If $\varphi \geq 0$, then φ is convex and $\varphi(0) = \varphi'(0) = 0$.*

Before presenting next results,which were recently proved by Oguntuase et al.[22], it is necessary to introduce some further notation: we use bold letters to denote n-tuples of real numbers, e.g. $\mathbf{x} = (x_1, \ldots, x_n)$. Also, we set $\mathbf{0} = (0, \ldots, 0) \in \mathbb{R}^n$ and $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^n$. Further more, the relations $<, \leq, >$ and $\geq$ are, as usual, defined componentwise, for example, for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we write $\mathbf{x} < \mathbf{y}$ if $x_i < y_i$, $i \in \{1, \ldots, n\}$. Finally, we denote $(\mathbf{0}, \mathbf{b}) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{0} < \mathbf{x} < \mathbf{b}\}$ and $(\mathbf{b}, \infty) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{b} < \mathbf{x} < \infty\}$.

Proposition A. *Let $\boldsymbol{b} \in (\mathbf{0}, \boldsymbol{\infty})$, $u : (\mathbf{0}, \boldsymbol{b}) \to \mathbb{R}$ is a weight function which is locally integrable in $(\mathbf{0}, \boldsymbol{b})$ and v is defined by*

$$v(\mathbf{t}) = t_1 \ldots t_n \int\limits_{t_1}^{b_1} \cdots \int\limits_{t_n}^{b_n} \frac{u(\boldsymbol{x})}{x_1^2 \cdots x_n^2} d\boldsymbol{x}, \quad \boldsymbol{t} \in (\mathbf{0}, \mathbf{b}).$$

Suppose $I = (a, c)$, $0 \leq a < c \leq \infty$, $\varphi : I \to \mathbb{R}$, *and* $f : (\mathbf{0}, \boldsymbol{b}) \to \mathbf{R}$ *is an integrable function, such that* $f(\boldsymbol{x}) \in I$, *for all* $\boldsymbol{x} \in (\mathbf{0}, \boldsymbol{b})$.

(i) *If* φ *is superquadratic, then the following inequality holds*

$$\int_0^{b_1} \cdots \int_0^{b_n} u(\boldsymbol{x}) \varphi \left(\frac{1}{x_1 \cdots x_n} \int_0^{x_1} \cdots \int_0^{x_n} f(\boldsymbol{t}) d\boldsymbol{t} \right) \frac{d\boldsymbol{x}}{x_1 \cdots x_n} +$$
$$+ \int_0^{b_1} \cdots \int_0^{b_n} \int_{t_1}^{b_1} \cdots \int_{t_n}^{b_n} \varphi \left(\left| f(\boldsymbol{t}) - \frac{1}{x_1 \dots x_n} \int_0^{x_1} \cdots \int_0^{x_n} f(\boldsymbol{t}) d\boldsymbol{t} \right| \right) \cdot$$
$$\cdot \frac{u(\boldsymbol{x})}{x_1^2 \cdots x_n^2} d\boldsymbol{x} d\boldsymbol{t} \leq \int_0^{b_1} \cdots \int_0^{b_n} v(\boldsymbol{x}) \varphi(f(\boldsymbol{x})) \frac{d\boldsymbol{x}}{x_1 \cdots x_n}. \tag{1.1}$$

(ii) *If* φ *is subquadratic, then* (1.1) *holds in the reversed direction.*

Proposition B. *Let* $\boldsymbol{b} \in [\mathbf{0}, \boldsymbol{\infty})$, $u : (\boldsymbol{b}, \boldsymbol{\infty}) \to \mathbb{R}$ *is a weight function which is locally integrable in* $(\mathbf{0}, \boldsymbol{b})$ *and define* v *by*

$$v(\mathbf{t}) = \frac{1}{t_1 \cdots t_n} \int_{b_1}^{t_1} \cdots \int_{b_n}^{t_n} u(\boldsymbol{x}) d\boldsymbol{x} \ < \infty, \ \boldsymbol{t} \in (\mathbf{b}, \boldsymbol{\infty}).$$

Suppose $I = (a, c)$, $0 \leq a < c \leq \infty$, $\varphi : I \to \mathbb{R}$, *and* $f : (\boldsymbol{b}, \boldsymbol{\infty}) \to \mathbb{R}$ *is an integrable function, such that* $f(\boldsymbol{x}) \in I$, *for all* $\boldsymbol{x} \in (\boldsymbol{b}, \boldsymbol{\infty})$.

