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# THERMOSTABILITY OF PRETWISTED SHELLS OF REVOLUTION, CLOSE BY THEIR FORM TO CYLINDRICAL ONES, WITH AN ELASTIC FILLER

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Abstract. The stability of shells of revolution which by their form are close to cylindrical ones, with an elastic filler, under the action of torques, applied to the shell end-walls, external pressure and heating is investigated. Temperature in the shell body is distributed uniformly. The shells of positive and negative gaussian curvature are considered. Formulas for finding critical load and forms of wave formation depending on temperature, rigidity of an elastic base and amplitude of deviation of the shell from cylindrical form, are derived.

**რეზიუმე.** შესწავლილია ბრუნვითი გარსების თერმომდგრადობა, რომლებიც ფორმით მიახლოებულია ცილინდრულთან, დრეკადი შევსებით, მგრეხავი მომენტების, ნორმალური წნევისა და ტემპერატურის მოქმედების ქვეშ. ბრუნვითი გარსები შესწავლილია როგორც დადებითი, ასევე უარყოფითი გაუსის სიმრუდით. მიღებულია ფორმულები კრიტიკული მგრეხავი მომენტების და ტალღური ფორმის განსაზღვრისათვის, რომელიც დამოკიდებულია ტემპერატურისაგან, ნორმალური წნევისაგან, დრეკადი ფუძის სიმტკიცისა და ცილინდრული გარსის გადახრის ამპლიტუდისაგან.

In the present paper we consider the stability of shells of revolution, close by their form to cylindrical ones, with elastic filler, which are under the action of torques $M$ applied to the shell end-walls in terms of uniformly distributed tangential forces, external pressure, uniformly distributed over the whole lateral surface of the shell, and heating. The shell is assumed to be thin and elastic. Temperature in the shell body is distributed uniformly. An elastic filler is modeled by Winkler's base; its extension due to heating is not taken into account. We consider the shells of middle length whose midsurface generatrix is described by a parabolic function. The shells of positive and negative gaussian curvature are also investigated. The boundary conditions at the shell end-walls correspond to a free support admitting certain radial displacement in a subcritical state.

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We consider the shell whose midsurface is formed by the rotation of square parabola about the $z$-axis of the rectangular system of coordinates $x, y, z$ with origin in the midsegment of the axis of revolution (Figure 1). It is assumed that radius $R$ of the midsurface cross-section of the shell is defined by the equality $R = r + \delta_0[1 - \xi^2(r/L)^2]$, where $r$ is radius of the end-wall section, $\delta_0$ is maximal deviation (the shell is convex for $\delta_0 > 0$ and concave for $\delta_0 < 0$), $L = 2\ell$ is the shell length, $\xi = z/r$. It is assumed that

$$(\delta_0/r)^2, \quad (\delta_0/\ell)^2 \ll 1. \tag{1}$$

Temperature in the shell body is assumed to be uniformly distributed. An elastic filler is modeled by Winkler's base and its extension due to heating is not taken into account.

As the basic equations of stability, we have used equations of the theory of shallow shells [1]. For the shells of middle length [2], the form of stability loss goes along with a weakly expressed longitudinal wave formation as compared with a circumferential one, therefore the relation

$$\frac{\partial^2 f}{\partial \xi^2} \ll \frac{\partial^2 f}{\partial \varphi^2} \quad (f = w, \psi), \tag{2}$$

is valid, where $w$ and $\psi$ are the functions, respectively, of radial displacement and stress. As a result, the system of equations for the shell under consideration is reduced to the following equation [7] (by our assumption, temperature terms are equal to zero [6]):

$$\begin{gathered}
\varepsilon \frac{\partial^8 w}{\partial \varphi^8} + \frac{\partial^4 w}{\partial \xi^4} + 4\delta \frac{\partial^4 w}{\partial \xi^2 \partial \varphi^2} + 4\delta^2 \frac{\partial^4 w}{\partial \varphi^4} + \\
+ \frac{\partial^4}{\partial \varphi^4}\left[\frac{\partial}{\partial \xi}\left(t_1^0 \frac{\partial w}{\partial \xi}\right) + \frac{\partial}{\partial \varphi}\left(t_2^0 \frac{\partial w}{\partial \varphi}\right) + \frac{\partial}{\partial \xi}\left(s^0 \frac{\partial w}{\partial \varphi}\right) + \frac{\partial}{\partial \varphi}\left(s^0 \frac{\partial w}{\partial \xi}\right)\right] + \\
+ \overline{\beta} \frac{\partial^4 w}{\partial \varphi^4} = 0, \quad \varepsilon = h^2/12\, r^2(1 - v^2), \quad \overline{\beta} = \beta r^2/Eh,
\end{gathered} \tag{3}$$

