Proceedings of A. Razmadze
Mathematical Institute
Vol. **162** (2013), 79–84

A NOTE ON GENERALIZED CAUCHY SINGULAR INTEGRALS IN WEIGHTED VARIABLE EXPONENT LEBESGUE SPACES

V. KOKILASHVILI AND S. SAMKO

Abstract. We prove the boundedness of the Cauchy singular intgeral operator S_Γ and, as a corollary, the boundedness of the generalized singular integral operator, known in the theory of Vekua's generalized analytic functions, in weighted variable exponent spaces $L^{p(\cdot)}_w(\Gamma)$ on Lyapunov curves or curves of bounded rotation without cusps and Muckenhoupt $A_{p(\cdot)}$-weights.

რეზიუმე. ნაშრომში დამტკიცებულია კოშის სინგულარული ინტეგრალური ოპერატორის შემოსაზღვრულობა ცვლადმაჩვენებლიან ლებეგის წონიან სივრცეებში. ამ შედეგზე დაყრდნობით დადგენილია ლიაპუნოვის წირებზე ან უკუქცევის წერტილების გარეშე სასრული ბრუნვის წირებზე განსაზღვრული (ი. ვეკუას აზრით) განზოგადებული სინგულარული ინტეგრალური ოპერატორების შემოსაზღვრულობა ცვლადმაჩვენებლიან ლებეგის წონიან სივრცეებში მაკენჰაუპტის ტიპის $A_{p(\cdot)}$ კლასის წონებით.

1. Introduction

Let Γ be a rectifiable curve in the complex plane with length ℓ, $0<\ell\le\infty$ and $L^{p(\cdot)}_w(\Gamma)$ be the weighted variable exponent Lebesgue space with the norm $\|f\|_{L^{p(\cdot)}_w(\Gamma)}=\|wf\|_{L^{p(\cdot)}(\Gamma)}$, where

$$\|f\|_{L^{p(\cdot)}(\Gamma)}=\inf\left\{\lambda>0:\ \int\limits_0^\ell\left|\frac{f[t(s)]}{\lambda}\right|^{p[t(s)]}ds\le 1\right\}.$$

Everywhere in the sequel we assume that $p(t)$ satisfies the log-condition

$$\big|p(t)-p(\tau)\big|\le\frac{A}{\ln\frac{2}{|t-\tau|}}\quad\text{for all}\quad t,\tau\in\Gamma\quad\text{with}\quad t-\tau|\le 1,$$

and $1<p_-\le p_+<\infty$, where $p_-=\inf_{t\in\Gamma}p(t),\ p_+=\sup_{t\in\Gamma}p(t)$.

Let

$$(S_\Gamma f)(t)=\frac{1}{\pi i}\int\limits_\Gamma\frac{f(\tau)}{\tau-t}d\tau$$

be the Cauchy singular integral operator along a curve Γ.

In the paper [4] there was in fact proved the following result:

If the maximal operator on a Carleson curve Γ of infinite length is bounded in the weighted space $L^{p(\cdot)}_w(\Gamma)$ for some weight w, then the Cauchy singular operator S_Γ is also bounded in the same weighted space.

Note that in formulations of statements in [4] one can see power weights, but with respect to the weight all the proofs in [4] use only the fact that the maximal operator is bounded with this weight; mentioning of powe weights there is due only to the fact that the boundedness of the maximal operator with such weights was known at that time.

Based on our previous results on the boundedness of the maximal operator on Carleson curves in the space $L^{p(\cdot)}_w(\Gamma)$ with power weights

$$w(t) = |t-t_0|^\lambda \prod_{j=1}^{n} |t-t_j|^{\lambda_j}, \quad t_j \in \Gamma, \quad t_0 \notin \Gamma,$$

then in [4] we derived the boundedness of the operator S_Γ in the space $L^{p(\cdot)}_w(\Gamma)$ with such weights. The attention payed there to power weights was also caused by the well known fact that the boundedness of the singular operator namely with power weights plays a crucial role in boundary value problems for analytic and harmonic functions, see e.g. [3].

