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# LOCALIZED BOUNDARY-DOMAIN INTEGRAL EQUATIONS APPROACH FOR ROBIN TYPE PROBLEM FOR SECOND ORDER STRONGLY ELLIPTIC SYSTEMS WITH VARIABLE COEFFICIENTS

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Abstract. The paper deals with the three–dimensional Robin type boundary-value problem (BVP) for a second order strongly elliptic system of partial differential equations in the divergence form with variable coefficients and develops the generalized potential method based on the localized parametrix method. Using Green's third representation formula and properties of the localized layer and volume potentials we reduce the Robin type BVP to the localized boundary-domain integral equations (LBDIE) system. The equivalence between the Robin type boundary value problem and the LBDIE system is studied. It is established that the localized boundary-domain integral operator obtained belongs to the Boutet de Monvel algebra and with the help of the Wiener-Hopf factorization method, we investigate corresponding Fredholm properties and prove invertibility of the localized operator in appropriate function spaces.

**რეზიუმე.** ნაშრომი ეძღვნება ლოკალიზებული პარამეტრიქსის მეთოდის განვითარებას სამგანზომილებიანი მეორე რიგის ძლიერად ელიფსური ცვლადკოეფიციენტებიანი დივერგენციული ფორმით ჩაწერილი დიფერენციალურ განტოლებათა სისტემისათვის დასმული რობენის ტიპის სასაზღვრო ამოცანის შემთხვევაში. გრინის წარმოდგენის ფორმულისა და ლოკალიზებული პოტენციალების თვისებების გამოყენებით ამოცანა დაიყვანება ლოკალიზებულ სასაზღვრო-სივრცულ ინტეგრალურ განტოლებათა სისტემაზე. შესწავლილია რობენის ტიპის სასაზღვრო ამოცანისა და მიღებულ ლოკალიზებულ სასაზღვრო-სივრცულ ინტეგრალურ განტოლებათა სისტემის ეკვივალენტობა. ვინერ-ჰოფის ფაქტორიზაციის მეთოდის გამოყენებით ნაჩვენებია, რომ ლოკალიზებული სასაზღვრო-სივრცული ინტეგრალური ოპერატორი, რომელიც ეკუთვნის ბუტე დე მონველის ალგებრას, არის ფრედჰოლმური და დადგენილია მისი შებრუნებადობა შესაბამის სივრცეებში.

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## 1. Introduction

We consider the Robin type boundary-value problem (BVP) for second order strongly elliptic systems of partial differential equations in the divergence form with *variable coefficients* and develop the generalized potential method based on the *localized parametrix method.*

The BVP treated in the paper is well investigated in the scientific literature by the variational and also by the usual classical potential methods when the corresponding fundamental solution is available in explicit form, e.g. in the case of constant coefficients (see, e.g., [18], [22], [24]).

Our goal here is to show that solutions of the problem can be represented by *localized potentials* and that the corresponding *localized boundary-domain integral operator* (LBDIO) is invertible, which seems very important from the point of view of numerical analysis, since they lead to very convenient numerical schemes in applications (for details see [25], [29], [30], [32], [33]).

The LBDIE approach for the Dirichlet type BVP for the second order strongly elliptic systems of partial differential equations is analyzed in [14].

Using Green's representation formula and properties of the localized layer and volume potentials we reduce the Robin type BVP to the *localized boundary-domain integral equations* (*LBDIE*) *system.* First we establish the equivalence between the original boundary value problem and the corresponding LBDIE system which proved to be a quite nontrivial problem and plays a crucial role in our analysis. Afterwards we establish that the localized boundary domain integral operator obtained belongs to the Boutet de Monvel algebra of pseudo-differential operators and with the help of the Vishik-Eskin theory, based on the factorization method (Wiener-Hopf method), we investigate corresponding Fredholm properties and prove invertibility of the localized operator in appropriate function spaces. This paper develops methods and results of [6]–[15], [26].

## 2. Reduction to LBDIE System and the Equivalence Theorems

2.1. **Formulation of the boundary value problem and Green's third formula.** Consider a self-adjoint uniformly strongly elliptic second order matrix partial differential operator

$$A(x,\partial_x)=\big[A_{pq}(x,\partial_x)\big]_{3\times 3}=\bigg[\frac{\partial}{\partial x_k}\Big(a_{kj}^{pq}(x)\,\frac{\partial}{\partial x_j}\Big)\bigg]_{3\times 3}, \tag{2.1}$$

where $\partial_x=(\partial_1,\partial_2,\partial_3)$, $\partial_j=\partial_{x_j}=\partial/\partial x_j$, $a_{kj}^{pq}=a_{jk}^{qp}=a_{pj}^{kq}\in C^\infty$, $j,k,p,q=1,2,3$. Here and in what follows by repeated indices summation from 1 to 3 is meant if not otherwise stated.

