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EXTREME DIFFERENTIALS IN A WIDE SENSE OF FUNCTIONS OF TWO VARIABLES

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Abstract. For the functions of two variables, the notions of upper and lower differentials in a wide sense are introduced and the sufficient conditions for their existence are established.

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Introduction

It is well-known that the notion of an upper variable ([1], p. 219; [2], p. 108) and that of upper semicontinuity ([2], p. 42; [3], p. 385) is a consequence of replacement of a limit by an upper limit in the corresponding definitions. Analogously, the notion of a total differentiability of functions of two variables ([2], p. 300) may result in two notions involving upper and lower limits which will be called below as upper and lower differentiability in a wide sense.

In 1932, U. S. Haslam-Jones introduced for functions of two variables the notion of upper differentiability and called it upper differentiability in the HJ sense consisting of two conditions ([4]; [2], p. 309). The second condition is naturally obtained from the notion of total differentiability, but it does not guarantee the uniqueness of the pair $\{A, B\}$. The first condition is a geometric characteristic of functions of two variables.

But nowadays we are well aware of the necessary and sufficient condition of differentiability ([5]; [6]; [7], pp. 70–73), and this allows us to proceed directly from the notion of total differentiability. In addition, the condition of upper differentiability in a wide sense imposed on the function is not harder than that of upper differentiability in the HJ sense.

The basic results of the present paper have been announced in [8].

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1. Preliminaries

For a finite in the neighborhood of the point $p_0 = (x_0, y_0) \in \mathbb{R}^2$ function of two variables $F(x, y)$, the notion of an upper differential consists in the following ([2], p. 309).

A pair of finite numbers $\{A, B\}$ is said to be an upper differential of the function F at the point (x_0, y_0), if for $z_0 = F(x_0, y_0)$ the following conditions are fulfilled:

(i) the plane $z - z_0 = A(x - x_0) + B(y - y_0)$ is an extreme intermediate tangent plane ([2], p. 263) of the graph of the function F at the point (x_0, y_0, z_0);

(ii) the equality

$$\varlimsup_{(x,y)\to(x_0,y_0)} \frac{F(x,y) - F(x_0,y_0) - A(x-x_0) - B(y-y_0)}{|x-x_0| + |y-y_0|} = 0. \tag{1.1}$$

is fulfilled.

The notion of an upper differential has been introduced by U.S. Haslam-Jones ([4], p. 309), and we shall call it an upper differential in the HJ sense, and the function F itself will be called upper differentiable in the HJ sense at the point (x_0, y_0).

The lower differential is defined analogously. Upper and lower differentials are called extreme differentials.

If the function F is differentiable at the point (x_0, y_0), then its extreme differentials coincide at the point (x_0, y_0). Conversely, if the function F has equal extreme differentials at the point (x_0, y_0), then they are differential of the function F at the same point. There following question is quite natural: what is the way the pair $\{A, B\}$ is connected with the function F satisfying the condition (1.1)?

To resolve the question, we take in the equality (1.1) a particular value $y = y_0$ and obtain the inequality [1]

$$\varlimsup_{x\to x_0} \frac{F(x,y_0) - F(x_0,y_0) - A(x-x_0)}{|x-x_0|} \le 0. \tag{1.2}$$

Consider two cases.

(I) $x > x_0$. Then the rather that

$$\varlimsup_{\substack{x\to x_0\\ x>x_0}} \frac{F(x,y_0) - F(x_0,y_0) - A(x-x_0)}{x-x_0} \le 0,$$

that is,

$$\varlimsup_{\substack{x\to x_0\\ x>x_0}} \left(\frac{F(x,y_0) - F(x_0,y_0)}{x-x_0} - A\right) \le 0,$$

[1] The upper limit with respect to the subset is not more than that with respect to the basic set.

whence by virtue of the known equality,

$$\overline{\lim_{t\to t_0}}\,[f(t)-g(t)] = \overline{\lim_{t\to t_0}}\, f(t) + \lim_{t\to t_0} g(t), \tag{1.3}$$

when a finite limit $\lim\limits_{t\to t_0} g(t)$ exists ([1], p. 146), we obtain

$$\overline{\lim_{\substack{x\to x_0\\ x>x_0}}}\,\frac{F(x,y_0)-F(x_0,y_0)}{x-x_0} \le A. \tag{1.4}$$

