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A NOTE ON THE STRONG CONVERGENCE OF TWO–DIMENSIONAL WALSH-FOURIER SERIES

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Abstract. Goginava and Gogoladze proved that the following result is true

$$\sum_{n=1}^{\infty}\frac{\left\|S_{n,n}f\right\|_{L_1(G^2)}}{n\log^2(n+1)}\leq c\left\|f\right\|_{H_1(G^2)},$$

where $f\in H_1\left(G^2\right)$ and c is absolute constant. The main aim of this paper is to prove that the rate of the deviant behavior of the $L_1(G^2)$ norm of (n,n)-th partial sum is exactly $[n\log^2(n+1)]_{n-1}^{\infty}$.

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$$\sum_{n=1}^{\infty}\frac{\left\|S_{n,n}f\right\|_{L_1(G^2)}}{n\log^2(n+1)}\leq c\left\|f\right\|_{H_1(G^2)},$$

სადაც $f\in H_1\left(G^2\right)$ და c რაღაც დამოუკიდებელი მუდმივია. ნაშრომში დამტკიცებულია, რომ (n,n)-ური კერძო ჯამის $L_1(G^2)$ ნორმის მნიშვნელის რიგი არის ზუსტად $[n\log^2(n+1)]_{n-1}^{\infty}$.

Let $\mathbf{N}_+$ denote the set of positive integers, $\mathbf{N}:=\mathbf{N}_+\cup\{0\}$. Denote by Z_2 the discrete cyclic group of order 2, that is $Z_2=\{0,1\}$, where the group operation is the modulo 2 addition and every subset is open. The Haar measure on Z_2 is given such that the measure of a singleton is 1/2. Let G be the complete direct product of the countable infinite copies of the compact group Z_2. The elements of G are of the form $x=(x_0,x_1,\ldots,x_k,\ldots)$ with $x_k\in\{0,1\}\,(k\in\mathbf{N})$. The group operation on G is the coordinate-wise addition, the measure (denote by μ) and the topology are the product measure and topology. The compact Abelian group G is called the Walsh group. A base for the neighborhoods of G can be given in the following way:

$$I_0(x):=G,$$
$$I_n(x):=I_n(x_0,\ldots,x_{n-1}):=\{y\in G:\ y=(x_0,\ldots,x_{n-1},y_n,y_{n+1},\ldots)\},$$

where $x\in G$ and $n\in\mathbf{N}_+$. Denote $I_n:=I_n(0)$, for $n\in\mathbf{N}$.

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If $n \in \mathbf{N}$, then $n = \sum_{i=0}^{\infty} n_i 2^i$, where $n_i \in \{0,1\}$ $(i \in \mathbf{N})$, i. e. n is expressed in the number system of base 2. Denote $|n| := \max\{j \in \mathbf{N} : n_j \neq 0\}$, that is, $2^{|n|} \leq n < 2^{|n|+1}$.

Define the variation of an $n \in \mathbb{N}$ with binary coefficients $(n_k, k \in \mathbb{N})$ by

$$V(n) = n_0 + \sum_{k=1}^{\infty} |n_k - n_{k-1}|.$$

For $k \in \mathbf{N}$ and $x \in G$ let us denote by

$$r_k(x) := (-1)^{x_k} \quad (x \in G,\ k \in \mathbf{N})$$

the k-th Rademacher function.

The Walsh-Paley system is defined as the sequence of Walsh-Paley functions:

$$w_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_k} \qquad (x \in G,\ n \in \mathbf{N}_+).$$

The Walsh-Dirichlet kernel is defined by

$$D_n(x) = \sum_{k=0}^{n-1} w_k(x).$$

Recall that (see [9, p. 7])

$$D_{2^n}(x) = \begin{cases} 2^n, & x \in I_n, \\ 0, & x \notin I_n, \end{cases} \tag{1}$$

and

$$D_{m+2^l}(x) = D_{2^l}(x) + w_{2^l}(x) D_m(x), \text{ when } m \leq 2^l. \tag{2}$$

Denote by $L_p(G^2)$, $(0 < p < \infty)$ the two-dimensional Lebesgue space, with corresponding norm $\|\cdot\|_p$.

The number $\|D_n\|_1$ is called n-th Lebesgue constant. Then (see [9])

$$\frac{1}{8} V(n) \leq \|D_n\|_1 \leq V(n). \tag{3}$$

The rectangular partial sums of the two-dimensional Walsh-Fourier series of a function $f \in L_1(G^2)$ are defined as follows:

$$S_{M,N} f(x,y) := \sum_{i=0}^{M-1} \sum_{j=0}^{N-1} \widehat{f}(i,j) w_i(x) w_j(y),$$

where the numbers $\widehat{f}(i,j) := \int_{G^2} f(x,y) w_i(x) w_j(y) d\mu(x,y)$ is said to be the (i,j)-th Walsh-Fourier coefficient of the function f.