$$\delta = \delta_0 r/\ell^2, \quad t_i^0 = T_i^0/Eh \quad (i = 1, 2), \quad s^0 = S^0/Eh, \tag{4}$$

where $E$ is the elasticity modulus and $v$ is the Poisson coefficient of the shell material; $\varphi$ is the angular coordinate; $T_1^0$ and $T_2^0$ are, respectively, axial and circumferential normal contractive forces in a subcritical state; $S^0$ is shearing stress in a subcritical state; $h$ is the shell thickness; $\beta$ is the "bed" coefficient of an elastic filler in a supercritical state. The subcritical state is assumed to be momentless. On the basis of the appropriate solution and inequalities (1) and (2), it is not difficult to show that

$$\begin{gathered}
\frac{\partial}{\partial \xi}\left(T_1^0 \frac{\partial w}{\partial \xi}\right) \ll T_2^0 \frac{\partial^2 w}{\partial \varphi^2}, \\
T_2^0 \approx r(q - \beta_0 w_0), \quad S^0 \approx M/2\pi r^2, \quad q > 0,
\end{gathered} \tag{5}$$

where $q$ is external pressure; $w_0$ and $\beta_0$ are, respectively, deflection and "bed" coefficient of the filler in a subcritical state. Total subcritical displacement equals

$$w_0 = w_{0_q} - w_{0_T}, \tag{6}$$

where $w_{0_q}$ and $w_{0_T}$ are deflections due to the action of pressure and temperature, respectively. They are expressed through the stresses $\sigma_{0_q}$ and $\sigma_{0_T}$ by the formulas

$$w_{0_q} = \frac{\sigma^0_{\varphi_q}(1-v^2)R}{E}, \quad w_{0_T} = \left[\alpha T - \frac{\sigma^0_{\varphi_T}(1-v^2)}{E}\right]R, \tag{7}$$

where $\sigma^0_{\varphi_q}$ is a circumferential normal stress in the shell due to external pressure; $\sigma^0_{\varphi_T}$ is a circumferential normal stress due to temperature and filler constraint; $T$ is temperature; $\alpha$ is the coefficient of linear extension of the shell material. Substituting (7) into (6) and (5), we obtain

$$\begin{gathered} T^0_2 \approx \frac{r}{g}(q + \alpha T r \beta_0), \quad g = 1 + (1-v^2)\frac{\beta_0 r^2}{Eh}, \\ T^0_2 = \sigma^0_\varphi h, \quad \sigma^0_\varphi = \sigma^0_{\varphi_q} + \sigma^0_{\varphi_T}. \end{gathered} \tag{8}$$

In addition, taking into account the fact that $R$ is close to $r$ just as above, in the expression for the force (8) we take $R \approx r$.

Like cylindrical shells, we have considered only the principal boundary conditions whose fulfilment allowed us to get good enough approximation of values of critical load for freely supported edges [2,3].

Taking into account the relations (5) and (8), we find that equation (3) takes the form

$$\begin{aligned} &\varepsilon \frac{\partial^8 w}{\partial \varphi^8} + \frac{\partial^4 w}{\partial \xi^4} + 4\delta \frac{\partial^4 w}{\partial \xi^2 \partial \varphi^2} + \left(4\delta^2 + \frac{r^2\beta}{Eh}\right)\frac{\partial^4 w}{\partial \varphi^4} + \\ &+\left(\frac{qr}{Eh} + \frac{\alpha T r^2 \beta_0}{Eh}\right) g^{-1} \frac{\partial^6}{\partial \varphi^6} - 2s^0 \frac{\partial^6 w}{\partial \xi\, \partial \varphi^5} = 0. \end{aligned} \tag{9}$$

Thus, on the basis of the above-said, determination of forms of stability loss for the shells under consideration is reduced to finding nonzero solutions of equation (9) under the boundary conditions

$$w(\ell/r, \varphi) = w(-\ell/r, \varphi) = 0. \tag{10}$$

A solution will be sought in the form of a series

$$\begin{gathered} w = \sum_{mn} \cos\lambda_m \xi \big(A_{mn} \sin n(\varphi - \gamma\xi) + B_{mn} \cos n(\varphi - \gamma\xi)\big), \\ \lambda_m = \frac{m\pi r}{2\ell} \quad (m = 2i+1, \ \ i = 0,1,2,\dots) \end{gathered} \tag{11}$$

satisfying the given boundary conditions.