We assume that the coefficients $a_{kj}^{pq}$ are real and the quadratic from $a_{kj}^{pq}(x)\,\eta_{kj}\,\eta_{pq}$ is uniformly positive definite with respect to symmetric variables $\eta_{kj}=\eta_{jk}\in\mathbb{R}$, which implies that the principal homogeneous symbol of the operator $A(x,\partial_x)$ with opposite sign, $A(x,\xi)=[a_{kj}^{pq}(x)\xi_k\,\xi_j]_{3\times 3}$ is uniformly positive definite, i.e. there are positive constants $c_1$ and $c_2$ such that

$$c_1\,|\xi|^2|\zeta|^2\le \big(A(x,\xi)\zeta\,,\,\zeta\big)\le c_2\,|\xi|^2|\zeta|^2,\ \ \forall\, x\in\mathbb{R}^3,\ \ \forall\,\xi\in\mathbb{R}^3,\ \ \forall\,\zeta\in\mathbb{C}^3, \quad (2.2)$$

where $(\cdot\,,\,\cdot)$ denotes the usual scalar product in $\mathbb{C}^3$.

Further, let $\Omega^+$ be a bounded domain in $\mathbb{R}^3$ with a simply connected boundary $\partial\Omega^+=S\in C^\infty$, $\overline{\Omega^+}=\Omega^+\cup S$. Throughout the paper $n=(n_1,n_2,n_3)$ denotes the unit normal vector to $S$ directed outward the domain $\Omega^+$. Set $\Omega^-:=\mathbb{R}^3\setminus\overline{\Omega^+}$.

By $H^r(\Omega)=H_2^r(\Omega)$ and $H^r(S)=H_2^r(S)$, $r\in\mathbb{R}$, we denote the Bessel potential spaces on a domain $\Omega$ and on a closed manifold $S$ without boundary, while $\mathcal{D}(\mathbb{R}^3)$ stands for $C^\infty$ functions in $\mathbb{R}^3$ with compact support and $\mathcal{S}(\mathbb{R}^3)$ denotes the Schwartz space of rapidly decreasing functions in $\mathbb{R}^3$. Recall that $H^0(\Omega)=L_2(\Omega)$ is a space of square integrable functions in $\Omega$.

For a vector $u=(u_1,u_2,u_3)^\top$ the inclusion $u=(u_1,u_2,u_3)^\top\in H^r$ means $u_j\in H^r$.

Let us denote $u^\pm\equiv\{u\}^\pm=\gamma^\pm u$, where $\gamma^+$ and $\gamma^-$ are the trace operators on $S$ from the interior and exterior of $\Omega^+$ respectively. We also need the following subspace of $H^1(\Omega)$,

$$H^{1,0}(\Omega;A):=\big\{u=(u_1,u_2,u_3)^\top\in H^1(\Omega)\ :\ A(x,\partial)u\in H^0(\Omega)\big\}\,. \quad (2.3)$$

The *co-normal derivative operator* associated with the differential operator $A(x,\partial_x)$ reads as

$$T(x,\partial_x)=\big[T_{pq}(x,\partial_x)\big]_{3\times3}:=\big[\,a_{kj}^{pq}(x)\,n_k(x)\,\partial_{x_j}\,\big]_{3\times 3}. \quad (2.4)$$

For a smooth vector-function $u$, say $u\in H^2(\Omega^+)$, we have

$$\begin{aligned}\big[\,T^\pm(x,\partial_x)\,u(x)\,\big]_p&=\big[\,\{T(x,\partial_x)\,u(x)\}^\pm\,\big]_p:=\\&=a_{kj}^{pq}(x)\,n_k(x)\,\{\partial_{x_j}u_q(x)\}^\pm,\quad x\in S,\quad p=1,2,3,\end{aligned} \quad (2.5)$$

which is understood in the usual traces sense.

Note that due to the decomposition $\partial_{x_j}=n_j(x)\partial_n+\mathcal{D}_j$, where $x\in S$, $\partial_n$ is the normal derivative, and $\mathcal{D}_j$ is the Stokes-Günter tangential derivative operator (see, e.g., [17]), we can represent the co-normal derivative operator (2.4) as

$$\begin{aligned}T(x,\partial_x)=\big[T_{pq}(x,\partial_x)\big]_{3\times3}&=\big[a_{kj}^{pq}(x)\,n_k(x)\,n_j(x)\big]_{3\times3}\,\partial_n+\\&\quad+\big[a_{kj}^{pq}(x)\,n_k(x)\,\mathcal{D}_j\big]_{3\times3}.\end{aligned} \quad (2.6)$$