(II) $x < x_0$. Now, likewise from (1.2), we have

$$\overline{\lim_{\substack{x\to x_0\\ x<x_0}}}\,\frac{F(x,y_0)-F(x_0,y_0)-A(x-x_0)}{-(x-x_0)} \le 0,$$

that is,

$$\overline{\lim_{\substack{x\to x_0\\ x<x_0}}}\left(-\frac{F(x,y_0)-F(x_0,y_0)}{x-x_0}\right) \le -A.$$

To the left-hand side of the latter inequality we apply the following equality (see, for e.g., [9], p. 17)

$$\overline{\lim_{\substack{p\to p_0\\ p\in E}}}\,(-u(p)) = -\underline{\lim_{\substack{p\to p_0\\ p\in E}}}\, u(p). \tag{1.5}$$

As a result, we get

$$\underline{\lim_{\substack{x\to x_0\\ x<x_0}}}\,\frac{F(x,y_0)-F(x_0,y_0)}{x-x_0} \ge A. \tag{1.6}$$

Inequalities (1.4) and (1.6) yield

$$\overline{\lim_{\substack{x\to x_0\\ x>x_0}}}\,\frac{F(x,y_0)-F(x_0,y_0)}{x-x_0} \le A \le \underline{\lim_{\substack{x\to x_0\\ x\in E}}}\,\frac{F(x,y_0)-F(x_0,y_0)}{x-x_0}. \tag{1.7}$$

The left-hand side of the inequality (1.4), symbolically $\overline{\partial}_x^{\,+} F(p_0)$, is called with respect to the variable x a right upper partial derivative of the function F at the point p_0 ([2], pp. 108 and 298).

Analogously, the left-hand side of the inequality (1.6), symbolically $\underline{\partial}_x^{\,-} F(p_0)$, is called with respect to the variable x a left lower partial derivative of the function F at the point p_0.

Thus, we have the correlations

$$\overline{\partial}_x^{\,+} F(p_0) \le A \le \underline{\partial}_x^{\,-} F(p_0).$$

The correlation

$$\overline{\partial}_y^{\,+} F(p_0) \le B \le \underline{\partial}_y^{\,-} F(p_0)$$

is obtained analogously.

Consequently, we have the following

Proposition 1.1 ([2], p. 313)**.** *If the function $F(x,y)$ satisfies the equality* (1.1)*, then the inequalities with finite terms*

$$\overline{\partial}_x^{+} F(p_0) \leq A \leq \underline{\partial}_x^{-} F(p_0), \tag{1.8}$$

$$\overline{\partial}_y^{+} F(p_0) \leq B \leq \underline{\partial}_y^{-} F(p_0) \tag{1.9}$$

hold.

Clearly, the condition (i) introduced by Haslam-Jones is to eliminate such a non-uniqueness of the pair $\{A, B\}$.

Below, we shall need certain definitions for the finite in the neighborhood $u(p_0,\delta) = \{(x,y) \in \mathbb{R}^2 : |x-x_0|+|y-y_0| < \delta\}$, $\delta > 0$ of the point $p_0(x_0,y_0)$ function $\varphi(p)$, $p = (x,y)$.

2. The Upper Differential in a Wide Sense

2.1. Further we will need some definitions.

Definition 2.1 ([8])**.** The function $\varphi(x,y)$ is said to be upper differentiable in a wide sense at the point (x_0,y_0), if there exists a unique pair of finite numbers $A = (A_1, A_2)$, such that the equality

$$\overline{\lim_{(x,y)\to(x_0,y_0)}} \frac{\varphi(x,y) - \varphi(x_0,y_0) - A_1(x-x_0) - A_2(y-y_0)}{|x-x_0| + |y-y_0|} = 0 \tag{2.1}$$

holds.

In this case the upper differential in a wide sense of the function $\varphi(x,y)$ at the point $p_0 = (x_0,y_0)$, symbolically $\overline{d}\varphi(p_0)$, we define by the equality

$$\overline{d}\,\varphi(p_0) = A_1 dx + A_2 dy. \tag{2.2}$$

Clearly, the conditions (i) and (ii) are not uniformly weak than the requirement for the uniqueness of the pair (A_1, A_2) in the equality (2.1). Just therefore the phrase "in a wide sense" appears in Definition 2.1.