(i) *If* φ *is superquadratic, then the following inequality holds:*

$$\int_{b_1}^{\infty} \cdots \int_{b_n}^{\infty} u(\boldsymbol{x}) \varphi \left(x_1 \cdots x_n \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} f(\boldsymbol{t}) \frac{d\boldsymbol{t}}{t_1^2 \cdots t_n^2} \right) \frac{d\boldsymbol{x}}{x_1 \cdots x_n}$$
$$+ \int_{b_1}^{\infty} \cdots \int_{b_n}^{\infty} \int_{b_1}^{t_1} \cdots \int_{b_n}^{t_n} \varphi \left(\left| f(\boldsymbol{t}) - x_1 \cdots x_n \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} f(\boldsymbol{t}) \frac{d\boldsymbol{t}}{t_1^2 \cdots t_n^2} \right| \right) \cdot$$
$$\cdot u(\boldsymbol{x}) d\boldsymbol{x} \frac{d\boldsymbol{t}}{t_1^2 \cdots t_n^2} \leq \int_{b_1}^{\infty} \cdots \int_{b_n}^{\infty} v(\boldsymbol{x}) \varphi(f(\boldsymbol{x})) \frac{d\boldsymbol{x}}{x_1 \cdots x_n}. \tag{1.2}$$

(ii) *If* φ *is subquadratic, then the inequality sign in (1.2) is reversed.*

1.3. **On Hardy inequality.** In 1920. G.H. Hardy ([12]) announced and proved in [13] (see also [14] and [20]) the following result: Let $p > 1$ and $f \in L^p(0, \infty)$ be a non-negative function, then

$$\int_0^{\infty} \left(\frac{1}{x} \int_0^x f(t) dt \right)^p dx \leq \left(\frac{p}{p-1} \right)^p \int_0^{\infty} f^p(x) dx \tag{1.3}$$

holds. This interesting result is today referred to as the classical Hardy's integral inequality. Inequality (1.3) has an interesting prehistory and history (see e.g. [14], [19], [20] and [21] and the references given there).

Other important inequalities are the following: if $p > 1$ and f is a non-negative function such that $f \in L^p(0,\infty)$, then

$$\int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y}\,dx \right)^p dy \le \left(\frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \right)^p \int_0^\infty f^p(y)\,dy, \tag{1.4}$$

and if in addition $g \in L^q(0,\infty)$ where $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\,dx\,dy \le \frac{\pi}{\sin\frac{\pi}{p}} \left(\int_0^\infty f^p(x)\,dx \right)^{\frac{1}{p}} \left(\int_0^\infty g^q(y)\,dy \right)^{\frac{1}{q}}. \tag{1.5}$$

Moreover, (1.5) is sometimes called Hilbert's or Hardy-Hilbert's inequality even if Hilbert himself only considered the case $p = 2$ (L^p-spaces were defined much later).

We also note that (1.3) shows that the Hardy operator H, defined by setting

$$(Hf)(x) := \frac{1}{x} \int_0^x f(t)\,dt,$$

maps L^p into itself with operator norm $\frac{p}{p-1}$. Similarly, (1.4) shows that the operator A, defined by setting

$$(Af)(y) := \int_0^\infty f(x)(x+y)^{-1}\,dx,$$

maps L^p into itself with operator norm $\frac{\pi}{\sin\frac{\pi}{p}}$.

It is now natural to generalize the operators above to the following ones:

$$H_k f(x) := \frac{1}{K(x)} \int_0^x f(t)k(x,t)\,dt,$$

where

$$K(x) := \int_0^x k(x,t)\,dt < \infty$$

and (more generally)

$$A_k f(x) := \frac{1}{K(x)} \int_0^\infty f(t)k(x,t)\,dt, \tag{1.6}$$

where now

$$K(x) := \int_0^\infty k(x,t)\,dt < \infty.$$

Here $k(x,y)$ is a general measurable and non-negative function, a so called kernel.

Now, let $(\Omega_1, \Sigma_1, \mu_1)$, $(\Omega_2, \Sigma_2, \mu_2)$ be σ-finite measure spaces and let A_k from (1.6) be generalized as follows:

$$A_k f(x) := \frac{1}{K(x)} \int_{\Omega_2} k(x,y)f(y)\,d\mu_2(y), \tag{1.7}$$

where $f : \Omega_2 \to \mathbb{R}$ is a measurable function, $k : \Omega_1 \times \Omega_2 \to \mathbb{R}$ is measurable and non-negative kernel and

$$K(x) := \int_{\Omega_2} k(x,y)\,d\mu_2(y) < \infty,\ x \in \Omega_1. \tag{1.8}$$

Recently, S. Abramovich, K. Krulić, J. Pečarić and L-E. Persson proved in [4] that for superquadratic function φ and an integral operator $A_k f$ defined by (1.7), the following theorem holds.

Theorem E. *Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with positive σ-finite measures, u be a weight function on Ω_1, k a non-negative measurable function on $\Omega_1 \times \Omega_2$, and K be defined on Ω_1 by* (1.8).