We represent the expression (11) as follows:

$$w=\sum_{mn}\frac{A_{mn}}{2}\Big(\sin\big[n(\varphi-\gamma\xi)-\lambda_m\xi\big]+\sin\big[n(\varphi-\gamma\xi)+\lambda_m\xi\big]\Big)+$$
$$+\frac{B_{mn}}{2}\Big(\cos\big[n(\varphi-\gamma\xi)-\lambda_m\xi\big]+\cos\big[n(\varphi-\gamma\xi)+\lambda_m\xi\big]\Big).\qquad(12)$$

Substituting (12) into equation (9), we obtain

$$\sum_{mn}\Big\{A_{mn}F(n,-m)\sin\big[n(\varphi-\gamma\xi)-\lambda_m\xi\big]+$$
$$+A_{mn}F(n,m)\sin\big[n(\varphi-\gamma\xi)+\lambda_m\xi\big]+$$
$$+B_{mn}F(n,-m)\cos\big[n(\varphi-\gamma\xi)-\lambda_m\xi\big]+$$
$$+B_{mn}F(n,m)\cos\big[n(\varphi-\gamma\xi)+\lambda_m\xi\big]\Big\}=0,\qquad(13)$$
$$F(n,\pm m)=\varepsilon n^8+\mu_\pm^4+4\delta\mu_\pm^2n^2+4(\delta^2+\omega/4)n^4-$$
$$-(t^0+\omega_0\alpha T)g^{-1}n^6+2s^0\mu_\pm n^5,$$
$$\mu_\pm=-n\gamma\pm\lambda_m,\quad \omega_0=r^2\beta_0/Eh,\quad \omega=r^2\beta/Eh,$$
$$t^0=qr/Eh,\quad s^0=M/2\pi r^2Eh.$$

From which it follows that

$$A_{mn}\big[F(n,-m)+F(n,m)\big]=0,\quad A_{mn}\big[F(n,-m)-F(n,m)\big]=0,$$
$$B_{mn}\big[F(n,-m)+F(n,m)\big]=0,\quad B_{mn}\big[F(n,-m)-F(n,m)\big]=0.$$

Thus, for the existence of a nontrivial solution of equation (9) under the boundary conditions (10), it is necessary and sufficient that there exist the integers $m$ and $n$ satisfying the conditions

$$F(n,m)=0.\quad F(n,-m)=0.\qquad(14)$$

The relations (14) in expanded form are, in fact, the following conditions:

$$\varepsilon n^4+(-\gamma+m\pi r/nL)^4+4\delta(-\gamma+m\pi r/nL)^2+4(\delta^2+\omega/4)+$$
$$-2s^0(-\gamma+m\pi r/nL)n^2-(t^0+\omega_0\alpha T)g^{-1}n^2=0,\quad t^0=qr/Eh,\qquad(15)$$
$$\varepsilon n^4+(-\gamma-m\pi r/nL)^4+4\delta(-\gamma-m\pi r/nL)^2+4(\delta^2+\omega/4)+$$
$$+2s^0(-\gamma-m\pi r/nL)n^2-(t^0+\omega_0\alpha T)g^{-1}n^2=0,\quad t^0=qr/Eh,\qquad(16)$$

from which it is not difficult to show that the least value $s^0$, depending on $m$, is realized for $m=1$. Therefore, in the sequel we put $m=1$.

Introduce the notation

$$\rho=\lambda_1\varepsilon_*^{-1/4}/n,\quad \theta=\gamma\varepsilon_*^{-1/4},\quad \varepsilon_*=hr/L^2(1-v^2)^{1/2},\qquad(17)$$
$$\lambda_1=\pi r/L,\ \ \delta_*=\delta\varepsilon_*^{-1/2},\ \ \omega_*=\omega\varepsilon_*^{-1},\ \ \omega_0=r\beta_0/Eh,\ \ \omega=r\beta/Eh,$$
$$S=s^0/s_*,\quad Q=t^0/t_0,\quad s_*=0,74(1-v^2)^{-5/8}(h/r)^{5/4}(r/L)^{1/2},\qquad(18)$$

$$t_* = 0,855(1-v^2)^{-3/4}(h/r)^{3/2}r/L, \quad \overline{T} = \big[Q + (\omega_0 \alpha T/t_*)\big]g^{-1}.$$