In our analysis we need also the following boundary operator depending on the parameter $t\in[0,1]$,

$$T_t(x,\partial_x)=\big[a_{kj}^{pq}(x)\,n_k(x)\,n_j(x)\big]_{3\times3}\,\partial_n+t\,\big[a_{kj}^{pq}(x)\,n_k(x)\,\mathcal{D}_j\big]_{3\times3}. \quad (2.7)$$

Clearly, the matrix $\big[a_{kj}^{pq}(x)\, n_k(x)\, n_j(x)\big]_{3\times 3} = A(x,n(x))$ is positive definite, in particular,

$$\det \big[a_{kj}^{pq}(x)\, n_k(x)\, n_j(x)\big]_{3\times 3} > 0, \quad \forall x \in S. \tag{2.8}$$

The co-normal derivative operator defined in (2.5) can be extended by continuity to the space $H^{1,0}(\Omega^+;A)$ with the help of Green's first identity,

$$\langle T^+ u\,,\, g\rangle_S := \int\limits_{\Omega^+} A(x,\partial_x)\, u(x)\, v(x)\, dx + \int\limits_{\Omega^+} E(u(x),v(x))\, dx, \tag{2.9}$$

where $E(u(x),v(x)) = a_{kj}^{pq}(x)\, \partial_{x_j} u_q(x)\, \partial_{x_k} v_p(x)$, $g \in H^{1/2}(S)$ is an arbitrary vector-function and $v \in H^1(\Omega^+)$ is an arbitrary extension of $g$ from $S$ onto the whole of $\Omega^+$, i.e., $v^+ = g$ on $S$, while $\langle\cdot\,,\,\cdot\rangle_S$ denotes the duality between the adjoint spaces $H^{-\frac12}(S)$ and $H^{\frac12}(S)$ which extends the usual bilinear $L_2(S)$ inner product. The definition (2.9) does not depend on the extension operator.

The Robin type boundary-value problem reads as follows:

Find a vector-function $u = (u_1,u_2,u_3)^\top \in H^{1,0}(\Omega^+,A)$ satisfying the differential equation

$$A(x,\partial_x)u = f \quad \text{in} \quad \Omega^+ \tag{2.10}$$

and the Robin type boundary condition

$$T^+u + \beta u^+ = \psi_0 \quad \text{on} \quad S, \tag{2.11}$$

where $\psi_0 = (\psi_{01},\psi_{02},\psi_{03})^\top \in H^{-1/2}(S)$, $f = (f_1,f_2,f_3)^\top \in H^0(\Omega^+)$ and $\beta = [\beta_{jk}]_{3\times 3}$ is a positive definite constant matrix.

Equation (2.10) is understood in the distributional sense, while the Robin type boundary condition (2.11) is understood in the functional sense defined in (2.9).

*Remark* 2.1. From the condition (2.2) it follows that the quadratic form

$$E(u(x),u(x)) = a_{kj}^{pq}(x)\, \varepsilon_{qj}(x)\, \varepsilon_{pk}(x) \quad \text{with} \quad \varepsilon_{qj}(x) = 2^{-1}\big(\partial_j u_q(x) + \partial_q u_j(x)\big)$$

is positive definite in the symmetric variables $\varepsilon_{qj}$. Therefore Green's first formula (2.9) along with the Lax-Milgram lemma imply that the above formulated Robin type BVP is uniquely solvable in the space $H^{1,0}(\Omega^+;A)$ (see, e.g., [22], [18], [24]).

Let us define the following classes of cut-off functions (see [9]).

**Definition 2.2.** We say $\chi \in X^k$ for integer $k \geq 0$ if $\chi(x) = \breve\chi(|x|)$, $\breve\chi \in W_1^k(0,\infty)$ and $\varrho\breve\chi(\varrho) \in L_1(0,\infty)$. We say $\chi \in X_+^k$ for integer $k \geq 1$ if

$\chi \in X^k$, $\chi(0) = 1$ and $\sigma_\chi(\omega) > 0$ for all $\omega \in \mathbb{R}$, where

$$\sigma_\chi(\omega) := \begin{cases} \dfrac{\hat{\chi}_s(\omega)}{\omega} > 0 & \text{for} \quad \omega \in \mathbb{R} \setminus \{0\}, \\ \displaystyle\int\limits_0^\infty \varrho \breve{\chi}\,(\varrho)\, d\varrho & \text{for} \quad \omega = 0, \end{cases} \tag{2.12}$$

and $\hat{\chi}_s(\omega)$ denotes the sine-transform of the function $\breve{\chi}$

$$\hat{\chi}_s(\omega) := \int\limits_0^\infty \breve{\chi}\,(\varrho)\, \sin(\varrho\,\omega)\, d\varrho. \tag{2.13}$$

We say $\chi \in X^k_{1+}$ for integer $k \geq 1$ if $\chi \in X^k_+$ and

$$\omega \hat{\chi}_s(\omega) \leq 1, \quad \forall\, \omega \in \mathbb{R}. \tag{2.14}$$

Evidently, we have the following imbeddings: $X^{k_1} \subset X^{k_2}$ and $X^{k_1}_+ \subset X^{k_2}_+$, $X^{k_1}_{1+} \subset X^{k_2}_{1+}$ for $k_1 > k_2$. The class $X^k_+$ is defined in terms of the sine-transform. The following lemma provides an easily verifiable sufficient condition for non-negative non-increasing functions to belong to this class (for details see [9]).