Suppose that $K(x) > 0$ for all $x \in \Omega_1$ and that the function $x \mapsto u(x)\dfrac{k(x,y)}{K(x)}$ is integrable on Ω_1 for each fixed $y \in \Omega_2$. Define v by

$$v(y) := \int_{\Omega_1} \frac{k(x,y)}{K(x)} u(x)d\mu_1(x) < \infty.$$

Suppose $I = (0, c)$*,* $c \leq \infty$*,* $\varphi : I \to \mathbb{R}$*. If* φ *is a superquadratic function, then the inequality*

$$\int_{\Omega_1} u(x)\varphi(A_k f(x))\, d\mu_1(x) + $$
$$+ \int_{\Omega_2}\int_{\Omega_1} \varphi\big(\,|f(y) - A_k f(x)|\,\big)\frac{u(x)k(x,y)}{K(x)}\, d\mu_1(x)\, d\mu_2(y) \leq$$
$$\leq \int_{\Omega_2} v(y)\varphi(f(y))\, d\mu_2(y) \tag{1.9}$$

holds for all measurable functions $f : \Omega_2 \to \mathbb{R}$*, such that* $Imf \subseteq I$*, where* A_k *is defined by* (1.7)*. If* φ *is subquadratic, then the inequality sign in* (1.9) *is reversed.*

2. Inequalities with General Kernel

In the rest of the article we will need the following hypothesis.

(H1) Let $(\Omega, \mathcal{M}, \mu_\Delta)$ and $(\Lambda, \mathcal{L}, \lambda_\Delta)$ be two time scale measure spaces.

(H2) Define

$$A_k f(x) := \frac{1}{K(x)} \int_\Lambda k(x,y) f(y)\, \Delta y, \quad x \in \Omega,$$

where $f : \Lambda \to \mathbb{R}$ is a λ_Δ - measurable function, $k : \Omega \times \Lambda \to \mathbb{R}$ is a non-negative kernel and $K(x) := \int_\Lambda k(x,y)\Delta y < \infty, \quad x \in \Omega$.

(H3) Let $\xi : \Omega \to \mathbb{R}_+$ be μ_Δ-integrable function and denote

$$w(y) = \int_\Omega \frac{k(x,y)\xi(x)}{K(x)} \Delta x, \quad y \in \Lambda.$$

In the next theorem we give an analogue of the Theorem E for arbitrary time scale measure spaces.

Theorem 2.1. *Suppose that hypothesis* (H1)–(H3) *are valid. Let* $I = [a, c)$*,* $0 \leq a < c \leq \infty$ *and* $\varphi : I \to \mathbb{R}$*. If* $\varphi \in C(I, \mathbb{R})$ *is superquadratic, then*

$$\int_\Omega \varphi(A_k f(x))\xi(x)\, \Delta x + \int_\Lambda \int_\Omega \varphi\big(\,|f(y) - A_k f(x)|\,\big)\frac{\xi(x)k(x,y)}{K(x)}\, \Delta x\, \Delta y \leq$$
$$\leq \int_\Lambda \varphi(f(y))w(y)\Delta y \tag{2.1}$$

holds for all $\lambda_\triangle$*-integrable functions* $f : \Lambda \to \mathbb{R}$ *such that* $f(\Lambda) \subset I$*. If* φ *is subquadratic, then the inequality* (2.1) *is reversed.*

Proof. Using the Jensen's inequality for superquadratic functions on time scales given in [8] and the Fubini theorem on time scales, we have

$$\begin{aligned}
&\int_{\Omega} \varphi(A_k f(x))\xi(x)\,\Delta x = \\
&= \int_{\Omega} \xi(x)\varphi\bigg(\frac{1}{K(x)}\int_{\Lambda} k(x,y)f(y)\,\Delta y\bigg)\,\Delta x \le \\
&\le \int_{\Omega} \xi(x)\frac{1}{K(x)}\bigg(\int_{\Lambda} k(x,y)\varphi(f(y))\Delta y\bigg)\Delta x - \\
&\quad - \int_{\Omega} \frac{\xi(x)}{K(x)}\int_{\Lambda} k(x,y)\varphi\big(\,|f(y)-A_k f(x)|\,\big)\Delta y\,\Delta x = \\
&= \int_{\Lambda} \varphi(f(y))\bigg(\int_{\Omega} \frac{k(x,y)}{K(x)}\xi(x)\Delta x\bigg)\Delta y - \\
&\quad - \int_{\Lambda}\int_{\Omega} \varphi\big(\,|f(y)-A_k f(x)|\,\big)\frac{\xi(x)k(x,y)}{K(x)}\,\Delta x\,\Delta y,
\end{aligned}$$

which is equivalent to (2.1). If the function φ is subquadratic then the Jensen's inequality for superquadratic functions on time scales will be reversed which implies, according to the conclusions made above, that reversed sign in (2.1) will hold. □