Later in the paper [5] we proved the boundedness theorem on Carleson curves for the generalized Cauchy singular integrals appeared in the theory of generalized analytic functions in the sense of I. N. Vekua [7], in the case of more general radial oscillating weights.

Recently, in [1] a complete description of weights ensuring the the boundedness of the classical Hardy-Littlewood maximal function in the space $L^{p(\cdot)}_w(\mathbb{R}^n)$. We formulate it below in the form we need to apply.

Theorem A. *The Hardy-Littlewwod operator is bounded in the space $L^{p(\cdot)(\mathbb{R})}_w$ if and only if*

$$\sup \frac{1}{|I|} \left\| w\chi_I \right\|_{L^{p(\cdot)}} \left\| w^{-1}\chi_I \right\|_{L^{p'(\cdot)}} < \infty, \tag{1}$$

where supremum is taken over all intervals $I \subset \mathbb{R}$.

2. The Main Statements

Recall that a curve $\Gamma = \{t = t(s),\ 0 \le s \le \ell\}$ is called a *Lyapunov curve*, if $t'(s) \in \mathrm{Lip}^\gamma$ for some $\gamma \in (0,1)$, and *a curve of bounded rotation*, if $t'(s)$ is a function of bounded variation.

In the sequel we use the notation

$$\widetilde{f}(s) = f[t(s)],\ \widetilde{p}(s) = p\big[t(s),\ \widetilde{w}(s) = w[t(s)]\big]$$

for brevity.

Theorem 1. *Let* Γ *be a closed Lyapunov curve or a curve of bounded rotation without cusps. Then the singular integral operator* S_Γ *is bounded in the space* $L_w^{p(\cdot)}(\Gamma)$*, if the weight* $w[t(s)]$, $0 \le s \le \ell < \infty$, *is a restriction of a weight on* $\mathbb{R}$, *satisfying the condition* (1).

Proof. It is well known that for Lyapunov curves and curves of bounded rotation without cusps we have (see [2], p.20)

$$\big|t(s) - t(\sigma)\big| \approx |s - \sigma|. \tag{2}$$

Let us first prove that the singular operator

$$(S_\Gamma f)(t) = \frac{1}{\pi i} \int\limits_\Gamma \frac{f(\tau)}{\tau - t} d\tau$$

is bounded in $L_w^{p(\cdot)}(\Gamma)$. It is known that

$$\frac{t'(\sigma)}{t(\sigma) - t(s)} = \frac{1}{\sigma - s} + O\left(\frac{1}{|\sigma - s|^{1-\gamma}}\right)$$

on Lyapunov curves. To derive the weighted boundedness of the singular operator

$$(H\widetilde{f})(s) = \frac{1}{\pi i} \int\limits_0^\ell \frac{\widetilde{f}(\sigma)}{\sigma - s} d\sigma,$$

from the boundedness of the weighted maximal operator, we continue the exponent $\widetilde{p}(s)$ beyond the interval $[0, \ell]$ by a constant value $\widetilde{p}(0) = \widetilde{p}(\ell)$. Then te arguments follow the known scheme via the norm inequality

$$\|\widetilde{f}\,\|_{L_{\widetilde{w}}^{\widetilde{p}(\cdot)}(\mathbb{R})} \le c \big\|\mathcal{M}^{\#} \widetilde{f}\,\big\|_{L_{\widetilde{w}}^{\widetilde{p}(\cdot)}(\mathbb{R})} \tag{3}$$

and the pointwise estimate

$$\mathcal{M}^{\#}\big(\big|H\widetilde{f}\,\big|^r\big)(s) \le c\big[Mf(s)\big]^r, \quad 0 < r < 1, \tag{4}$$

assuming that $\widetilde{f}(s)$ is extended by zero beyond $[0, \ell]$, where $\mathcal{M}^{\#} f$ is the sharp maximal function (see details e.g. in [4], the proof of Theorem A).