**Lemma 2.3.** *Let $k \geq 1$. If $\chi \in X^k$, $\breve{\chi}(0) = 1$, $\breve{\chi}(\varrho) \geq 0$ for all $\varrho \in (0, \infty)$, and $\breve{\chi}$ is a non-increasing function on $[0, +\infty)$, then $\chi \in X^k_+$.*

The following examples for $\chi$ are presented in [9],

$$\chi_1(x) = \begin{cases} \left[1 - \dfrac{|x|}{\varepsilon}\right]^k & \text{for} \quad |x| < \varepsilon, \\ 0 & \text{for} \quad |x| \geq \varepsilon, \end{cases} \tag{2.15}$$

$$\chi_2(x) = \begin{cases} \exp\left[\dfrac{|x|^2}{|x|^2 - \varepsilon^2}\right] & \text{for} \quad |x| < \varepsilon, \\ 0 & \text{for} \quad |x| \geq \varepsilon, \end{cases} \tag{2.16}$$

One can observe that $\chi_1 \in X^k_+$, while $\chi_2 \in X^\infty_+$ due to Lemma 2.3.

Moreover, $\chi_1 \in X^k_{1+}$ for $k = 2$ and $k = 3$, while $\chi_1 \notin X^1_{1+}$ and $\chi_2 \notin X^\infty_{1+}$ (for details see [9]).

Define a *localized matrix parametrix* corresponding to the fundamental solution $F_1(x) := -[\,4\,\pi\,|x|\,]^{-1}$ of the Laplace operator, $\Delta = \partial_1^2 + \partial_2^2 + \partial_3^2$,

$$P(x) \equiv P_\chi(x) := F_\chi(x)\, I = \chi(x)\, F_1(x)\, I = -\frac{\chi(x)}{4\,\pi\,|x|}\, I \quad \text{with} \quad \chi(0) = 1, \tag{2.17}$$

where $I$ is the identity $3 \times 3$ matrix and $\chi$ is a localizing function

$$\chi \in X^k_{1+}, \quad k \geq 4. \tag{2.18}$$

Throughout the paper we assume that the condition (2.18) is satisfied and $\chi$ has a compact support if not otherwise stated.

Further we introduce the singular integral operator, in the sense of Cauchy principal value,

$$\mathcal{A}\, u(y) := \text{v.p.} \int\limits_{\Omega^+} [\, A(x, \partial_x) P(x-y)\,]\, u(x)\, dx. \tag{2.19}$$

If the domain of integration in (2.19) is the whole space $\mathbb{R}^3$, we employ the notation $\mathcal{A}\, u \equiv \mathbf{A}\, u$,

$$\mathbf{A}\, u(y) := \text{v.p.} \int\limits_{\mathbb{R}^3} [\, A(x, \partial_x) P(x-y)\,]\, u(x)\, dx\,. \tag{2.20}$$

For $u \in H^{1,0}(\Omega^+, A)$ the following representation formula holds

$$\begin{aligned} \mathbf{b}(y)\, u(y) + \mathcal{A}\, u(y) - V(T^+ u)(y) + W(u^+)(y) = \\ = \mathcal{P}\big(A(x, \partial_x) u\big)(y), \quad y \in \Omega^+, \end{aligned} \tag{2.21}$$

where $\mathcal{A}$ is a *localized singular integral operator* given by (2.19), while $V$, $W$, and $\mathcal{P}$ are the *localized single layer, double layer and Newtonian volume potentials*,

$$V(g)(y) := -\int\limits_{S} P(x-y)\, g(x)\, dS_x, \tag{2.22}$$

$$W(g)(y) := -\int\limits_{S} \big[\, T(x, \partial_x)\, P(x-y)\,\big]^\top\, g(x)\, dS_x, \tag{2.23}$$

$$\mathcal{P}(h)(y) := \int\limits_{\Omega^+} P(x-y)\, h(x)\, dx. \tag{2.24}$$

If the domain of integration in the Newtonian volume potential (2.24) is the whole space $\mathbb{R}^3$, we employ the notation $\mathcal{P}\, h \equiv \mathbf{P}\, h$,

$$\mathbf{P}(h)(y) := \int\limits_{\mathbb{R}^3} P(x-y)\, h(x)\, dx. \tag{2.25}$$

Mapping properties of the above potentials are investigated in [9].