Remark 2.2. In [10] authors proved that, with the same assumptions as given in hypothesis (H1)–(H3), for convex function $\varphi \in C(I,\mathbb{R})$, where $I \subset \mathbb{R}$ is an interval, inequality

$$\int_{\Omega} \xi(x)\varphi\bigg(\frac{1}{K(x)}\int_{\Lambda} k(x,y)f(y)\ \Delta y\bigg)\Delta x \le \int_{\Lambda} w(y)\varphi(f(y))\ \Delta y, \tag{2.2}$$

holds for all λ_Δ - measurable functions $f : \Lambda \to \mathbb{R}$ such that $f(\Lambda) \subset I$. Now, it is easy to see that in the case when function φ in Theorem 2.1 is non-negative (then, according to Lemma A, φ is convex) Theorem 2.1 gives the refinement of inequality (2.2), since

$$\int_{\Lambda}\int_{\Omega} \varphi\big(|f(y)-A_k f(x)|\big)\frac{\xi(x)k(x,y)}{K(x)}\,\Delta x\,\Delta y \ge 0.$$

Corollary 2.3. *Assume* (H1)–(H3) *are true. If* $p \geq 2$*, then*

$$\int_{\Omega} A_k^p f(x)\xi(x)\,\Delta x + \int_{\Lambda}\int_{\Omega} \left|f(y) - A_k f(x)\right|^p \frac{\xi(x)k(x,y)}{K(x)}\,\Delta x\,\Delta y \leq \leq \int_{\Lambda} f^p(y)w(y)\Delta y \tag{2.3}$$

holds for all λ_Δ*-integrable* $f : \Lambda \to \mathbb{R}_+$*. If* $0 < p \leq 2$*, then* (2.3) *holds in the reversed direction.*

Proof. Use $\varphi(x) = x^p$ in Theorem 2.1. □

Remark 2.4. In particular, if $p = 2$ in Corollary 2.3, we get the following identity

$$\int_{\Omega} A_k^2 f(x)\xi(x)\,\Delta x + \int_{\Lambda}\int_{\Omega} \left|f(y) - A_k f(x)\right|^2 \frac{\xi(x)k(x,y)}{K(x)}\,\Delta x\,\Delta y = = \int_{\Lambda} f^2(y)w(y)\Delta y.$$

Corollary 2.5. *Assume* (H1)–(H3)*. Then*

$$\int_{\Omega} \exp(A_k g(x))\xi(x)\,\Delta x + I \leq \int_{\Lambda} g(y)w(y)\Delta y, \tag{2.4}$$

holds for all λ_Δ*-integrable* $g : \Lambda \to (0,\infty)$ *with*

$$A_k g(x) := \frac{1}{K(x)} \int_{\Lambda} k(x,y)\ln g(y)\,\Delta y$$

and

$$I = \int_{\Lambda}\int_{\Omega} \Big(\exp\big(|\ln g(y) - A_k g(x)|\big) - |\ln g(y) - A_k g(x)|\Big) \frac{\xi(x)k(x,y)}{K(x)}\,\Delta x\,\Delta y + + \int_{\Lambda} \ln g(y)w(y)\Delta y - \int_{\Omega} (1 + A_k g(x))\Delta x.$$

Proof. Use $\varphi(x) = e^x - x - 1$ and $f(x) = \ln g(x)$ in Theorem 2.1. □

Corollary 2.6. *Let assumptions* (H1)–(H3) *hold and denote* $\int_\Omega \Delta y = |\Omega|$, $\int_\Lambda \Delta y = |\Lambda|$, *such that* $|\Omega|$, $|\Lambda| < \infty$. *If* $\varphi \in C(I, \mathbb{R})$ *is superquadratic, then*

$$\int_\Omega \varphi\left(\frac{1}{|\Lambda|}\int_\Lambda f(y)\Delta y\right)\Delta x + \frac{1}{|\Lambda|}\int_\Lambda\int_\Omega \varphi\left(\left|f(y) - \frac{1}{|\Lambda|}\int_\Lambda f(y)\Delta y\right|\right)\Delta x\Delta y \leq$$
$$\leq \frac{|\Omega|}{|\Lambda|}\int_\Lambda \varphi(f(y))\Delta y$$

holds for all λ_Δ*-integrable* $f : \Lambda \to \mathbb{R}$ *such that* $f(\Lambda) \subset I$. *If the function* φ *is subquadratic, then the above inequality is reversed.*

Proof. Follows directly from Theorem 2.1 taking $k(x, y) = 1$ and $\xi(x) = 1$, since in this case $K(x) = \int_\Lambda \Delta y = |\Lambda|$ and $w(y) = \int_\Omega \frac{1}{|\Lambda|}\Delta x = \frac{|\Omega|}{|\Lambda|}$. □

3. Inequalities with Special Kernels

Throughout this section and in the next section, we assume the following hypothesis holds.

(H4) Let $\Omega = \Lambda = [a_1, b_1)_\mathbb{T} \times [a_2, b_2)_\mathbb{T} \times \cdots \times [a_n, b_n)_\mathbb{T}$, $0 \leq a_i < b_i \leq \infty$ for all $i \in \{1, \ldots, n\}$, where $\mathbb{T}$ is an arbitrary time scale.