The term

$$T\widetilde{f}(s) = \int\limits_0^\ell \frac{|\widetilde{f}(\sigma)|}{|\sigma - s|^{1-\gamma}} d\sigma$$

with potential kernel, is also controlled by the maximal function:

$$l\big|(T\widetilde{f})(s)\big| \le CM\widetilde{f}(s). \tag{5}$$

Indeed, with $\widetilde{f}_0(\sigma) = \widetilde{f}(\sigma)$ for $\sigma \in [0,\ell]$ and $\widetilde{f}_0(\sigma) = 0$ for $\sigma \notin [0,\ell]$ we have

$$\int\limits_0^\ell \frac{|\widetilde{f}(\sigma)|d\sigma}{|\sigma - s|^{1-\gamma}} = \int\limits_{-\ell}^{2\ell} \frac{|\widetilde{f}_0(\sigma)|d\sigma}{|\sigma - s|^{1-\gamma}} = \sum_{j=0}^{\infty} \int\limits_{\frac{\ell}{2^{j+1}}<|\sigma-s|<\frac{\ell}{2^j}} \frac{|\widetilde{f}_0(\sigma)d\sigma|}{|\sigma - s|^{1-\gamma}} \le$$
$$\le \sum_{j=0}^{\infty} \frac{1}{\left(\frac{\ell}{2^{j+1}}\right)^{1-\gamma}} \int\limits_{|\sigma-s|<\frac{\ell}{2^j}} |\widetilde{f}_0(\sigma)|d\sigma \le CM\widetilde{f}_0(s).$$

Then by (5) and Theorem A,

$$\big\|T\widetilde{f}\big\|_{L^{\widetilde{p}(\cdot)}_{\widetilde{w}}(0.\ell)} \le C\big\|T\widetilde{f}_0\big\|_{L^{\widetilde{p}(\cdot)}_{\widetilde{w}}(0,\ell)} \le C\big\|M\widetilde{f}_0\big\|_{L^{\widetilde{p}(\cdot)}_{\widetilde{w}}(0,\ell)} \le$$
$$\le C\big\|M\widetilde{f}_0\big\|_{L^{\widetilde{p}(\cdot)}_{\widetilde{w}}(\mathbb{R})} \le C\big\|\widetilde{f}_0\big\|_{L^{\widetilde{p}(\cdot)}(0,\ell)}.$$

This competes the proof of the boundedness of the Cauchy singular operator S_Γ in the space $L^{p(\cdot)}_w(\Gamma)$ in the case of Lyapunov curves.

The boundedness of the maximal singular operator S^*_Γ in $L^{p(\cdot)}_w(\Gamma)$ for Lyapunov curves follows then from the known pointwise inequality

$$(S^*_\Gamma)(t) \le CM(S_\Gamma f)(t) + C(Mf)(t). \tag{6}$$

Let now Γ be a curve of bounded rotation and V be the total variation of the function $t'(s)$ on $[0,\ell]$. Then we have

$$\left|\int\limits_{|\sigma-s|>\varepsilon} \frac{\widetilde{f}(\sigma)\; t'(\sigma)}{t(\sigma)-t(s)} d\sigma\right| \le C \left|\int\limits_{|\sigma-s|>\varepsilon} \frac{\widetilde{f}(\sigma)}{\sigma - s} d\sigma\right| +$$
$$+ \int\limits_{|\sigma-s|>\varepsilon} |\widetilde{f}(\sigma)| \cdot \frac{V(\sigma)-V(s)}{\sigma - s} d\sigma, \tag{7}$$

which follows from the relation

$$\frac{t'(\sigma)}{t(\sigma)-t(s)} - \frac{1}{\sigma - s} = \frac{1}{t(\sigma)-t(s)} \int\limits_0^1 \big[t'(\sigma) - t'(s + \xi(\sigma - s))\big] d\xi,$$

in view of (2). From (7) we obtain the pointwise estimate

$$(S^*_\Gamma)(t) \le c\Big[\big(H^* f\big)(s) + V(s)\big(H^*|\widetilde{f}\,|\big)(s) + \big(H^* V|\widetilde{f}\,|\big)(s)\Big],$$

where

$$(H^*\widetilde{f})(s) = \sup_{\varepsilon>0} \int\limits_{|\sigma-s|>\varepsilon} \frac{\widetilde{f}(\sigma)d\sigma}{\sigma - s}$$

is the maximal Hilbert singular operator on $[0,\ell]$. By the above obtained boundedness of the maximal singular operator (see (6)), the operator H^* is bounded in the space $L^{\widetilde{p}(\cdot)}_{\widetilde{w}}(0,\ell)$. Consequently, the operator S^*_Γ is bounded in the space $L^{p(\cdot)}_w(\Gamma)$. □