Denote by $\ell_0$ the extension operator by zero from $\Omega^+$ onto $\Omega^-$. It is evident that for a function $u \in H^1(\Omega^+)$ we have

$$\big(\mathcal{A}\, u\big)(y) = \big(\mathbf{A} \ell_0 u\big)(y) \quad \text{for} \quad y \in \Omega^+.$$

Introduce the notation

$$(\mathbf{K} \ell_0 u)(y) := (\, \mathbf{b}(y) - \mathbf{I}\,)\, u(y) + \big(\mathbf{A} \ell_0 u\big)(y) \quad \text{for} \quad y \in \Omega^+, \tag{2.26}$$

and rewrite Green's third formula (2.21) in a more convenient form for our further purposes

$$[\,\mathbf{I}+\mathbf{K}\,]\,\ell_0 u(y) - V(T^+u)(y) + W(u^+)(y) = \\ = \mathcal{P}\big(A(x,\partial_x)u\big)(y), \quad y\in\Omega^+, \tag{2.27}$$

where $\mathbf{I}$ is the identity operator.

The principal homogeneous symbols of the singular integral operators $\mathbf{K}$ and $\mathbf{I}+\mathbf{K}$ read as

$$\mathfrak{S}(\mathbf{K})(y,\xi) = |\xi|^{-2}A(y,\xi) - I, \quad \forall\, y\in\overline{\Omega^+}, \quad \forall\,\xi\in\mathbb{R}^3\setminus\{0\}, \tag{2.28}$$

$$\mathfrak{S}(\mathbf{I}+\mathbf{K})(y,\xi) = |\xi|^{-2}A(y,\xi), \quad \forall\, y\in\overline{\Omega^+}, \quad \forall\,\xi\in\mathbb{R}^3\setminus\{0\}. \tag{2.29}$$

It is evident that the symbol matrix (2.29) is positive definite due to (2.2),

$$\big(\mathfrak{S}(\mathbf{I}+\mathbf{K})(y,\xi)\,\zeta,\zeta\big) = |\xi|^{-2}\big(A(y,\xi)\,\zeta,\zeta\big) \geq c_1\,|\zeta|^2, \tag{2.30}$$

$$\forall\, y\in\overline{\Omega^+}, \quad \forall\,\xi\in\mathbb{R}^3\setminus\{0\}, \quad \forall\,\zeta\in\mathbb{C}^3, \tag{2.31}$$

where $c_1$ is the same positive constant as in (2.2). If $\chi\in X^k$ with integer $k\geqslant r+2$, then

$$r_{_{\Omega^+}}\mathbf{K}\,\ell_0\;:\;H^r(\Omega^+)\to H^r(\Omega^+), \quad r\geqslant 0 \tag{2.32}$$

is bounded, since the symbol (2.28) is rational (see, e.g., [2], [18, Theorem 8.6.1]). Here and throughout the paper $r_{_\Omega}$ denotes the restriction operator to $\Omega$.

Assuming that $u\in H^2(\Omega^+)$ and applying the differential operator $T(x,\partial)$ to Green's formula (2.27) and using the properties of localized potentials we arrive at the relation:

$$(T\mathbf{K})^+\ell_0 u + (\,\mathbf{I}-\mathbf{d})(T^+u) - \mathcal{W}'(T^+u) + \mathcal{L}(u^+) = \\ = (T\mathcal{P})^+\big(A(x,\partial_x)u\big) \quad \text{on} \quad S, \tag{2.33}$$

where the localized boundary integral operators $\mathcal{W}'$ and $\mathcal{L}$ are defined as follows

$$\mathcal{W}'\,g(y) := -\int\limits_S \big[\,T(y,\partial_y)\,P(x-y)\,\big]\,g(x)\,dS_x, \quad y\in S,$$

$$\mathcal{L}g(y) := [T(y,\partial_y)\,Wg(y)]^+, \quad y\in S,$$

while

$$(T\mathbf{K})^+\ell_0 u \equiv \gamma^+T(\mathbf{K}\ell_0 u) := \{\,T(\mathbf{K}\,\ell_0 u)\,\}^+ \quad \text{on} \quad S,$$

$$\mathcal{P}^+(f) \equiv \gamma^+\mathcal{P}(f) := \{\,\mathcal{P}(f)\}^+ \quad \text{on} \quad S,$$

$$\mathbf{d}(y) = [\mathbf{d}^{pq}(y)]_{3\times 3} := \frac{1}{2}\,\big[\,a_{kj}^{pq}(y)\,n_k(y)\,n_j(y)\,\big]_{3\times 3}, \quad y\in S.$$