In what follows, $\overline{T}$ will be called a reduced circumferential force. As a result, equations (15) and (16) can be represented in the following dimensionless form:

$$\frac{\pi^2\rho^{-2}}{12} + \pi^{-2}\big[\rho^2(-\theta \pm \rho)^4 + 4\delta_*\rho^2(-\theta \pm \rho)^2 + 4\overline{\delta}{}_*^{\,2}\rho^2\big] + \\ +1,485\,S(-\theta \pm \rho) - 0,855\,\overline{T} = 0, \quad \overline{\delta}{}_*^{\,2} = \delta_*^2 + \omega_*/4, \tag{19}$$

from which we get the equalities

$$1,48\,S = \frac{\pi^2}{12}\,\frac{\theta}{\rho^2(\theta^2-\rho^2)} + \\ +\pi^{-2}\left[\rho^2\theta(\theta^2+3\rho^2) + 4\delta_*\rho^2\theta + \frac{4\overline{\delta}{}_*^{\,2}\rho^2\theta}{\theta^2-\rho^2}\right] + \frac{0,855\,\theta\overline{T}}{\theta^2-\rho^2}\,, \tag{20}$$

$$\frac{\pi^2}{12}\,\frac{\theta}{\rho^2(\theta^2-\rho^2)} = \pi^{-2}\left[\rho^2\theta(3\theta^2-\rho^2) + 4\delta_*\rho^2\theta - \frac{4\overline{\delta}{}_*^{\,2}\rho^2\theta}{\theta^2-\rho^2}\right] - \frac{0,855\,\theta\overline{T}}{\theta^2-\rho^2}\,. \tag{21}$$

Substituting (21) into (20), we have

$$S = 0,274\,\rho^2\theta(\theta^2+\rho^2+2\delta_*). \tag{22}$$

Equality (21) results in the equation

$$3\Lambda^4 - B\Lambda^2 - C = 0, \quad \Lambda = \theta/\rho, \quad B = 2 - 4\delta_*\rho^{-2}, \tag{23}$$

$$C = 1 + 4\delta_*\rho^{-2} + 4\overline{\delta}{}_*^{\,2}\rho^{-4} + \pi^2\left(\frac{\pi^2\rho^{-8}}{12} - 0,855\,\overline{T}\rho^{-6}\right). \tag{24}$$

Since $\rho > 0$, $\gamma \geq 0$, of our interest are only positive or zero roots of that equation. Depending on the values $B$ and $C$, the positive roots of equation (23) have the form

$$\Lambda_1 = \Big[\big(\sqrt{B^2+12\,C}+B\big)/6\Big]^{1/2}, \quad \Lambda_2 = \Big[\big(B-\sqrt{B^2+12\,C}\big)/6\Big]^{1/2}. \tag{25}$$

Inserting the variable $\Lambda$ into formula (22), we obtain

$$S = 0,274\,\rho^3\Lambda\big[\rho^2(1+\Lambda^2) + 2\delta_*\big]. \tag{26}$$

Substituting the expression $\Lambda_i$ $(i = 1,2)$, according to equalities (25), into formula (26), we obtain $S$ as the function of one dimensionless value $\rho$ and of three dimensionless parameters $\delta_*$, $\omega_*$, $\overline{T}$ ($\overline{T} = (t_0/t_*) + \omega_*\alpha T/t_*$, $t^0 > 0$ is external pressure; $T$ is temperature, $T > 0$). Defining the least value $S$ of $\rho$ for fixed $\delta_*$, $\omega_*$, $\overline{T}$ we obtain the corresponding critical value of $S$.

For $C > 0$, we have $\Lambda_1$ is the real root, whereas $\Lambda_2$ is an imaginary one. For $C = 0$, we have $\Lambda_1 = (B/3)^2$, $\Lambda_2 = 0$ and for $C < 0$, $B > 0$ we have $\Lambda_2 < \Lambda_1$.

Note that for $C = 0$, the least value $S = 0$ results in the root $\Lambda_2 = 0$ (i.e., there is the case of action of one reduced pressure).