Theorem 3.1. *Suppose* $\xi : \Omega \to \mathbb{R}_+$ *is a* μ_Δ*-integrable function and denote*

$$\widetilde{w}(y_1, \ldots, y_n) = \int_{y_1}^{b_1}\cdots\int_{y_n}^{b_n} \frac{\xi(x_1, \ldots, x_n)}{\prod_{i=1}^n (\sigma(x_i) - a_i)}\Delta x_1 \cdots \Delta x_n.$$

If $\varphi \in C(I, \mathbb{R})$, $I \subset \mathbb{R}$, *is superquadratic function, then*

$$\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} \xi(x_1, \ldots, x_n)\varphi\left(\widetilde{A_k}f(x_1, \ldots, x_n)\right)\Delta x_1 \cdots \Delta x_n +$$
$$+\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\int_{y_1}^{b_1}\cdots\int_{y_n}^{b_n} \frac{\xi(x_1, \ldots, x_n)}{\prod_{i=1}^n(\sigma(x_i) - a_i)}\cdot$$
$$\cdot\varphi\left(\left|f(y_1, \ldots, y_n) - \widetilde{A_k}f(x_1, \ldots, x_n)\right|\right)\Delta x_1 \cdots \Delta x_n \Delta y_1 \cdots \Delta y_n \leq$$
$$\leq \int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} \widetilde{w}(y_1, \ldots, y_n)\varphi(f(y_1, \ldots, y_n))\Delta y_1 \cdots \Delta y_n, \tag{3.1}$$

holds for all λ_Δ*-integrable* $f:\Lambda\to\mathbb{R}$ *such that* $f(\Lambda)\subset I$*, where*

$$(\widetilde{A_k}f)(x_1,\ldots,x_n)=\frac{1}{\prod_{i=1}^{n}(\sigma(x_i)-a_i)}\int_{a_1}^{\sigma(x_1)}\cdots\int_{a_n}^{\sigma(x_n)}f(y_1,\ldots,y_n)\Delta y_1\cdots\Delta y_n.$$

If φ *is subquadratic then* (3.1) *is reversed.*

Proof. Statement follows from Theorem 2.1 using

$$k(x_1,\ldots,x_n,y_1,\ldots,y_n)=\left\{\begin{array}{ll}1 & \text{if } a_i\le y_i<\sigma(x_i)\le b_i,\ i\in\{1,\ldots,n\}\\ 0 & \text{otherwise,}\end{array}\right\},$$

since in this case

$$K(x_1,\ldots,x_n)=\int_{a_1}^{\sigma(x_1)}\cdots\int_{a_n}^{\sigma(x_n)}\Delta y_1\cdots\Delta y_n=\prod_{i=1}^{n}(\sigma(x_i)-a_i),$$

and thus $A_k=\widetilde{A}_k$, $w=\widetilde{w}$. □

Corollary 3.2. *Let in (H4)* $a_i=0$ *for all* $i\in\{1,\ldots,n\}$*. If* $\varphi\in C(I,\mathbb{R})$ *is superquadratic for* $I\subset\mathbb{R}$ *then*

$$\begin{aligned}&\int_0^{b_1}\cdots\int_0^{b_n}\varphi\left(A_kf(x_1,\ldots,x_n)\right)\frac{\Delta x_1\cdots\Delta x_n}{\prod_{i=1}^{n}x_i}+\\&+\int_0^{b_1}\cdots\int_0^{b_n}\int_{y_1}^{b_1}\cdots\int_{y_n}^{b_n}\varphi(|f(y_1,\ldots,y_n)-\\&-A_kf(x_1,\ldots,x_n)|)\frac{\Delta x_1\cdots\Delta x_n}{\prod_{i=1}^{n}x_i\sigma(x_i)}\Delta y_1\cdots\Delta y_n\le\\&\le\int_0^{b_1}\cdots\int_0^{b_n}w(y_1,\ldots,y_n)\varphi(f(y_1,\ldots,y_n))\Delta y_1\cdots\Delta y_n,\end{aligned}\tag{3.2}$$

holds for all λ_Δ*-integrable* $f:\Lambda\to\mathbb{R}$ *such that* $f(\Lambda)\subset I$*, where*

$$(A_kf)(x_1,\ldots,x_n):=\frac{1}{\prod_{i=1}^{n}\sigma(x_i)}\int_0^{\sigma(x_1)}\cdots\int_0^{\sigma(x_n)}f(y_1,\ldots,y_n)\Delta y_1\cdots\Delta y_n.$$