2.2. **LBDIE formulation of the Robin type problem. Equivalence theorem.** Let $u \in H^2(\Omega^+)$ be a solution to the Robin type BVP (2.10)–(2.11) with $\psi_0 \in H^{\frac{1}{2}}(S)$ and $f \in H^0(\Omega^+)$. As we have derived above there hold the relations (2.27) and (2.33), which now can be rewritten in the form

$$[\mathbf{I}+\mathbf{K}]\,\ell_0 u + W(\varphi) + V(\beta\varphi) = \mathcal{P}(f) + V(\psi_0) \quad \text{in} \quad \Omega^+, \tag{2.34}$$

$$(T\mathbf{K})^+\ell_0 u + \mathcal{L}(\varphi) + (\,\mathbf{d}-\mathbf{I}\,)\,\beta\,\varphi + \mathcal{W}'\beta\varphi = \\ = (T\mathcal{P})^+(f) + (\mathbf{d}-\mathbf{I})\psi_0 + \mathcal{W}'(\psi_0) \quad \text{on} \quad S, \tag{2.35}$$

where $\varphi := u^+ \in H^{\frac{3}{2}}(S)$.

One can consider these relations as a LBDIE system with respect to the segregated unknown vector-functions $u$ and $\varphi$. Now we prove the following equivalence theorem.

**Theorem 2.4.** *Let $\chi \in X^4_{1+}$. The Robin type boundary value problem* (2.10)–(2.11) *is equivalent to LBDIE system* (2.34)–(2.35) *in the following sense:*

(i) *If a vector-function $u \in H^2(\Omega^+)$ solves the Robin type BVP* (2.10)–(2.11)*, then it is unique and the pair $(u,\varphi) \in H^2(\Omega^+) \times H^{\frac{3}{2}}(S)$ with*

$$\varphi = u^+\,, \tag{2.36}$$

*solves the LBDIE system* (2.34)–(2.35) *and, vice versa.*

(ii) *If a pair $(u,\varphi) \in H^2(\Omega^+) \times H^{\frac{3}{2}}(S)$ solves the LBDIE system* (2.34)–(2.35)*, then it is unique and the vector-function $u$ solves the Robin type BVP* (2.10)–(2.11)*. and relation* (2.36) *holds.*

## 3. Invertibility of the LBDIO corresponding to the Robin type BVP

From Theorem 2.4 it follows that the LBDIE system (2.34)–(2.35), which has a special right hand side, is uniquely solvable in the class $H^2(\Omega^+, A) \times H^{3/2}(S)$. Here we investigate Fredholm properties of the localized boundary-domain integral operator generated by the left hand side expressions in (2.34)–(2.35) and show that it is invertible in appropriate functional spaces.

The LBDIE system (2.34)–(2.35) with an arbitrary right hand side vector-functions from the space $H^2(\Omega^+) \times H^{1/2}(S)$ can be written as

$$(\mathbf{I}+\mathbf{K})\ell_0 u + W(\varphi) + V(\beta\varphi) = F_1 \quad \text{in} \quad \Omega^+, \tag{3.1}$$

$$(T\mathbf{K})^+\ell_0 u + \mathcal{L}(\varphi) + (\mathbf{d}-\mathbf{I})\beta\varphi + \mathcal{W}'(\beta\varphi) = F_2 \quad \text{on} \quad S, \tag{3.2}$$

where $F_1 \in H^2(\Omega^+)$ and $F_2 \in H^{1/2}(S)$.

Denote

$$\mathbf{B} := \mathbf{I}+\mathbf{K}. \tag{3.3}$$

The principal homogeneous symbol matrix of the operator $\mathbf{B}$ reads as (see (2.29))

$$\mathfrak{S}(\mathbf{B})(y,\xi) = |\xi|^{-2}A(y,\xi) \quad \text{for} \quad y \in \overline{\Omega^+}, \quad \xi \in \mathbb{R}^3 \setminus \{0\}. \tag{3.4}$$

The entries of the matrix $\mathfrak{S}(\mathbf{B})(y,\xi)$ are even rational homogeneous functions of order 0 in $\xi$. Moreover, due to (2.2) the matrix $\mathfrak{S}(\mathbf{B})(y,\xi)$ is positive definite,

$$\big(\mathfrak{S}(\mathbf{B})(y,\xi)\zeta,\zeta\big) \geq c_1\,|\zeta|^2 \quad \text{for all} \quad y \in \overline{\Omega^+}, \quad \xi \in \mathbb{R}^3 \setminus \{0\} \ \text{ and } \zeta \in \mathbb{C}^3.$$

Consequently, $\mathbf{B}$ is a strongly elliptic pseudodifferential operator of zero order (i.e., a singular integral operator) and the partial indices of factorization of the symbol (3.4) equal to zero (cf. [28], [3], [5]).