Consequently, the upper differentiability in the HJ sense implies that in a wide sense.

Inequalities (1.8) and (1.9) and the notions of angular partial $\pm$ derivatives (see [10]) are considered worthwhile to introduce the following

Definition 2.2 ([8])**.** We say that the function $\varphi(x,y)$ at the point p_0 has the right upper angular derivative with respect to the variable x, symbolically $\overline{\partial}_{\hat{x}}^{+}\varphi(p_0)$, if for every constant $c \geq 0$ there exists an independent of c finite or infinite upper limit

$$\overline{\partial}_{\hat{x}}^{+}\varphi(p_0) = \overline{\lim_{\substack{x\to x_0^+ \\ |y-y_0|\leq c(x-x_0)}}} \frac{\varphi(x,y) - \varphi(x_0,y)}{x - x_0}. \tag{2.3}$$

The right upper angular derivative with respect to y for $\varphi(x,y)$ at p_0 is defined as

$$\overline{\partial}^{\,+}_{\widehat{y}}\varphi(p_0) = \overline{\lim_{\substack{x\to y_0^+\\ |x-x_0|\le \ell(y-y_0)}}} \frac{\varphi(x,y)-\varphi(x_0,y)}{y-y_0}, \tag{2.4}$$

if that upper limit, finite or infinite, exists for every constant $\ell \geq 0$ and does not depend on ℓ.

By means of the lower limit, we analogously define for $\varphi(x,y)$ at the point p_0 the left lower angular derivative $\underline{\partial}^{\,-}_{\widehat{x}}\varphi(p_0)$ with respect to y and the left lower angular derivative $\underline{\partial}^{\,-}_{\widehat{y}}\varphi(p_0)$ with respect to y (see also equality (2.13), below).

The existence of $\overline{\partial}^{\,+}_{\widehat{x}}\varphi(p_0)$ implies that of $\overline{\partial}^{\,+}_{x}\varphi(p_0)$, and their equality. The converse statement is invalid.

In the case if $\overline{\partial}^{\,+}_{\widehat{x}}\varphi(p_0)$ and $\overline{\partial}^{\,+}_{\widehat{y}}\varphi(p_0)$ exist, using the equality

$$^{+}\overline{\operatorname{anggrad}}\,\varphi(p_0) = \big(\overline{\partial}^{\,+}_{\widehat{x}}\varphi(p_0), \overline{\partial}^{\,+}_{\widehat{y}}\varphi(p_0)\big) \tag{2.5}$$

we introduce the upper right angular gradient of the function $\varphi(x,y)$ at the point p_0.

Analogously, we define the left lower angular gradient

$$^{-}\underline{\operatorname{anggrad}}\,\varphi(p_0) = \big(\underline{\partial}^{\,-}_{\widehat{x}}\varphi(p_0), \underline{\partial}^{\,-}_{\widehat{y}}\varphi(p_0)\big) \tag{2.6}$$

of the function $\varphi(x,y)$ at the point p_0.

2.2. The sufficient conditions for the existence of $\overline{d}\varphi(p_0)$ are given in the following

Theorem 2.1. *For the upper differentiability in a wide sense of the function $\varphi(x,y)$ at the point p_0 it suffices to fulfil the equality $^{-}\underline{\operatorname{anggrad}}\,\varphi(p_0) = {}^{+}\overline{\operatorname{anggrad}}\,\varphi(p_0)$ or, what is the same thing, to fulfil the equalities*

$$\underline{\partial}^{\,-}_{\widehat{x}}\varphi(p_0) = \overline{\partial}^{\,+}_{\widehat{x}}\varphi(p_0), \tag{2.7}$$

$$\underline{\partial}^{\,-}_{\widehat{y}}\varphi(p_0) = \overline{\partial}^{\,+}_{\widehat{y}}\varphi(p_0) \tag{2.8}$$

under the condition that all their terms are finite, and in these conditions we have

$$\begin{aligned}\overline{d}\varphi(p_0) &= \overline{\partial}^{\,+}_{\widehat{x}}\varphi(p_0)dx + \overline{\partial}^{\,+}_{\widehat{y}}\varphi(p_0)dy = \\ &= \overline{\partial}^{\,+}_{x}\varphi(p_0)dx + \overline{\partial}^{\,+}_{y}\varphi(p_0)dy, \end{aligned}\tag{2.9}$$