Proof. The statement follows from Theorem 3.1 using $\xi(x_1,\dots,x_n) = \frac{1}{x_1\cdots x_n}$, since in this case

$$w(y_1,\dots,y_n) = \int_{y_1}^{b_1}\cdots\int_{y_n}^{b_n} \frac{1}{\prod_{i=1}^{n} x_i\sigma(x_i)}\Delta x_1\cdots\Delta x_n = \prod_{i=1}^{n}\left(\frac{1}{y_i}-\frac{1}{b_i}\right). \qquad \square$$

Remark 3.3. If $b_i=\infty$ for all $i\in\{1,\dots,n\}$, then inequality (3.2) takes the form

$$\int_0^\infty\cdots\int_0^\infty \varphi\left(A_k f(x_1,\dots,x_n)\right)\frac{\Delta x_1\cdots\Delta x_n}{\prod_{i=1}^{n} x_i} + \int_0^\infty\cdots\int_0^\infty\int_{y_1}^\infty\cdots$$

$$\cdots\int_{y_n}^\infty \varphi\left(|f(y_1,\dots,y_n)-A_k f(x_1,\dots,x_n)|\right)\frac{\Delta x_1\cdots\Delta x_n}{\prod_{i=1}^{n} x_i\sigma(x_i)}\Delta y_1\cdots\Delta y_n \le$$

$$\le \int_0^\infty\cdots\int_0^\infty \varphi(f(y_1,\dots,y_n))\frac{\Delta y_1\cdots\Delta y_n}{\prod_{i=1}^{n} y_i}.$$

If φ is subquadratic then above inequality is reversed.

Theorem 3.4. *Let $a\ge 0, b=\infty$ in $(H4)$ and $v:\Lambda\to\mathbb{R}_+$ be defined by*

$$v(y_1,\dots,y_n) = \frac{1}{\prod_{i=1}^{n}\sigma(y_i)}\int_{a_1}^{\sigma(y_1)}\cdots\int_{a_n}^{\sigma(y_n)}\Delta x_1\cdots\Delta x_n = \prod_{i=1}^{n}\left(1-\frac{a_i}{\sigma(y_i)}\right).$$

If $\varphi\in C(I,\mathbb{R})$ is superquadratic, then

$$\int_{a_1}^\infty\cdots\int_{a_n}^\infty \varphi\left(\prod_{i=1}^{n} x_i\int_{x_1}^\infty\cdots\int_{x_n}^\infty \frac{f(y_1,\dots,y_n)}{\prod_{i=1}^{n} y_i\sigma(y_i)}\Delta y_1\cdots\Delta y_n\right)\frac{\Delta x_1\cdots\Delta x_n}{\prod_{i=1}^{n} x_i}+$$

$$+\int_{a_1}^\infty\cdots\int_{a_n}^\infty\int_{a_1}^{\sigma(y_1)}\cdots\int_{a_n}^{\sigma(y_n)} \varphi\left(|W|\right)\Delta x_1\cdots\Delta x_n\frac{\Delta y_1\cdots\Delta y_n}{\prod_{i=1}^{n} y_i\sigma(y_i)} \le$$

$$\le \int_{a_1}^\infty\cdots\int_{a_n}^\infty v(y_1,...,y_n)\varphi(f(y_1,\dots,y_n))\frac{\Delta y_1\cdots\Delta y_n}{\prod_{i=1}^{n} y_i}$$

holds for all λ_Δ-integrable $f : \Lambda \to \mathbb{R}$ such that $f(\Lambda) \subset I$ and W is a label which replaces the following expression

$$f(y_1,\dots,y_n) - \prod_{i=1}^{n} x_i \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} \frac{f(y_1,\dots,y_n)}{\prod\limits_{i=1}^{n} y_i \sigma(y_i)} \Delta y_1 \cdots \Delta y_n.$$

If φ is subquadratic then above inequality is reversed.

Proof. Statement follows from Theorem 3.1, using

$$k(x_1,\dots,x_n,y_1,\dots,y_n) = \left\{ \begin{array}{l} \prod\limits_{i=1}^{n} \dfrac{1}{y_i \sigma(y_i)},\ y_i \geq x_i \text{ for all } i \in \{1,\dots,n\}, \\ 0 \quad \text{otherwise}, \end{array} \right\},$$

since in this case $K(x_1,\dots,x_n) = \frac{1}{\prod_{i=1}^{n} x_i}$, and we replace $\xi(x_1,\dots,x_n)$ with $\frac{1}{\prod_{i=1}^{n} x_i}$, therefore we obtain $w(y_1,\dots,y_n) = \frac{v(y_1,\dots,y_n)}{\prod_{i=1}^{n} y_i}$. □

Remark 3.5. Let $\varphi(u) = u^p$ in Theorem 3.1 and Theorem 3.4. Then for $p \geq 2$ the corresponding inequalities are preserved. However, for $p \in (0,2]$ the corresponding inequalities are reversed.