We need some auxiliary assertions in our further analysis. To formulate them, let $y_0 \in \partial\Omega^+$ be some fixed point and consider the frozen symbol $\mathfrak{S}(\mathbf{B})(y_0,\xi) \equiv \mathfrak{S}(\mathbf{B})(\xi)$. Further, let $\widehat{\mathbf{B}}$ denote the pseudodifferential operator with the symbol

$$\widehat{\mathfrak{S}}(\mathbf{B})(\xi',\xi_3) := \mathfrak{S}(\mathbf{B})\big((1+|\xi'|)\omega,\xi_3\big) \quad \text{with}$$

$$\omega = \frac{\xi'}{|\xi'|}, \quad \xi = (\xi',\xi_3), \quad \xi' = (\xi_1,\xi_2).$$

The principal homogeneous symbol matrix $\mathfrak{S}(\mathbf{B})(\xi)$ of the operator $\widehat{\mathbf{B}}$ can be factorized with respect to the variable $\xi_3$,

$$\mathfrak{S}(\mathbf{B})(\xi) = \mathfrak{S}^{-}(\mathbf{B})(\xi)\ \mathfrak{S}^{+}(\mathbf{B})(\xi), \tag{3.5}$$

where

$$\mathfrak{S}^{\pm}(\mathbf{B})(\xi) = \frac{1}{\xi_3 \pm i\,|\xi'|}\, A^{\pm}(\xi',\xi_3),$$

$A^{\pm}(\xi',\xi_3)$ are the "plus" and "minus" polynomial matrix factors of the first order in $\xi_3$ of the positive definite polynomial symbol matrix $A(\xi',\xi_3) \equiv A(y_0,\xi',\xi_3)$ (see [19], [20], [21]), i.e.

$$A(\xi',\xi_3) = A^{-}(\xi',\xi_3)\ A^{+}(\xi',\xi_3) \tag{3.6}$$

with $\det A^{+}(\xi',\tau) \neq 0$ for $\operatorname{Im}\tau > 0$ and $\det A^{-}(\xi',\tau) \neq 0$ for $\operatorname{Im}\tau < 0$. Moreover, the entries of the matrices $A^{\pm}(\xi',\xi_3)$ are homogeneous functions in $\xi = (\xi',\xi_3)$ of order 1.

Denote, by $a^{\pm}(\xi')$ the coefficients at $\xi_3^3$ in the determinants $\det A^{\pm}(\xi',\xi_3)$. Evidently,

$$a^{-}(\xi')\,a^{+}(\xi') = \det\ A(0,0,1) > 0 \quad \text{for} \quad \xi' \neq 0. \tag{3.7}$$

It is easy to see that the factor-matrices $A^{\pm}(\xi',\xi_3)$ have the following structure

$$\big[A^{\pm}(\xi',\xi_3)\big]^{-1} = \frac{1}{\det A^{\pm}(\xi',\xi_3)}\,[\,p^{\pm}_{ij}(\xi',\xi_3)\,]_{3\times 3}, \tag{3.8}$$

where $[\,p^{\pm}_{ij}(\xi',\xi_3)\,]_{3\times 3}$ are the matrix of co-factors corresponding to the matrix $A^{\pm}(\xi',\xi_3)$. They can be written in the form

$$p^{\pm}_{ij}(\xi',\xi_3)=c^{\pm}_{ij}(\xi')\,\xi_3^2+b^{\pm}_{ij}(\xi')\,\xi_3+d^{\pm}_{ij}(\xi'). \tag{3.9}$$

Here $c^{\pm}_{ij}$, $b^{\pm}_{ij}$ and $d^{\pm}_{ij}$, $i,j=1,2,3$, are homogeneous functions in $\xi'$ of order 0, 1, and 2, respectively.

Denote by $\Pi^+$ the Cauchy type integral operator

$$\Pi^+(f)(\xi)=\frac{i}{2\pi}\lim_{t\to 0+}\int\limits_{-\infty}^{+\infty}\frac{f(\xi',\eta_3)\,d\eta_3}{\xi_3+i\,t-\eta_3},\quad f\in S(\mathbb{R}^3), \tag{3.10}$$

where $\xi=(\xi',\xi_3)$, $\xi'=(\xi_1,\xi_2)$.

The following lemmata hold (see [14]).

**Lemma 3.1.** *Let $\chi\in X^k_{1+}$ with integer $k\geqslant s+2$ and let $\ell_0$ be the extension operator by zero from $\mathbb{R}^3_+$ onto the half-space $\mathbb{R}^3_-$. The operator*

$$r_{_{\mathbb{R}^3_+}}\widehat{\mathbf{B}}\ell_0\ :\ H^s(\mathbb{R}^3_+)\to H^s(\mathbb{R}^3_+)$$

*is invertible for all $s\geq 0$, where $r_{_{\mathbb{R}^3_+}}$ is the restriction operator to the half-space $\mathbb{R}^3_+$.*