$$\begin{aligned}\overline{d}\varphi(p_0) &= \underline{\partial}^{\,-}_{\widehat{x}}\varphi(p_0)dx + \underline{\partial}^{\,-}_{\widehat{y}}\varphi(p_0)dy = \\ &= \underline{\partial}^{\,-}_{x}\varphi(p_0)dx + \underline{\partial}^{\,+}_{y}\varphi(p_0)dy. \end{aligned}\tag{2.10}$$

Proof. Since the right-hand side of the equality (2.3) is finite for every constant $c \geq 0$, we have

$$\varlimsup_{\substack{x \to x_0^+ \\ |y-y_0| \leq (x-x_0)}} \frac{\varphi(x,y) - \varphi(x_0,y)}{x - x_0} = \overline{\partial}\,_{\widehat{x}}^{+} \varphi(p_0). \tag{2.11}$$

Introduce the set $M_1 = \{(x,y) \in \mathbb{R}^2 : |y - y_0| \leq (x - x_0),\ x > x_0\}$. The set of all points $(x,y) \in M_1$ from the η-neighborhood $U(p_0,\eta)$ of the point p_0 we denote by M_1^η.

Applying to (2.11) the equality (1.3), we obtain

$$\varlimsup_{\substack{(x,y) \to p_0 \\ (x,y) \in M_1^\eta}} \frac{\varphi(x,y) - \varphi(x_0,y) - (x - x_0)\overline{\partial}\,_{\widehat{x}}^{+} \varphi(p_0)}{x - x_0} = 0,$$

which can be rewritten (since $x > x_0$) in the form

$$\varlimsup_{\substack{(x,y) \to p_0 \\ (x,y) \in M_1^\eta}} \frac{\varphi(x,y) - \varphi(x_0,y) - (x - x_0)\overline{\partial}\,_{\widehat{x}}^{+} \varphi(p_0)}{|x - x_0|} = 0. \tag{2.12}$$

Using now equalities (2.7), we can show that the equality (2.12) holds likewise in the case $x < x_0$, symbolically $(x,y) \in M_2$, where $M_2 = \{(x,y) \in \mathbb{R}^2 : |y - y_0| \leq (x_0 - x),\ x < x_0\}$. Indeed, we write the equality

$$\underline{\partial}\,_{\widehat{x}}^{-} \varphi(p_0) = \varliminf_{\substack{x \to x_0^- \\ |y-y_0| \leq (x-x_0)}} \frac{\varphi(x,y) - \varphi(x_0,y)}{x - x_0} \tag{2.13}$$

in the form

$$\varliminf_{\substack{x \to x_0^- \\ (x,y) \in M_2^\eta}} \frac{\varphi(x,y) - \varphi(x_0,y) - (x - x_0)\underline{\partial}\,_{\widehat{x}}^{-} \varphi(p_0)}{x - x_0} = 0$$

or, what is the same thing, in the form

$$\varliminf_{\substack{(x,y) \to p_0 \\ (x,y) \in M_2^\eta}} \left[- \frac{\varphi(x,y) - \varphi(x_0,y) - (x - x_0)\underline{\partial}\,_{\widehat{x}}^{-} \varphi(p_0)}{|x - x_0|} \right] = 0.$$

Thus, taking into account the equalities (1.5) and (2.7), we obtain

$$\varlimsup_{\substack{(x,y) \to p_0 \\ (x,y) \in M_2^\eta}} \left[\frac{\varphi(x,y) - \varphi(x_0,y) - (x - x_0)\underline{\partial}\,_{\widehat{x}}^{+} \varphi(p_0)}{|x - x_0|} \right] = 0. \tag{2.14}$$

We can now combine the equalities (2.12) and (2.14) in the form of the following equality:

$$\varlimsup_{\substack{(x,y) \to p_0 \\ (x,y) \in M^\eta}} \frac{\varphi(x,y) - \varphi(x_0,y) - (x - x_0)\overline{\partial}\,_{\widehat{x}}^{+} \varphi(p_0)}{|x - x_0|} = 0, \tag{2.15}$$

where $M = M_1 \cup M_2$.