The above proved theorem may be applied to obtain teh boundedness of the generalized singular integral operator, widely used in the theory of Vekua's generalized analytic functions ([6]-[7]). Generalized analytic functions in the sense of I. N. Vekua, are regular solutions of the equation

$$\partial_{\bar z}\Phi(z) + A(z)\Phi(z) + B(z)\overline{\Phi}(z) = 0 \tag{8}$$

where $\partial_{\bar z} = \frac{1}{2}\big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\big)$ and $A, B \in L^r(G),\ r > 2$. Recall that the integral It is known that the integral

$$\Phi(z) = \frac{1}{2\pi i}\int\limits_\Gamma \Omega_1(z,\tau)f(\tau)\,d\tau - \Omega_2(z,\tau)\overline{f}(\tau)\,d\overline{\tau}, \qquad f \in L^1(\Gamma), \tag{9}$$

where Ω_1 and Ω_2 are the so called basic normalized kernels, is a regular solution of (8), see details in [7], [6]. The integral in (9) is called the generalized Cauchy type integral. The corresponding generalized singular integral is introduced as

$$\widetilde{S}_\Gamma f(t) = \lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int\limits_{\Gamma_\varepsilon} \Omega_1(t,\tau)f(\tau)\,d\tau - \Omega_2(t,\tau)\overline{f}(\tau)\,d\overline{\tau} \tag{10}$$

and the corresponding maximal singular operator is

$$(\mathbb{S}^*_\Gamma f)(t) = \sup_{\varepsilon>0}\frac{1}{2\pi}\Bigg|\int\limits_{|\tau-t|>\varepsilon} \Omega_1(t,\tau)f(\tau)\,d\tau - \Omega_2(t,\tau)\overline{f}(\tau)\,d\overline{\tau}\Bigg|.$$

Theorem 2. *Let Γ be a closed Lyapunov curve or a curve of bounded rotation without cusps. Then the generalized maximal singular integral operator $\mathbb{S}^*_\Gamma$ is bounded in the space $L^{p(\cdot)}_w(\Gamma)$, if the weight $w[t(s)],\ 0 \le s \le \ell < \infty$, is a restriction of a weight on $\mathbb{R}$, satisfying the condition* (1).

Proof. To obtain the boundedness of the generalized Cauchy singular integral $\mathbb{S}^*$, it suffices to use of the known representaions

$$\Omega_1(t,\tau) = \frac{1}{\tau - t} + O\left(\frac{1}{|\tau - t|^\alpha}\right), \qquad \Omega_2(t,\tau) = O\left(\frac{1}{|\tau - t|^\alpha}\right)$$

of the functions $\Omega_1(t,\tau)$ and $\Omega_2(t,\tau)$, where $0 < \alpha < 1$, see [7]. □

Open problem. Let $A_{p(\cdot)}(a,b)$ be the class of weights w satisfying the condition (1) where sup is taken with respect to all intervals $I \subset [a,b]$, $-\infty < a < b < \infty$. Is every weight $w \in A_{p(\cdot)}(a,b)$ a restriction of a certain weight in $A_{p(\cdot)}(\mathbb{R})$?

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(Received 26.02.2013)

Authors' addresses:

V. Kokilashvili

A. Razmadze Mathematical Institute
I. Javakhishvili Tbilisi State University
6, Tamarashvili St., Tbilisi 0177, Georgia
E-mail: kokil@rmi.ge

S. Samko

Universidade do Algarve, FCT
Campus de Gambelas, Faro, 8000, Portugal
E-mail: ssamko@ualg.pt