*Moreover, for $f\in H^s(\mathbb{R}^3_+)$ with $s\geq 0$, the unique solution of the equation*

$$r_{_{\mathbb{R}^3_+}}\widehat{\mathbf{B}}\ell_0 u=f, \tag{3.11}$$

*can be represented in the following form*

$$u_+:=\ell_0 u=\mathcal{F}^{-1}\Big\{[\widehat{\mathfrak{S}}^+(\mathbf{B})]^{-1}\Pi^+\Big([\widehat{\mathfrak{S}}^-(\mathbf{B})]^{-1}\mathcal{F}(\ell f)\Big)\Big\}, \tag{3.12}$$

*where $\ell f\in H^s(\mathbb{R}^3)$ is an arbitrary extension of $f$ onto the whole space $\mathbb{R}^3$.*

**Lemma 3.2.** *Let the factor matrix $A^+(\xi',\tau)$ be as in* (3.6)*, and $a^+$ and $c^+_{ij}$ be as in* (3.7) *and* (3.9) *respectively. Then the following equality holds*

$$\frac{1}{2\pi i}\int\limits_{\gamma^-}\big[A^+(\xi',\tau)\big]^{-1}d\tau=\frac{1}{a^+(\xi')}\big[\,c^+_{ij}(\xi')\,\big]_{3\times 3}, \tag{3.13}$$

*and*

$$\det\,[\,c^+_{ij}(\xi')\,]_{3\times 3}\neq 0\quad \text{for}\quad \xi'\neq 0. \tag{3.14}$$

*Here $\gamma^-$ is a contour in the lower complex half-plane enclosing all the roots of the polynomial* $\det A^+(\xi',\tau)$ *with respect to $\tau$.*

It is well known that the differential operator $T(x,\partial_x)$ covers the operator $A(x,\partial_x)$ on the boundary $S$ (see, e.g., [1], [4], [27], [31]), i. e., the problem

$$A\Big(\xi',i\frac{d}{dt}\Big)\,v(\xi',t)=0,\qquad 0<t<\infty, \tag{3.15}$$

$$T\Big(\xi',i\frac{d}{dt}\Big)\,v(\xi',t)\Big|_{t=0}=0 \tag{3.16}$$

has only the trivial solution in the Schwartz space $\mathcal{S}(\mathbb{R}_+)$ of infinitely smooth, rapidly decreasing vector-functions at infinity. Here $A(\xi',\xi_3):=A(y_0,\xi',\xi_3)$ and $T(\xi',\xi_3):=T(y_0,\xi',\xi_3)$ correspond respectively to the "frozen" differential and co-normal operators at the point $y_0\in\partial\Omega^+$.

The above covering condition and Lemma 3.2 implies the following assertion.

**Lemma 3.3.** *Let $\gamma^-$ be the same as in Lemma 3.2. The matrix*

$$\int\limits_{\gamma^-} T(\xi',\tau)\big[A^+(\xi',\tau)\big]^{-1}d\tau \tag{3.17}$$

*is non-degenerated for all $\xi'\neq 0$.*

Now, with the above auxiliary results in hand, we can investigate the invertibility of the localized boundary-domain integral operator generated by the left hand side expressions in the system (3.1)–(3.2). Denote this operator by $\mathfrak{R}$,

$$\mathfrak{R}:=\begin{bmatrix} r_{_{\Omega^+}}\mathbf{B}\ell_0 & -r_{_{\Omega^+}}W+r_{_{\Omega^+}}V\beta\\ (T\mathbf{K})^+\ell_0 & \mathcal{L}+(\mathbf{d}-\mathbf{I})+\mathcal{W}'\beta \end{bmatrix}.$$

Applying the local principal technique (cf., e.g. [16], §19 and §22) we investigate Fredholm properties of the operator $\mathfrak{R}$ and prove the following basic theorem.

**Theorem 3.4.** *Let a cut-off function $\chi\in X^{\infty}_{1+}$, $r\geq 1$, and the following condition be satisfied*

$$\det T_t(\xi',-i\,|\xi'|)\equiv\det T(t\,\xi',-i\,|\xi'|)\neq 0 \tag{3.18}$$
$$\textit{for all}\ \ \xi'\neq 0\quad \textit{and}\quad \textit{for all}\ \ t\in[0,\,1],$$

*where the matrix $T_t$ is defined in* (2.7). *Then the operator*

$$\mathfrak{R}\ :H^{r+1}(\Omega^+)\times H^{r+1/2}(S)\to H^{r+1}(\Omega^+)\times H^{r-1/2}(S) \tag{3.19}$$

*is invertible.*

**Corollary 3.5.** *Let a cut-off function $\chi\in X^4_{1+}$ and the condition* (3.18) *be fulfilled. Then the operator*

$$\mathfrak{R}\ :\ H^2(\Omega^+)\times H^{3/2}(S)\to H^2(\Omega^+)\times H^{1/2}(S)$$

*is invertible.*

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