Take an arbitrarily small number $\varepsilon > 0$. As follows from the notion of the upper limit, to the number ε there corresponds the number $\delta_1 = \delta_1(p_0, \varepsilon, \varphi) > 0$ such that the inequality

$$\varphi(x, y) - \varphi(x_0, y) - (x - x_0)\overline{\partial}^{\,+}_{\hat{x}}\varphi(p_0) < \varepsilon|x - x_0| \tag{2.16}$$

holds for all points $(x, y) \in M^{\delta_1}$, by virtue of (2.15).

Reasoning analogously, we obtain the inequality

$$\varphi(x, y) - \varphi(x, y_0) - (y - y_0)\overline{\partial}^{\,+}_{\hat{y}}\varphi(p_0) < \varepsilon|y - y_0| \tag{2.17}$$

for all points $(x, y) \in N^{\delta_2}$, where the set N is obtained in the same way as we used for the set $M = M_1 \cup M_2$. Obviously, combination of the sets M and N covers the deleted neighborhood $U(p_0)\backslash\{p_0\}$ of the point p_0.

To prove the theorem, it suffices to assume that tending of the point (x, y) to the point p_0 is realized along the sets M^{δ_1} or N^{δ_2} (since the deleted neighborhood of the point p_0 is represented as a combination of a finite number of sets). For the sake of definiteness we assume that the point (x, y) belongs to the set M^{δ_1} and hence tends to the point p_0.

The inequality (2.17) holds, particularly, for all points $(x, y) \in N^{\delta_2}$. Thus, we have

$$\varphi(x_0, y) - \varphi(x_0, y_0) - (y - y_0)\overline{\partial}^{\,+}_{\hat{y}}\varphi(p_0) < \varepsilon|y - y_0|; \tag{2.18}$$

$$\begin{aligned}\varphi(x, y) - \varphi(x_0, y_0) - (x - x_0)\overline{\partial}^{\,+}_{\hat{x}}\varphi(p_0) - (y - y_0)\overline{\partial}^{\,+}_{\hat{y}}\varphi(p_0) = \\ = \big[\varphi(x, y) - \varphi(x_0, y) - (x - x_0)\overline{\partial}^{\,+}_{\hat{x}}\varphi(p_0)\big] + \\ + \big[\varphi(x_0, y) - \varphi(x_0, y_0) - (y - y_0)\overline{\partial}^{\,+}_{\hat{y}}\varphi(p_0)\big].\end{aligned} \tag{2.19}$$

Taking into account inequalities (2.16) and (2.18), the inequality (2.19) yields

$$\begin{aligned}\varphi(x, y) - \varphi(x_0, y_0) - (x - x_0)\overline{\partial}^{\,+}_{\hat{x}}\varphi(p_0) > -(y - y_0)\overline{\partial}^{\,+}_{\hat{y}}\varphi(p_0) < \\ < \varepsilon(|x - x_0| + |y - y_0|)\end{aligned} \tag{2.20}$$

for all points $(x, y) \in M^{\delta}$, where $\delta = \min\{\delta_1, \delta_2\}$.

To complete the proof of the theorem, we need a well-defined inequality appearing in the notion of the upper limit.

On the one hand, $\overline{\partial}^{\,+}_{\hat{x}}\varphi(p_0) = \overline{\partial}^{\,+}_{x}\varphi(p_0)$. On the other hand, for $\overline{\partial}^{\,+}_{x}\varphi(p_0)$ there exists the convergent to the point p_0 sequence of the points (x_k, y_0), $x_k > x_0$ such that

$$\varphi(x_k, y_0) - \varphi(x_0, y_0) - (x_k - x_0)\overline{\partial}^{\,+}_{\hat{x}}\varphi(p_0) > -\varepsilon|x_k - x_0|. \tag{2.21}$$

This implies that for the convergent to the point p_0 sequence (x_k, y_0), the right-hand side of the equality (2.19), and hence its left-hand side, is more than $-\varepsilon(|x_k - x_0| + |y_0 - y_0|)$.