Equating the expression (24) for $C$ to zero, we obtain the equation

$$0,855\,\pi^2\overline{T} = \frac{\pi^4}{12}\,N + N^{-3} + 4\,\delta_* N^{-2} + 4\,\overline{\delta}_* N^{-1}, \quad N^{-1} = \rho^2. \qquad (27)$$

The investigation for finding a minimum of the above expression has been performed in [7]. It was found that under the condition $6(\delta_*^2 + \omega_*/4)/\pi^4 \ll 1$ the minimum of the expression (27) is realized for

$$N_\pm = \left(\frac{12}{\pi^4}\right)^{1/4}\left[\sqrt{\sqrt{3} + 0,234\Big(\delta_*^2 + \frac{3}{4}\,\omega_*\Big)} \pm 0,635\,|\delta_*|\right], \qquad (28)$$

where the indices " $\pm$ " and " $-$ " correspond to $\delta_0 > 0$ and $\delta_0 < 0$, respectively. In particular, for $\delta_0 = 0$, $\omega_* = 0$, from which follows the well known formula for a critical number of waves of cylindrical shell [2]. Substituting (28) into formula (27), we obtain critical value for dimensionless reduced pressure.

Moreover, it should be noted that for $\Lambda = 0$, by virtue of (17), $\gamma = 0$, and consequently, the reduced solution (11) transforms into the solution for the freely supported shell.

Figure 1 for $S = 0$ shows dimensionless form of critical values $N_k$ and $\overline{T}_k$ $(k = 1, 2)$ depending on $\delta_*$ for $\omega_{0_*} = \omega_* = 0$ (continuous curves) $(k = 1)$ and for $\omega_{0_*} = \omega_* = 3,816$ (dotted curves) $(k = 2)$.

Figure 2 for $\overline{T} = 0$ shows dimensionless form of critical values $S = s^0/s_*$ depending on $\delta_*$ for the above-considered cases $(k = 1)$ and $(k = 2)$.

In Figure 3, we can see the values $\rho_0$ for which the least value $S$ is realized for those cases.

Further, defining on the basis of formula (26) the least values $S$ of $\rho$ for fixed $\omega_*$, $\delta_*$, $\overline{T}$ we obtain the corresponding critical values $S(\overline{T}, \delta_*, \omega_*)$.

Figure 4 shows curves of critical values $S_k(\overline{T}_k)$ for $\delta_* = 0,\ 0,781,\ -0,781$, $\omega_{0_*} = \omega_* = 0$ and $\omega_{0_*} = \omega_* = 3,816$ which are denoted, respectively, by $0_0$, $1_0$, $2_0$ and 0, 1, 2 (where the index corresponds to the values $\omega_*$ and the number itself corresponds to three values of $\delta_*$). Note that for $\omega_* = \delta_* = T = 0$, $q \neq 0$, $S \neq 0$ the curve $0_0$ coincides practically with the corresponding curve mentioned in [4] for a cylindrical shell.

Thus, given all parameters for the shell, filler, temperature and external pressure, on the basis of the above-obtained formulas and curves, it is not difficult to find the critical shearing force (critical torque).

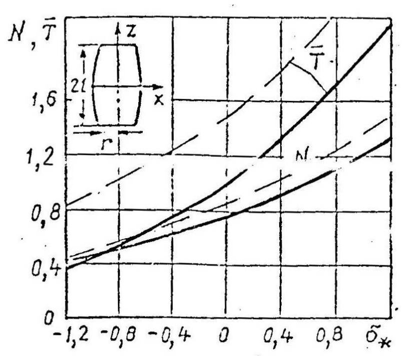


FIGURE 1

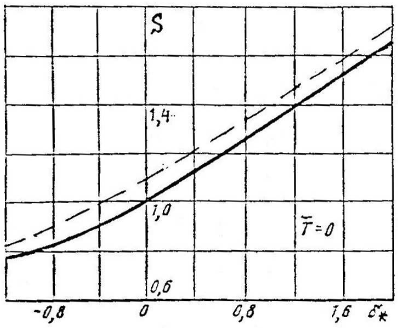


FIGURE 2

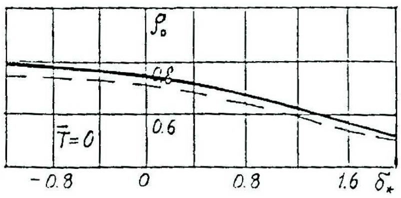


FIGURE 3

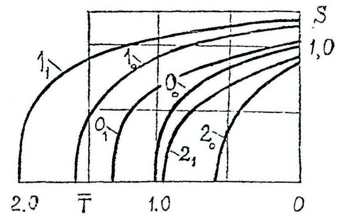


Figure 4

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