Let $f \in L_1\left(G^2\right)$. Then the dyadic maximal function is given by

$$f^*(x,y) = \sup_{n\in\mathbf{N}} \frac{1}{\mu\left(I_n(x)\times I_n(y)\right)} \left| \int\limits_{I_n(x)\times I_n(y)} f(s,t)\, d\mu(s,t) \right|.$$

The dyadic Hardy space $H_p(G^2)$ $(0<p<\infty)$ consists of all functions for which

$$\|f\|_{H_p} := \|f^*\|_p < \infty.$$

If $f \in L_1\left(G^2\right)$, then (see [14])

$$\|f\|_{H_1} = \left\| \sup_{k\in\mathbf{N}} \left| S_{2^k,2^k} f \right| \right\|_1 . \tag{4}$$

It is known [8, p. 125] that the Walsh-Paley system is not a Schauder basis in $L_1(G)$. Moreover, there exists a function in the dyadic Hardy space $H_1(G)$, the partial sums of which are not bounded in $L_1(G)$. However, Simon ([10] and [11]) proved that there is an absolute constant c_p, depending only on p, such that

$$\frac{1}{\log^{[p]} n} \sum_{k=1}^{n} \frac{\|S_k f\|_p^p}{k^{2-p}} \leq c_p \|f\|_{H_p}^p \tag{5}$$

for all $f \in H_p(G)$, where $0<p\leq 1$, $S_k f$ denotes the k-th partial sum of the Walsh-Fourier series of f and $[p]$ denotes integer part of p. (For the Vilenkin system when $p=1$ see in Gat [2]). When $0<p<1$ and $f\in H_p(G)$ the author [13] proved that sequence $\left\{1/k^{2-p}\right\}_{k=1}^{\infty}$ in (5) can not be improved.

For the two-dimensional Walsh-Fourier series some strong convergence theorems are proved in [12] and [15]. Convergence of quadratic partial sums was investigated in details in [3, 7]. Goginava and Gogoladze [6] proved that the following result is true:

Theorem G. *Let $f \in H_1\left(G^2\right)$. Then there exists absolute constant c, such that*

$$\sum_{n=1}^{\infty} \frac{\|S_{n,n} f\|_1}{n \log^2(n+1)} \leq c \|f\|_{H_1} . \tag{6}$$

The main aim of this paper is to prove that sequence $\left\{1/n\log^2(n+1)\right\}_{n=1}^{\infty}$ in (6) is essential too. In particular, the following is true:

Theorem 1. *Let $\Phi : \mathbf{N} \to [1,\infty)$ be any nondecreasing function, satisfying the condition $\lim_{n\to\infty}\Phi(n) = +\infty$. Then*

$$\sup_{\|f\|_{H_1}\leq 1} \sum_{n=1}^{\infty} \frac{\|S_{n,n} f\|_1 \Phi(n)}{n \log^2(n+1)} = \infty.$$

Proof. Let

$$f_{n,n}(x,y)=\left(D_{2^{n+1}}(x)-D_{2^n}(x)\right)\left(D_{2^{n+1}}(y)-D_{2^n}(y)\right).$$

It is easy to show that

$$\widehat{f_{n,n}}(i,j)=\begin{cases}1, \text{ if } \ (i,j)\in\left\{2^n,\ldots,\ 2^{n+1}-1\right\}^2,\\ 0, \text{ if } \ (i,j)\notin\left\{2^n,\ldots,\ 2^{n+1}-1\right\}^2.\end{cases}\tag{7}$$

Applying (1) and (4) we have

$$\left\|f_{n,n}\right\|_{H_1}=\left\|\sup_{k\in\mathbf{N}}\left|S_{2^k,2^k}f_{n,n}\right|\right\|_1=\left\|f_{n,n}\right\|_1=1.\tag{8}$$

Let $2^n<k\le 2^{n+1}$. Combining (2) and (7) we get

$$\begin{aligned}S_{k,k}f_{n,n}(x,y)&=\sum_{i=2^n}^{k-1}\sum_{j=2^n}^{k-1}w_i(x)\,w_j(y)=\\&=\left(D_k(x)-D_{2^n}(x)\right)\left(D_k(y)-D_{2^n}(y)\right)=\\&=w_{2^n}(x)\,w_{2^n}(y)\,D_{k-2^n}(x)\,D_{k-2^n}(y).\end{aligned}$$

Using (3) we have

$$\begin{aligned}\left\|S_{k,k}f_{n,n}(x,y)\right\|_1&\ge\int_{G^2}\left|D_{k-2^n}(x)\,D_{k-2^n}(y)\right|d\mu(x,y)\ge\\&\ge cV^2\left(k-2^n\right).\end{aligned}\tag{9}$$

Let $\Phi(n)$ be any nondecreasing, nonnegative function, satisfying condition $\lim_{n\to\infty}\Phi(n)=\infty$. Since (see Fine [1])

$$\frac{1}{n\log n}\sum_{k=1}^{n}V(k)=\frac{1}{4\log 2}+o(1),$$

using (8) and (9) and Cauchy-Schwarz inequality we obtain

$$\begin{aligned}\sup_{\|f\|_{H_1}\le 1}\sum_{k=1}^{2^{n+1}}\frac{\|S_{k,k}f\|_1\Phi(k)}{k\log^2(k+1)}&\ge\sum_{n=2^n+1}^{2^{n+1}}\frac{\|S_{k,k}f_{n,n}\|_1\Phi(k)}{k\log^2(k+1)}\ge\\&\ge\frac{c\Phi(2^n)}{n^2 2^n}\sum_{n=2^n+1}^{2^{n+1}}V^2\left(k-2^n\right)\ge\frac{c\Phi(2^n)}{n^2 2^n}\sum_{k=1}^{2^n}V^2(k)\ge\\&\ge c\Phi(2^n)\left(\frac{1}{n2^n}\sum_{k=1}^{2^n}V(k)\right)^2\ge c\Phi(2^n)\to\infty \text{ as } \ n\to\infty,\end{aligned}$$

which completes the proof of Theorem 1. □

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