This fact, with regard for the inequality (2.20), allows us to prove the equality (2.1) for the numbers

$$A_1 = \overline{\partial}^{+}_{\widehat{x}}\varphi(p_0) = \underline{\partial}^{-}_{\widehat{x}}\varphi(p_0), \tag{2.22}$$

$$A_2 = \overline{\partial}^{+}_{\widehat{y}}\varphi(p_0) = \underline{\partial}^{-}_{\widehat{y}}\varphi(p_0), \tag{2.23}$$

or in view of the above-said, for the numbers

$$A_1 = \overline{\partial}^{+}_{x}\varphi(p_0) = \underline{\partial}^{-}_{x}\varphi(p_0), \tag{2.24}$$

$$A_2 = \overline{\partial}^{+}_{y}\varphi(p_0) = \underline{\partial}^{-}_{y}\varphi(p_0), \tag{2.25}$$

By virtue of the equality (2.2), we obtain the equalities (2.9) and (2.10). Thus, the proof of Theorem 2.1 is completed. □

3. The Lower Differential in a Wide Sense

3.1. If the function $\varphi(x,y)$ satisfies the condition

$$\varliminf_{(x,y)\to p_0} \frac{\varphi(x,y)-\varphi(x_0,y_0)-M(x-x_0)-N(y-y_0)}{|x-x_0|+|y-y_0|} = 0 \tag{3.1}$$

for some finite numbers M and N, then the inequalities [2]

$$\varlimsup_{\substack{x\to x_0\\ x<x_0}} \frac{\varphi(x,y_0)-\varphi(x_0,y_0)}{x-x_0} \le M \le \varliminf_{\substack{x\to x_0\\ x>x_0}} \frac{\varphi(x,y_0)-\varphi(x_0,y_0)}{x-x_0}\,, \tag{3.2}$$

$$\varlimsup_{\substack{y\to y_0\\ y<y_0}} \frac{\varphi(x_0,y)-\varphi(x_0,y_0)}{y-y_0} \le N \le \varliminf_{\substack{y\to y_0\\ y>y_0}} \frac{\varphi(x_0,y)-\varphi(x_0,y_0)}{y-y_0} \tag{3.3}$$

hold.

The left-hand side of (3.2), symbolically $\overline{\partial}^{-}_{x}\varphi(p_0)$, is called with respect to the variable x a left upper partial derivative, and its right-hand side, symbolically $\underline{\partial}^{+}_{x}\varphi(p_0)$, is called with respect to x a right lower partial derivative of the function $\varphi(x,y)$ at the point p_0.

Therefore the relations (3.2) can be written in the form

$$\overline{\partial}^{-}_{x}\varphi(p_0) \le M \le \underline{\partial}^{+}_{x}\varphi(p_0). \tag{3.4}$$

For (3.3) we likewise have

$$\overline{\partial}^{-}_{y}\varphi(p_0) \le N \le \underline{\partial}^{+}_{y}\varphi(p_0). \tag{3.5}$$

Consequently, from the equality (3.1) we obtain the relations (3.4) and (3.5).

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[2] The lower limit with respect to the subset is not less than that with respect to the basic set.

3.2.

Definition 3.1 ([8])**.** The function $\varphi(x,y)$ is said to be lower differentiable in a wide sense at the point $p_0=(x_0,y_0)$, if there exists a unique pair of finite numbers $B=(B_1,B_2)$ such that

$$\varliminf_{(x,y)\to(x_0,y_0)} \frac{\varphi(x,y)-\varphi(x_0,y_0)-B_1(x-x_0)-B_2(y-y_0)}{|x-x_0|+|y-y_0|}=0. \tag{3.6}$$

In this case, the lower differential in a wide sense of the function $\varphi(x,y)$ at the point $p_0=(x_0,y_0)$, symbolically $\underline{\partial}\,\varphi(p_0)$, can be defined by the equality

$$\underline{\partial}\,\varphi(p_0)=B_1dx+B_2dy. \tag{3.7}$$

Introduce now the quantities

$$\underline{\partial}\,^{\pm}_{\widehat{x}}\varphi(p_0)=\varliminf_{\substack{x\to x_0^{\pm}\\ |y-y_0|\le c(x-x_0)}} \frac{\varphi(x,y)-\varphi(x_0,y)}{x-x_0}\,, \quad c\ge 0, \tag{3.8}$$

$$\underline{\partial}\,^{\pm}_{\widehat{y}}\varphi(p_0)=\varliminf_{\substack{y\to y_0^{\pm}\\ |x-x_0|\le \ell(y-y_0)}} \frac{\varphi(x,y)-\varphi(x,y_0)}{y-y_0}\,, \quad \ell\ge 0. \tag{3.9}$$