4. Some particular cases

Corollary 4.1. *Suppose time scale $\mathbb{T}$ consists of isolated points, $\Omega = \Lambda = [a,\infty)_\mathbb{T}$, $a \geq 0$ and $\xi : \Omega \to \mathbb{R}_+$ is a μ_Δ-integrable function. For $p \geq 2$ inequality*

$$\begin{aligned} &\sum_{x\in[a,\infty)_\mathbb{T}} \xi(x) \left(A_k f(x)\right)^p \mu(x) + \\ &+ \sum_{y\in[a,\infty)_\mathbb{T}} \sum_{x\in[y,\infty)_\mathbb{T}} \frac{\xi(x)}{(\sigma(x) - a)} \left(|f(y) - A_k f(x)|^p\right) \mu(x) \mu(y) \leq \\ &\leq \sum_{y\in[a,\infty)_\mathbb{T}} w(y) (f(y))^p \mu(y), \end{aligned} \tag{4.1}$$

holds for all λ_Δ-integrable $f : \Lambda \to \mathbb{R}_+$, where

$$A_k f(x) = \frac{1}{(\sigma(x) - a)} \sum_{y\in[a,\sigma(x))_\mathbb{T}} f(y) \mu(y)$$

and

$$w(y) = \sum_{x\in[y,\infty)_\mathbb{T}} \frac{\xi(x)}{(\sigma(x) - a)} \mu(x).$$

Proof. Use $\varphi(r) = r^p$ in Theorem 3.1 for $p \geq 2$ and $n = 1$. □

Corollary 4.2. *Suppose time scale $\mathbb{T}$ consists of isolated points, $\Omega = \Lambda = [a,\infty)_{\mathbb{T}}$, $a \geq 0$ and $\xi : \Omega \to \mathbb{R}_+$ is a μ_Δ-integrable function. Then*

$$\sum_{x\in[a,\infty)_{\mathbb{T}}} \xi(x)\left(A_k f(x)\right)\mu(x) + I \leq \sum_{y\in[a,\infty)_{\mathbb{T}}} w(y)f(y)\mu(y)$$

holds for all λ_Δ-integrable $f : \Lambda \to (0,\infty)$, where

$$I = \sum_{y\in[a,\infty)_{\mathbb{T}}} \sum_{x\in[y,\infty)_{\mathbb{T}}} \frac{\mu(x)\mu(y)\,\xi(x)}{(\sigma(x)-a)} \left(\exp\left|\ln\left(\frac{g(y)}{\bar{g}(x)}\right)\right| - \left|\ln\left(\frac{g(y)}{\bar{g}(x)}\right)\right|\right) +$$
$$+ \ln\left(\frac{\prod\limits_{y\in[a,(\sigma(x))_{\mathbb{T}}} (g(y))^{w(y)\mu(y)}}{\prod\limits_{x\in[a,\infty)_{\mathbb{T}}} e^{\mu(x)}\bar{g}(x)^{\mu(x)}}\right), \tag{4.2}$$

$\bar{g}(x) = \left(\prod\limits_{y\in[a,\sigma(x))_{\mathbb{T}}} (g(y))^{\mu(y)}\right)^{\frac{1}{(\sigma(x)-a)}}$ *for λ_Δ-integrable function $g : \Lambda \to (0,\infty)$ and $w(y)$ as in Corollary* 4.1.

Proof. Use $\varphi(x) = e^x - x - 1$ and $f(x) = \ln g(x)$ in Theorem 3.1. □

Example 4.3. If we consider time scale $\mathbb{T} = h\mathbb{N} = \{hn : n \in \mathbb{N}\}$ with fixed $h > 0$ and taking $\xi(x) = \dfrac{1}{\sigma(x)}$, (4.1) takes the form

$$\sum_{n=1}^{\infty} \frac{1}{n+1}\left(\frac{1}{n}\sum_{k=1}^{n} f(kh)\right)^p +$$
$$+ \sum_{k=1}^{\infty}\sum_{n=k}^{\infty} \frac{1}{n(n+1)}\left|f(kh) - \frac{1}{n}\sum_{k=1}^{n} f(kh)\right|^p \leq \sum_{k=1}^{\infty} \frac{(f(kh))^p}{k}.$$

Example 4.4. If we consider time scale $\mathbb{T} = \mathbb{N}^2 = \{n^2 : n \in \mathbb{N}\}$, $a = 1$, and

$$\xi(x) = \frac{2(\sigma(x)-1)}{(\sigma(x)-x)^2(2\sqrt{x}+3)},$$

(4.1) takes the form

$$\sum_{n=1}^{\infty} \frac{2\left(n(n+2)\right)^{1-p}}{(2n+1)(2n+3)}\left(\sum_{k=1}^{n}(2k+1)f(k^2)\right)^p +$$
$$+ \sum_{k=1}^{\infty}\sum_{n=k}^{\infty} \frac{2(2k+1)}{(2n+1)(2n+3)}\left|f(k^2) - \frac{1}{n(n+2)}\sum_{k=1}^{n}(2k+1)f(k^2)\right|^p \leq$$
$$\leq \sum_{k=1}^{\infty}\left(f(k^2)\right)^p.$$