Theorem 3.1. *For the function $\varphi(x,y)$ to be lower differentiable in a wide sense at the point p_0, it is sufficient that the equalities*

$$\underline{\partial}\,^{+}_{\widehat{x}}\varphi(p_0)=\overline{\partial}\,^{-}_{\widehat{x}}\varphi(p_0), \tag{3.10}$$

$$\underline{\partial}\,^{+}_{\widehat{y}}\varphi(p_0)=\overline{\partial}\,^{-}_{\widehat{y}}\varphi(p_0) \tag{3.11}$$

be fulfilled under the condition that their terms are finite, and in such a case

$$\begin{aligned}\underline{\partial}\,\varphi(p_0)&=\underline{\partial}\,^{+}_{\widehat{x}}\varphi(p_0)dx+\underline{\partial}\,^{+}_{\widehat{y}}\varphi(p_0)dy=\\&=\underline{\partial}\,^{+}_{x}\varphi(p_0)dx+\underline{\partial}\,^{+}_{y}\varphi(p_0)dy,\end{aligned} \tag{3.12}$$

$$\begin{aligned}\underline{\partial}\,\varphi(p_0)&=\overline{\partial}\,^{-}_{\widehat{x}}\varphi(p_0)dx+\overline{\partial}\,^{-}_{\widehat{y}}\varphi(p_0)dy=\\&=\overline{\partial}\,^{-}_{x}\varphi(p_0)dx+\overline{\partial}\,^{-}_{y}\varphi(p_0)dy.\end{aligned} \tag{3.13}$$

4. On the Total Differential

The upper and lower differentials in a wide sense are called extremal ones in a wide sense.

Theorem 4.1 ([8])**.** *For the total differential $d\varphi(p_0)$ to exist, it is necessary and sufficient that the extremal differentials in a wide sense be equal: $\overline{d}\varphi(p_0)=\underline{d}\varphi(p_0)$.*

Proof. The Necessity. Since the existence of the total differential $d\varphi(p_0)$ is equivalent to that of the finite angular partial derivatives $\varphi'_{\widehat{x}}(p_0)$, $\varphi'_{\widehat{y}}(p_0)$

and to the equality $d\varphi(p_0) = \varphi'_{\widehat{x}}(p_0)dx + \varphi'_{\widehat{y}}(p_0)dy$ ([7], p. 71), therefore the existence of $d\varphi(p_0)$ implies that of

$$\underline{d}^{\,-}_{\widehat{x}}\varphi(p_0) = \overline{d}^{\,+}_{\widehat{x}}\varphi(p_0)d\varphi(p_0) = \underline{d}^{\,+}_{\widehat{x}}\varphi(p_0) = \overline{d}^{\,-}_{\widehat{x}}\varphi(p_0) = \partial_{\widehat{x}}\varphi(p_0),$$
$$\underline{d}^{\,-}_{\widehat{y}}\varphi(p_0) = \overline{d}^{\,+}_{\widehat{y}}\varphi(p_0)d\varphi(p_0) = \underline{d}^{\,+}_{\widehat{y}}\varphi(p_0) = \overline{d}^{\,-}_{\widehat{y}}\varphi(p_0) = \partial_{\widehat{y}}\varphi(p_0),$$
$$\underline{d}\varphi(p_0) = d\varphi(p_0) = \overline{d}\varphi(p_0).$$

The Sufficiency. If $\overline{d}\varphi(p_0) = \underline{d}\varphi(p_0)$, then for the coefficients appearing in (2.1) and (3.6) we have the equalities $A_1 = B_1$ and $A_2 = B_2$ from which we can easily get the differentiability of the function φ at the point p_0 and the equalities $\varphi'_{\widehat{x}}(p_0) = A_1 = B_1$ and $\varphi'_{\widehat{y}}(p_0) = A_2 = B_2$. Thus, $\underline{d}\varphi(p_0) = d\varphi(p_0) = \overline{d}\varphi(p_0)$. $\square$

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