Example 4.5. Consider $\mathbb{T} = q^{\mathbb{N}} = \{q^n : n \in \mathbb{N}\}$ with fixed $q > 1$ and

$$\xi(x) = \frac{2(\sigma(x)-1)}{(\sigma(x)-x)^2(2\sqrt{x}+3)},$$

(4.1) takes the form

$$\sum_{n=1}^{\infty} \frac{(q-1)^p(q^n-1)^{1-p}}{q^n}\left(\sum_{k=1}^{n} q^{k-1} f(q^k)\right)^p + \\ + \sum_{k=1}^{\infty}\sum_{n=k}^{\infty} \frac{q^{k-1}(q-1)}{q^n}\left| f(q^k) - \frac{q-1}{q^n-1}\sum_{k=1}^{n} q^{k-1} f(q^k)\right|^p \leq \sum_{k=1}^{\infty}\left(f(q^k)\right)^p.$$

Example 4.6. Consider $\mathbb{T}$ and $\xi(x)$ as in Example 4.3. (4.2) takes the form

$$\sum_{n=1}^{\infty} \frac{1}{n+1}\left(\prod_{k=1}^{n} g(kh)\right)^{\frac{1}{n}} + \\ + \sum_{k=1}^{\infty}\sum_{n=k}^{\infty} \frac{1}{n(n+1)}\left(\exp\left|\ln\left(\frac{g(kh)}{\prod_{k=1}^{n}(g(kh))^{\frac{1}{n}}}\right)\right| - \left|\ln\left(\frac{g(kh)}{\prod_{k=1}^{n}(g(kh))^{\frac{1}{n}}}\right)\right|\right) + \\ + \ln\left(\frac{\prod_{k=1}^{\infty}(g(kh))^{\frac{1}{k}}}{e^h \prod_{n=1}^{\infty}\left(\prod_{k=1}^{n} g(kh)\right)^{\frac{h}{n}}}\right) \leq \sum_{k=1}^{\infty} \frac{g(kh)}{k}.$$

Example 4.7. Consider $\mathbb{T}$ and $\xi(x)$ as in Example 4.4. (4.2) takes the form

$$\sum_{n=1}^{\infty} \frac{2n(n+2)}{(2n+1)(2n+3)}\left(\prod_{k=1}^{n}\left(g(k^2)\right)^{2k+1}\right)^{\frac{1}{n(n+2)}} + \\ + \sum_{k=1}^{\infty}\sum_{n=k}^{\infty} \frac{2(2k+1)}{(2n+1)(2n+3)}\left(\begin{array}{l} \exp\left|\ln\left(\frac{g(k^2)}{\left(\prod_{k=1}^{n} g(k^2)^{2k+1}\right)^{\frac{1}{n(n+2)}}}\right)\right| \\ -\left|\ln\left(\frac{g(k^2)}{\left(\prod_{k=1}^{n} g(k^2)^{2k+1}\right)^{\frac{1}{n(n+2)}}}\right)\right| \end{array}\right) + \\ + \ln\left(\frac{\prod_{k=1}^{\infty} g(k^2)}{\prod_{n=1}^{\infty} e^{2n+1}\left(\prod_{k=1}^{n} g(k^2)^{2k+1}\right)^{\frac{2n+1}{n(n+2)}}}\right) \leq \sum_{k=1}^{\infty} g(k^2).$$

Example 4.8. Consider $\mathbb{T}$ and $\xi(x)$ as in Example 4.5. (4.2) takes the form

$$\sum_{n=1}^{\infty} q^{-n}(q^n-1)\left(\prod_{k=1}^{n}\left(g(q^k)\right)^{q^{k-1}}\right)^{\frac{q-1}{q^n-1}} +$$

$$+\sum_{k=1}^{\infty}\sum_{n=k}^{\infty}\frac{q^{k-1}(q-1)}{q^n}\left(\exp\left|\ln\left(\frac{g(q^k)}{\left(\prod_{k=1}^{n} g(q^k)^{q^{k-1}}\right)^{\frac{q-1}{q^n-1}}}\right)\right| - \right.$$

$$\left. -\left|\ln\left(\frac{g(q^k)}{\left(\prod_{k=1}^{n} g(q^k)^{q^{k-1}}\right)^{\frac{q-1}{q^n-1}}}\right)\right|\right) +$$

$$+\ln\left(\frac{\prod_{k=1}^{\infty} g(q^k)}{\prod_{n=1}^{\infty} e^{q^n(q-1)}\left(\prod_{k=1}^{n} g(q^k)^{q^{k-1}}\right)^{\frac{q^n(q-1)^2}{q^n-1}}}\right) \le \sum_{k=1}^{\infty} g(q^k).$$

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(Received 24.02.2014)

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